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R. J. FRITZ
Consulting Engineer,
General Electric Co.,
Schenectady, N. Y.
Mem. ASME

The Effect of Liquids on the Dynamic Motions of Immersed Solids

It is known that the presence of liquids can significantly affect the dynamic motions of immersed solids. This paper proposes a method for evaluating fluid forces for use in the dynamic analysis of moving systems in which solid bodies are completely immersed in incompressible, frictionless fluids. A damping parameter is suggested to determine whether a fluid system may be considered frictionless. The incompressibility requirement is also discussed. Experimental data are cited to support the proposed method. Formulas for hydrodynamic masses are tabulated.

Introduction

Whenever solids move in contact with liquids, the liquids must be displaced to accommodate these motions. Fluid pressures are generated as a result. Fluid forces occur on these solids due to the integrated effect of these pressures. In this paper the case of moving solids completely immersed in frictionless, incompressible liquids is considered. In this case, the fluid force is usually proportional to the relative accelerations of the moving solids, and therefore gives rise to an effective or hydrodynamic mass. Where the liquids must flow dynamically in small passages, the hydrodynamic masses may be many times larger than the solid masses, even though the solids may be of large specific gravity. For such systems, dynamic analyses of the solid motions must consider the presence of the liquids in order to provide meaningful results. It is expected that the results of this paper would be found useful in the dynamic analysis of nuclear reactor and steam generator internals subjected to seismic shock as well as in the dynamic analysis of some fluidic devices, including fluidic shock absorbers.

The concept of the hydrodynamic mass has been described by Stokes [1],[1] Lamb [2], Birkhoff [3], Patton [4], and others. These reports have generally considered the motion of a single body in a fluid. In this paper, existing information, particularly from Lamb [2], is applied to the dynamic analysis of systems with more than a single solid completely immersed in a liquid. The plan of presentation will be (a) analysis of two-body motions with liquid coupling, (b) theory of multiple-body motion with liquid coupling, (c) experimental data on two-body motions with liquid coupling, and (d) discussion of analysis of multiple-degrees-of-freedom systems with hydrodynamic effects.

[1] Numbers in brackets designate References at end of paper.

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Two-Body Motions With Fluid Coupling

Consider the case of two long concentric cylinders separated by a liquid annulus, see Fig. 1. The inner cylinder of radius a is surrounded by an outer concentric cylindrical container of inner radius b. The length of the annulus is L where L is much greater than b. The outer cylinder is assumed to have a velocity $\dot{x}_2$ and the inner cylinder $\dot{x}_1$. The relative displacement $x_2 - x_1$ is assumed small compared to $b - a$. A velocity potential ϕ may be defined (similar to Lamb [2, p. 76] who considers single cylinder motion):

$$V_r = -\frac{\partial \phi}{\partial r}; \quad V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{1}$$

where

V_r = radial fluid velocity
V_θ = tangential fluid velocity

The fluid is considered frictionless and to be at rest when the cylinders are at rest. Under such conditions the fluid is irrotational and ϕ will be single-valued. The boundary conditions are:

$$-\frac{\partial \phi}{\partial r} = \dot{x}_1 \cos\theta \quad \text{at} \quad r = a \tag{2}$$

$$\frac{\partial \phi}{\partial r} = \dot{x}_2 \cos\theta \quad \text{at} \quad r = b \tag{3}$$

The continuity equation is

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{4}$$

A form of solution was assumed

$$\phi = f(r) \cos\theta \tag{5}$$

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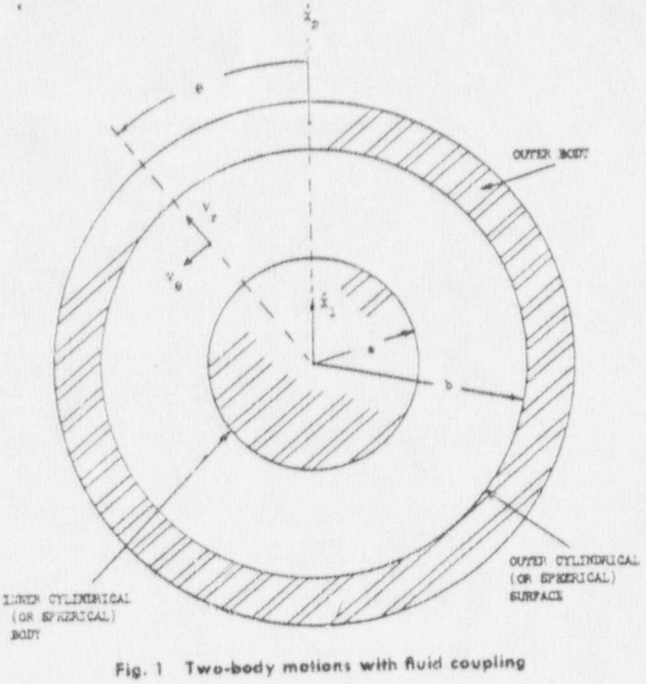


Fig. 1 Two-body motions with fluid coupling

From equations (4) and (5)

$$r^2 f'' + r f' - f = 0 \qquad (6)$$

where the prime indicates differentiation with respect to r.

The solution of this equation is reasonably straightforward. The final solution is

$$V_r = \left(\frac{B}{r^2} - A\right)\cos\theta \qquad (7)$$

$$V_\theta = \left(\frac{B}{r^2} + A\right)\sin\theta \qquad (8)$$

where

$$B = \frac{b^2 a^2}{b^2 - a^2}(\dot{x}_1 - \dot{x}_2) \qquad (9)$$

$$A = \frac{\dot{x}_1 a^2 - \dot{x}_2 b^2}{b^2 - a^2} \qquad (10)$$

Since the velocity of each fluid particle is uniquely determined by the generalized variables of motion x_1 and x_2 then Lagrange's equations of motion will apply. A frictionless, incompressible fluid in irrotational motion will possess no potential energy and may be called an inertial Lagrangian system. The fluid reaction force F_{fi} in such a system is given by Lagrange's equation,

$$F_{fi} = -\frac{d}{dt}\frac{\partial T_f}{\partial \dot{x}_i} + \frac{\partial T_f}{\partial x_i} \qquad (11)$$

where x_i are the generalized coordinates of motion and T_f is the fluid kinetic energy. In this paper x_i will generally be the translational motion of a solid body (body i), and F_{fi} will be the fluid reaction force on that solid body.

It is reasonable to neglect the contribution of the last term in equation (11) if the solid motions are assumed to be small with respect to fluid channel thicknesses. Such an assumption is made in this paper. Lamb [2] considers a few cases of single-body motion including the last term. Neglecting the last term of equation (11), the fluid reaction force is

$$F_{fi} = -\frac{d}{dt}\left(\frac{\partial T_f}{\partial \dot{x}_i}\right) \quad \text{(approximately)} \qquad (12)$$

The fluid kinetic energy is

$$T_f = \int_a^b \int_0^{2\pi} \frac{1}{2}\rho r L dr d\theta (V_r^2 + V_\theta^2) \qquad (13)$$

From equations (7), (8), (12), and (13)

$$F_{f1} = -M_H \ddot{x}_1 + (M_1 + M_H)\ddot{x}_2 \qquad (14)$$

$$F_{f2} = (M_1 + M_H)\ddot{x}_1 - (M_1 + M_2 + M_H)\ddot{x}_2 \qquad (15)$$

where F_{f1} and F_{f2} are the fluid reaction forces on the inner and outer cylinders, respectively, and

$M_1 = \pi a^2 L\rho$ = mass of fluid displaced by the inner cylinder

$M_2 = \pi b^2 L\rho$ = mass of fluid that could fill the outer cylindrical cavity in the absence of the inner cylinder

$$M_H = M_1 \frac{b^2 + a^2}{b^2 - a^2}$$

For the case of concentric spheres separated by a frictionless incompressible fluid (see Fig. 1), the fluid forces that result from a similar analysis are also given by equations (14) and (15) where F_{f1} and F_{f2} are the fluid reaction forces on the inner and outer spheres, respectively, and

$M_1 = \frac{4}{3}\pi a^3\rho$ = mass of fluid displaced by inner sphere

$M_2 = \frac{4}{3}\pi b^3\rho$ = mass of fluid that could fill the outer spherical cavity in the absence of the inner sphere

$$M_H = \frac{M_1}{2}\frac{b^3 + 2a^3}{b^3 - a^3}$$

Synthesis of Fluid Forces for Two-Body Problem

Equations (14) and (15) may be developed in a more general way. Consider the case where fluid motion is determined by the motion of immersed solids. Similar to Lamb [2, p. 188], the fluid kinetic energy is taken as a quadratic function

$$2T_f = A_{11}\dot{x}_1^2 + A_{22}\dot{x}_2^2 + \cdots + 2A_{12}\dot{x}_1\dot{x}_2 + \cdots \qquad (18)$$

or in matrix form

$$2T_f = \dot{x}^T A \dot{x} \qquad (19)$$

where $\dot{x}$ is a column vector and A is a square matrix. Since any quadratic form can be expressed in terms of a symmetric matrix [5] the mass matrix A may be considered symmetric so that $A_{ij} = A_{ji}$. For the two-body problem

$$2T_f = A_{11}\dot{x}_1^2 + 2A_{12}\dot{x}_1\dot{x}_2 + A_{22}\dot{x}_2^2 \qquad (20)$$

From equations (12) and (20)

$$F_{f1} = -A_{11}\ddot{x}_1 - A_{12}\ddot{x}_2 \qquad (21)$$

$$F_{f2} = -A_{12}\ddot{x}_1 - A_{22}\ddot{x}_2 \qquad (22)$$

where again, F_{f1} and F_{f2} are the fluid reaction forces on solid bodies 1 and 2. The coefficients A will now be determined. Assume for this example that body 2 surrounds body 1, similar to the condition of the problem above for the cylinders and spheres. Now equations (21) and (22) are generally true for all values of $\ddot{x}_1$ and $\ddot{x}_2$. If $\ddot{x}_1 = \ddot{x}_2$, then the fluid acceleration is $\ddot{x}_2$ at every point in an incompressible fluid and a pressure gradient exists throughout the fluid due to the fluid inertia,

$$-\frac{\partial P}{\partial x_i} = \rho\ddot{x}_2 \qquad (23)$$

This pressure distribution gives rise to a buoyancy force of an Archimedes type, so that

$$F_{f1} = -(A_{11} + A_{12})\ddot{x} = M_1\ddot{x}_2 \tag{24}$$

$$F_{f2} = -(A_{12} + A_{22})\ddot{x}_2 = -M_2\ddot{x}_2 \tag{25}$$

from which

$$A_{11} + A_{12} = -M_1 \tag{26}$$

$$A_{12} + A_{22} = M_2 \tag{27}$$

and

M_1 = the mass of fluid displaced by the inner body

M_2 = the mass of fluid that would fill the body 2 in the absence of the inner body

Equations (26) and (27) provide two relations. To evaluate the three unknowns A_{11}, A_{12}, A_{22}, a third relation is needed. Assume the containing body 2 to be static, $\ddot{x}_2 = 0$. From equation (21)

$$F_{f1} = -A_{11}\ddot{x}_1 = -M_H\ddot{x}_1 \qquad \text{(defines } M_H\text{)} \tag{28}$$

As indicated, equation (28) defines the term M_H. M_H may be evaluated by assuming the body 1 to have a velocity $\dot{x}_1$ and by the continuity of flow, the fluid velocity distribution may also be evaluated. The fluid force may be evaluated using the conservation of momentum or by using equation (12), which results in

$$M_H = \frac{2T_f}{\dot{x}_1^2} \qquad (\text{for } \dot{x}_2 = 0) \tag{29}$$

where T_f is the fluid kinetic energy. Since the momentum relation will give the fluid pressure which must be integrated to obtain a fluid force on an immersed body, the use of equation (29) will usually be simpler. From equations (21), (22), (26), (27), and (28), equations (14) and (15) follow. Thus, these relations which were derived from basic fluid mechanics are also obtainable by the method of synthesis described above.

The data in Table 1 are typical of some available information giving hydrodynamic mass relations where a single body is in motion and is surrounded either by an unbounded fluid initially at rest or by a static container. By use of the above procedure, these tabulated data are transformable into hydrodynamic mass relations where the single body is either surrounded by a moving container whose dimensions are large compared to the single body for the cases where a single body is shown in Table 1 or where the outer surface for Cases 8, 9, 10, 11, and 14 may be considered in motion.

The reader may note in equation (14) that when $\ddot{x}_1 = \ddot{x}_2$, the hydrodynamic mass M_1 is the displaced mass arising from buoyancy. The hydrodynamic mass M_H in equation (14) is associated with relative motion and may be considered an inertial squeeze-film effect. In general, the hydrodynamic mass can be considered to consist of these buoyancy and inertial squeeze-film components.

Multiple-Body Motions With Fluid Coupling

Where many bodies are immersed in a frictionless, incompressible liquid and are coupled by this liquid, the method of synthesis described above for the two-body problem should facilitate the determination of the hydrodynamic forces. This method is summarized for the multiple-body problem. From equations (19) and

$$-F_{fi} = A\ddot{x}_i \tag{30}$$

where F_f and $\ddot{x}$ are $(n \times 1)$ column vectors and A is a square symmetric matrix $(n \times n)$. F_{fi} is the vector component of fluid force on each solid body where $\ddot{x}_i$ is the instantaneous acceleration of each solid body. To determine the components of A, we observe that since equation (30) must be generally true, it must be true for any generalized input vector $\ddot{x}$. We assume first that all $\ddot{x}$'s are equal. For this condition the fluid forces are usually easily determinable, similar to the two-body case already described. With these fluid forces determinable, n equations are established involving the components of A. There are $n(n + 1)/2$ components of A which must be determined. The remaining equations may be established by setting all $\ddot{x}_i = 0$ except one, $\ddot{x}_j$, and letting $j = 1$ to n. The values of the fluid forces are most easily determined if only one body at any one time is allowed to move. It is suggested that these fluid forces would be determined from the continuity of flow and by use of equation (12). Although it is difficult to predict all the possible configurations that may be met in practice, it is suggested that in solving the continuity equation, some method of series and parallel flow impedances might be considered, analogous to an electric network analysis. Following this prescription, the components of the fluid mass matrix A in equation (30) are determined. These fluid forces are then considered along with other forces present, to arrive at the complete dynamic solution.

For the multiple-body problem the analyst may find it more convenient to synthesize the dynamic problem by considering the response of single channels. The pressure distribution of these channels can be written in terms of entrance and exit fluid velocities and channel wall motions determined by the motion of immersed solids. By considering continuity and momentum or continuity and Lagrange's equation, a series of equations result. The pressure distributions are then considered as elements of dynamic force generation in the equations of motion of the solids. An eigenvalue problem results, which can be solved to develop the solution for frequency and deflectional responses, given the necessary boundary conditions. In many cases, it may be necessary for a fluid specialist to work with the dynamics specialist to develop the solution for the complex fluid-solid problem.

In some cases, fluid compressibility acts in conjunction with fluid inertiance or hydrodynamic mass to cause frequency modes largely due to the fluid. An example is a Helmholz resonator formed by a tubesheet vibrating relative to an adjacent plenum. Oscillation of the tubesheet must be accompanied by displacement of the fluid. This displacement can be accommodated by the compressibility of the adjacent fluid and by the flow through the tubes which causes an inertiance effect. The author developed equations of motion for such systems by first assuming that compressibility was so low that only tube flow was important. The methods of analysis of this paper were then used to develop dynamic equations. Then, the tube flow impedance was assumed to be so high that compressibility effects predominated. In such a case it was easy to write the equation for a fluid spring. It was then an easy step to write a continuity requirement considering both effects. Another example of such a Helmholz resonator which interacts with the mechanical system occurs in the analysis of violins.

The Effect of Fluid Damping

The preceding analysis assumes a frictionless fluid. The engineering designer must have some guidance to judge when a fluid may be considered frictionless. Some guidance is presented here.

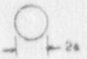
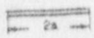
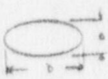
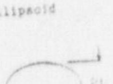
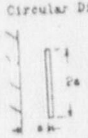
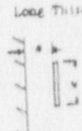
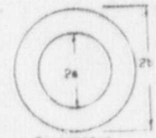
The frictional pressure drop is assumed to be based on the Darcy friction factors, obtained from steady-flow data,

$$\Delta P = \frac{fL}{D_H}\frac{V^2}{2}\rho \tag{31}$$

where

ΔP = frictional pressure drop
f = Darcy friction factor
L = length of channel
V = channel velocity, assumed uniform in channel
ρ = fluid mass density

Table 1 Hydrodynamic mass relations (where only one body is given, it is assumed that the dimensions of other surrounding bodies are large compared to the given body; small displacements are assumed)

Item	Direction of Motion	Hydrodynamic Mass	Ref.
1. Long Cylinder 	↕	$\pi\rho a^2$, per unit length	1
2. Thin Strip 	↕	$\pi\rho a^2$, per unit length	4
3. Rectangular Prism  Length L	↕	a/b — $K = \frac{M_H}{\pi\rho a^2 L}$ ∞ — 1 10 — 1.14 5 — 1.21 2 — 1.36 1 — 1.51 1/2 — 1.70 1/5 — 1.98 1/10 — 2.23	4
4. Diamond Section 	↕	a/b — $K = \frac{M_H}{\pi\rho a^2}$, per unit length 2 — .85 1 — .76 1/2 — .67 1/5 — .61	4
5. Elliptical Disk 	Normal to Disk	b/a — $K = \frac{M_H}{\frac{\pi}{[illegible]}\rho\,[illegible]}$ ∞ — 1.[illegible] 14.3 — .991 6.00 — .964 3.00 — .900 1.50 — [illegible] [illegible] — .637	4
6. Rectangular Plates 	Normal to Plate	b/a — $K = \frac{M_H}{\pi\rho\,[illegible]\,b}$ 1 — .478 2 — .840 3 — 1.000 ∞ — 1.000	4
7. Ellipsoid 	Vertical ↕ Axial ↔	$M_H = K\,4/3\,\pi\rho\,ab^2$ a/b — K_{axial} — $K_{vertical}$ 1.00 — .500 — .500 2.00 — .209 — .702 3.00 — .121 — .804 5.00 — .059 — .895 [illegible] — .029 — .945 ∞ — 0 — 1.000	4
8. Long Cylindrical Annulus of Length h 	Transverse to Axis	$\frac{b^2 + a^2}{b^2 - a^2}\rho\pi a^2 h$ $\frac{a}{c}\rho\pi a^2 h \quad (a \gg c = b - a)$	
9. Short Cylindrical Annulus - Axial Flow Annual Length h	Transverse to Axis	$\frac{\pi\rho h^3 a}{12c}$ $c = b - a$ $c \ll a$ Thin annulus discharges axially into plena at each end	
10. Finite Length Thin Annulus Combination of Items 8 and 9	Transverse to Axis	$M_H = \frac{a}{c}\,\frac{\rho\pi a^2 h}{1 + 12\frac{a^2}{b^2}}$ This approximate formula checks within 0 to 12% higher than the three dimensional solution of Ref. 12 for a/c = 9	
11. Long Thin Annulus Surrounding Rotating Inner Cylinder	Transverse to Axis	$\frac{\pi\rho a^3 h}{4c}$ (at critical speed) c = annulus thickness a = annulus radius, $a \gg c$ h = annulus length, $h \gg a$	
12. Circular Disk 	↔	$\frac{\pi\rho a^4}{[illegible]}$	
13. Long Thin Strip  Length L, $L \gg w$	↔	$\frac{\rho w^3 L}{[illegible]}$	
14.  Concentric Spherical Annulus	↔	$\frac{2}{3}\pi a^3\rho\,\frac{b^3 + 2a^3}{b^3 - a^3}$	

D_H = hydraulic diameter = $2c$ for a channel with parallel walls and separation thickness c, which will be used in this exposition

The frictional energy is

$$E_f = \int \Delta P A V dt = \frac{fL\rho A}{4c}\int V^3 dt \tag{32}$$

where

A = fluid area

t = time

Assume the fluid velocity to be cyclic,

$$V = V_0 \sin \omega t \tag{33}$$

Using equations (33), equation (32) may be integrated over a half-cycle to give

$$E_f = \frac{fL\rho A V_0^3}{3\omega c} \tag{34}$$

If the fluid damping force were linear and equal to bV, then the energy E_L over a half-cycle is

$$E_L = \int_0^\pi bV^2 dt = \frac{bV_0}{\omega}\int_0^\pi \sin^2 \omega t\, d\omega t = \frac{\pi b V_0^2}{2\omega} \tag{35}$$

The effective linear damping coefficient b may be determined by setting $E_L = E_f$, with the result

$$b = \frac{2}{3\pi} \frac{f V_0 \rho L A}{c} \quad (36)$$

A parameter 2ξ may be defined where

$$2\xi = \frac{b}{M_f \omega} \quad (37)$$

and M_f = fluid mass. 2ξ is the ratio of the damping impedance to the inertial impedance. ξ is similar to a fraction of critical damping for a one-degree-of-freedom system with linear damping b, mass M_f, and natural frequency ω. From equations (36) and (37) and with $M_f = \rho L A$,

$$2\xi = \frac{2}{3\pi} \frac{f V_0}{\omega c} \quad (38)$$

Further, if $V_0 = \omega x_0$, from equation (38)

$$2\xi = \frac{2}{3\pi} \frac{f x_0}{c} \quad \text{(turbulent flow)} \quad (39)$$

Equation (39) defines a dimensionless number, a damping parameter, that should provide a reasonable measure of the ratio of fluid friction to fluid inertia. We recall that in equation (39)

f = the Darcy friction factor for turbulent flow
x_0 = the distance that the fluid moves in an oscillatory cycle (amplitude of sinusoidal motion)
c = the fluid channel spacing

A similar analysis for laminar flow through a parallel plate channel gives the result

$$2\xi = \frac{12\nu}{\omega c^2} \quad \text{(laminar flow)} \quad (40)$$

where

ν = fluid kinematic viscosity
ω = angular frequency of oscillatory motion
c = fluid channel spacing

If the concept of this damping parameter 2ξ is reasonable, then the assumption of a frictionless fluid must require that 2ξ must be much smaller than 1. The quantity 2ξ will later be calculated for some test cases.

Fluid Compressibility

The preceding analysis assumed an incompressible fluid. Where the possibility of a fluid spring is present, it is usually a straightforward calculation to determine if the volume storage of a fluid spring will affect the continuity balance. The application of this paper is further restricted to cases of small Mach number (less than about 10 percent) and cases where the flow channel length is small compared to the wave length for propagating vibratory disturbances (less than about 10 percent), in order to avoid the possibility of standing-wave effects.

Comparison to Test Data

Keane's Data. Keane [6] vibrated a circular cantilever tube surrounded by an annular cavity. The empty space was filled with a liquid, which made the test consistent with item 8 of Table 1 (due originally to Stokes [1]). Keane examined flexural vibrations and analyzed the cantilever beam with the added hydrodynamic mass. For one set of tests, Keane's beam length-to-diameter ratio was 17 and b/a (in item 8, Table 1) ranged from 1.2 to 6.5. According to Fig. 4.3.2 of Keane's report, the hydrodynamic mass reduced the natural frequency from 56 cps (air in cavity) to as low as 29 cps (with water in cavity), which seemed to agree exactly with prediction. For other points of his graphed data, the variation between measured natural frequency and predicted natural frequency was typically less than 2 percent.

For b/a = 1.2 and a frequency of 29 cps, the maximum value of the Reynolds number during the vibratory cycle is estimated from Keane's data to be 24,000. Turbulence may be assumed to occur if the Reynolds number is greater than 3000. Thus, the water surrounding the beam can be considered turbulent. Relation (39) gives a value of 2ξ of 0.03, using a friction factor of 0.025. This friction factor is taken from Moody [7]. Since 2ξ = 0.03 is much smaller than 1, the concept of this damping parameter would imply that the fluid could be considered essentially frictionless. The fact that the data on natural frequency agreed so well with theory would tend to validate the assumption of a frictionless fluid. The amplification in Keane's test was about 15 at resonance, which can also be considered as evidence that the overall inertial impedance is considerably greater than the overall damping impedance.

Data of Fritz and Kiss. Fritz and Kiss [8] reported the results of a test on a solid aluminum cylinder flexibly supported within a rigid cylindrical container. The equipment was vibrated on a shake-table. The length-to-diameter of the cylinder was about 1.0. The cylinder was surrounded by a thin annular fluid which was free to flow axially as well as circumferentially. The natural frequency was taken as the frequency at which the vibrational amplitude of the cylinder reached its maximum value with a constant table amplitude. The axial and circumferential hydrodynamic masses were combined as shown in item 10 of Table 1. The natural frequency of the cylinder in air was 35.5 cps. With water surrounding the cylinder, the frequency was reduced to 17.0 cps, which gave very satisfactory agreement with the prediction of 16.9 cps.

In reference [8] the Reynolds number was estimated to be 4800 which was considered turbulent. The value of 2ξ is calculated from equation (38) to be 0.03, using the data from reference [8]: f = 0.04, V_0 = 3.2 fps, $\omega = 2\pi$ (17 cps), c = 0.090 in. Since 2ξ is much less than 1, the assumption of a frictionless fluid should be valid. This is validated in that the use of the hydrodynamic mass concept in reference [8] provided a very

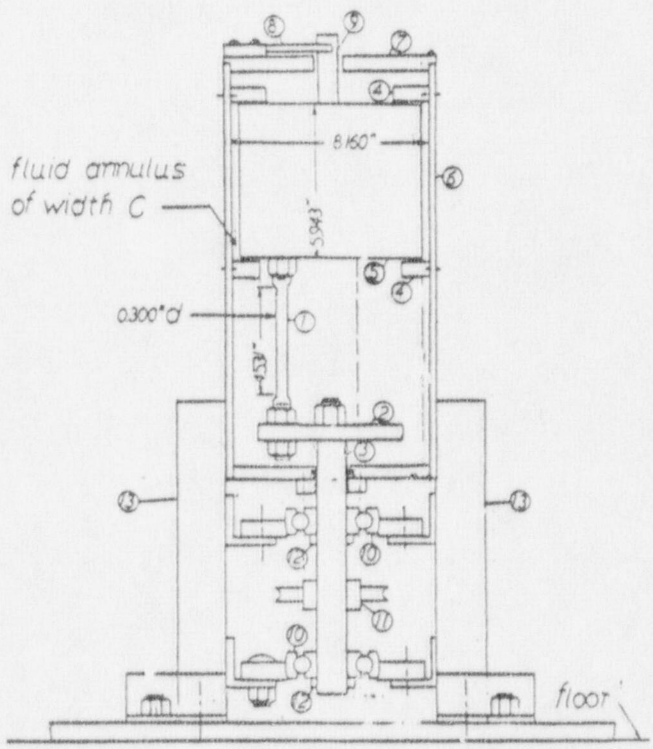


Fig. 2 Vertical section of test equipment

accurate estimate of the natural frequency in the presence of the liquid.

Free Vibration of a Concentric Cylinder. A concentric cylinder assembly shown in Fig. 2 was available for test. An aluminum cylinder (part 5) is flexibly supported by column struts (parts 1). The cylinder is surrounded by an annulus. Fluid leakage from the annulus is limited by close-clearance metal seals, parts 4. The motion of the cylinder was measured by use of a cantilever displacement gage, fitted with strain gages, part 8. The fluid annulus width c was varied by machining the cylinder diameter. Table 2 shows some numerical data from these tests. Fig. 3 shows some typical oscillograph records of the free vibrations taken during these tests. The vibrations with air and the glycerol solution were created by velocity-shocking the cylinder with a large mallet. The vibrations with water and oil were created by causing an initial large deflection.

The hydrodynamic weight was calculated from item 8, Table 1. The hydrodynamic weight was also calculated from the test frequencies. Due to the nature of the equipment it was felt that there was a small amount of leakage past the end seals which would cause the actual hydrodynamic weight to be smaller than the theoretical value. Even though the value of 2ξ was as high as 0.7, the calculated hydrodynamic weight was felt to be in reasonable agreement with the test values.

It is noted that this apparatus was also used to measure the hydrodynamic mass of a thin annulus around a rotating cylinder. The results were published in [11].

General Comments on These Comparisons to Test Data. The above comparisons to test data apply to the effect of the inertial squeeze film and the use of the damping parameter 2ξ. The phenomenon of the inertial squeeze film or, in other words, the virtual mass effect, was first predicted by Stokes [1], and has since been widely accepted. Therefore, confirmation of this effect is not new. However, both references [6] and [8] involve annular fluid spaces where the outer and inner boundaries of the fluid annulus both move. Both references [6] and [8] report forms of fluid motion equations (cf. equation (4.2.15) of [6] and equation (1) of [8]) that are consistent with equation (14) of this paper. These equations for two-body motions presented in this paper connect buoyancy and inertial squeeze-film effects. Therefore, references [6] and [8] are consistent with the model presented in this paper. In references [6] and [8] the buoyancy term implied negligible effects on natural frequency, but did imply a significant effect on the predicted amplifications. The effect on amplification was specifically noted in reference [8].

References [6] and [8] do result in some confirmation of the models proposed in this paper. Admittedly, more confirmation is desirable. However, since the information of this paper is based on basic principles, it is expected that the equations will be accurate for the specified conditions.

Table 2 Calculations of 2ξ, relations (39) or (40), for equipment shown in Fig. 2

Annulus Radius R in.	Annulus Clearance c in.	Kinematic Viscosity ν 10^{-3} in²/sec	Liquid	Natural Frequency cpm	R_{NR} (1)	f	2ξ	Rotor Weight lbs.	Calculated Hydrodynamic Weight lbs.	Hydrodynamic Weight From Test lbs.
4.0	.16	1.52	Water	570	19,000	.086	.07	89	285	180
5.9	.39	1.52	Water	580	65,000	.080	.09	85	100	75
4.0	.25	1.52	Water	425	34,000	.025	.06	86	170	127
4.0	.25	6.2	Glycerol Solution	390	6,000	.036	.06	28	190	150
4.0	.25	150	Oil	320	260	Laminar	.7	28	150	200

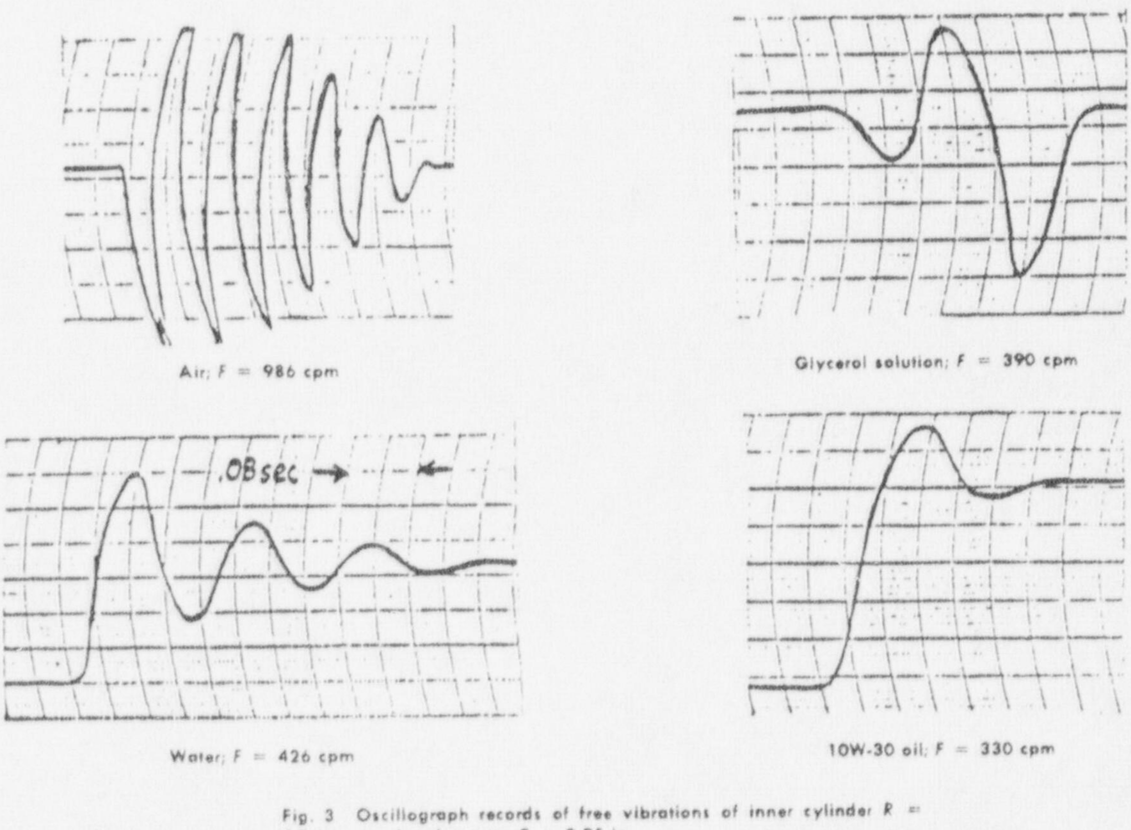
(1) R_{NR} = Reynolds Number = $\frac{VS}{\nu}$ = $\frac{\frac{\Delta}{2}\cdot\frac{R}{c}\cdot 2c}{\nu}$ = $\frac{\omega\Delta R}{\nu}$; where the cylinder deflection is $\Delta/2$ and the fluid displacement is $R/c\left(\frac{\Delta}{2}\right)$, ω is the angular frequency.

(2) Turbulent: $2\xi = \frac{2}{3\pi}\frac{f\,x_0}{c}$; $x_0 = \frac{\Delta}{2}\cdot\frac{R}{c}$ = R/2; f = friction factor taken from L.F. Moody[7] for smooth pipe [illegible]

Laminar: $2\xi = 12\nu/\omega c^2$

Comments of Multi-Degree-of-Freedom Dynamic Analysis

A salient feature of the most widely used multi-degree-of-freedom dynamic analyses of linear systems which are excited by some arbitrary base motion is the transformation of an arbitrary configuration of dynamic components into a decoupled array of simple oscillators which are excited by the base motion



Air; F = 986 cpm

Glycerol solution; F = 390 cpm

Water; F = 426 cpm

10W-30 oil; F = 330 cpm

Fig. 3 Oscillograph records of free vibrations of inner cylinder R = 4.0 in., annular clearance C = 0.25 in.

The motion of the complex array is then related to the motion of the simple oscillators. Many forms of this transformation have been reported in the literature in equations which are in most cases applicable without dynamic coupling, that is, for the case where the mass matrix is diagonal in the dynamic equations. The correct transformations to be used for the case of dynamic coupling, where the mass matrix is nondiagonal, is given by McCalley in reference [9]. The methods of hydrodynamic analysis of this paper generally result in dynamic coupling and should therefore use McCalley's relations (or equivalent) when treating fluid effects in a multi-degree-of-freedom analysis.

Summary

Some available relations are given in Table 1 for hydrodynamic masses for motions of a single solid body fully immersed in a frictionless incompressible fluid. This paper proposes a method of using these results for two-body motions. Where a single body is shown in Table 1, the second body is considered large compared to the single body. For cases 8, 9, 10, 11, and 14 the outer surfaces may be considered in arbitrary motion. Some guidelines are proposed to establish the conditions of frictionless, incompressible flow. The case of motions of multiple immersed solids is considered. Comparisons to test data indicated favorable agreement.

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