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THE APPLICATION OF GAME THEORY TO NUCLEAR MATERIAL ACCOUNTING

Final Report

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ABSTRACT

An approach based upon the theory of games is presented that determines an optimal alarm threshold for detecting unauthorized or deliberate diversion of nuclear material based upon material accounting data. A mathematical model is developed, solved, and applied to a generic nuclear facility. By considering a malevolent diverter as a basic ingredient of the analysis this approach offers advantages over conventional statistical hypothesis testing. The results show that periodic inventories and appropriate interpretation of MUF can provide a high assurance for indicating diversion in a nuclear material safeguards situation. The optimal policy is to select the alarm threshold by a mixed strategy rather than a pre-set single fixed value. Procedures for doing this are presented in the report. With this approach, MUF data by itself may be more useful in indicating possible unauthorized diversion of special nuclear material.

TABLE OF CONTENTS

	Page
ABSTRACT	i
1. Introduction	1
2. The Criterion Function Approach to Analysis	7
3. A Game Theoretic Model Formulation	11
4. Solution for the Fixed Alarm Threshold Model	19
5. Alarm Threshold as a Strategic Variable	35
6. Summary and Conclusions	46
REFERENCES	51
Appendix A - Sensitivity Analysis	A-1
Appendix B - Some Alarm Threshold Models	B-1
Appendix C - Mathematical Proof for Form of Diverter Optimal Strategy	C-1

LIST OF FIGURES

Page

Figures

1. Criterion Function Approach to Analysis.................. 12

2. A Decision Maker's Utility Balance........................ 13

3. Plant Characteristics and Model Parameters 20

4. Optimal Solution for Plant 1, Fixed Alarm Threshold Case.. 22

5. Range of Parameter Values For Sensitivity Analysis 27

6. Value of Game as a Function of Alarm...................... 29

7. Relationship of Alarm to Safeguards Objective............. 30

8. Relationship of Alarm to Recovery Coefficient............. 31

9. Relationship of Alarm to Error Coefficients............... 33

10. Relationship of Alarm to Equal Error Coeffients.......... 34

11. Value of The Game for Alternative Alarm Policies......... 39

12. Optimal Strategy for the Defender........................ 40

13. Optimal Strategy for the Diverter........................ 42

14. Probability Distribution for Alarm Selection............. 43

15. Comparison of Results for Alternative Alarm Policies...... 44

16. Expected Alarm Rates..................................... 45

17. Probability of Alarm..................................... 47

1. Introduction

The Nuclear Materials Accounting Study, which began in July 1976, had as its overall objective the examination of the accounting systems used in safeguarding special nuclear material. The purpose of this examination was to look for improvements in the procedures utilized in nuclear materials accounting so that the overall effectiveness of safeguards can be enhanced. More specifically, the purpose of the study was (a) to directly examine the implications of deliberate diversion on nuclear material accounting (b) to determine the validity of the MUF* concept to establish assurance concerning the possible unauthorized diversion of Special Nuclear Material (SNM) and (c) to provide tools for assessing licensee material accounting safeguards performance requirements.

The safeguards problem, by its very nature, implies an adversary situation in which someone seeks to divert nuclear material, and NRC or the licensee tries to prevent him from succeeding. The theory of games, which developed as a means of modeling just such competitive situations, is a natural candidate for evaluating accounting information. This study is a first step in investigating the feasibility of applying the game theoretic approach to nuclear materials accounting.

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*The term ID (Inventory Difference) has replaced MUF in NRC terminology. However, in order to keep this report consistent with NUREG-0290, we will continue to use the term MUF. There is no difference in meaning between MUF and ID.

The results of the first phase of the study were published in NUREG-0290 [1] in June, 1977. Volume 1 of NUREG-0290 described the preliminary work done on formulating a game theoretic model of materials accounting and applying it to three generic plants. The second phase of the study, reported in this volume, was in part a more detailed game theoretic analysis of one of the plants. A sensitivity analysis of the results of phase I was performed, and a second model was formulated and applied. The study was sponsored by the Test and Evaluation Branch of the Division of Safeguards, U.S. Nuclear Regulatory Commission. Mr. S. Moglewer was the Project Manager for NRC. The prime contractor was Lawrence Berkeley Laboratory. Dr. Melvin Dresher served as a mathematical consultant. This volume can be considered to be a companion volume to Volume I of NUREG-0290. The report presents the results of the game theory evaluation work as of January 31, 1978. (Additional work developed in Phase II is presented in [3] and [4]).

This report (like NUREG-0290), is primarily concerned with decision criteria for taking action based upon statistical information in a nuclear safeguards situation. Nuclear material accounting safeguards in the nuclear industry relies on material balance accounting. This accounting signals the occurrence of losses, if any, and may be the basis for subsequent inspection and recovery actions.

The accounting system keeps track of material inputs and outputs by taking inventory at regular intervals and locations. If there were no measurement errors, process errors, mistakes, or diversion the book inventory and the physical inventory would balance. Due to measuring errors, recording errors, bias in individuals and instruments, etc., the inventory does not usually balance leaving a quantity called MUF (Material Unaccounted For). MUF is a function of the realizations of the many errors involved and would include any diversion that may have occurred during the inventory period.

The decision-making problem is, given a MUF reading, what action should be taken to verify possible theft and/or recover material that may possibly have been diverted. The present practice in the licensed domestic nuclear industry follows NRC regulations which establish inventory periods and limits on measurement accuracy. The latter term is regulated by placing a limit on LEMUF (Limit of Error on MUF) which is defined as two standard deviations of a normal measurement error distribution. NRC has also established guidelines and operating suggestions for appropriate action limits. For instance, when MUF exceeds 2.0 times the LEMUF limit approved for the activity, the facility is shutdown and a clean-out inventory is conducted with an investigation of the cause initiated. The activity normally remains shutdown until the MUF calculated as a result of a clean-out inventory is within 1.5 times the applicable LEMUF limits. The operational procedure at present is to establish a fixed alarm threshold and to take action when MUF exceeds this threshold. A more complete discussion of present practice is presented in [1].

The essential safeguards material accounting decision problem is how to establish the alarm threshold in a manner to satisfy safeguards objectives. If the alarm threshold is too high, a potential diverter may not be detected. If it is too low, there will be an excessive number of false alarms with consequent additional inventory cost burdens and a desensitizing of the safeguards system from a "Cry Wolf" syndrome. Furthermore, as discussed later, if the alarm threshold is set at any value prior to an inventory, a potential diverter could utilize this information for his own advantage.

The decision problem described in the above paragraph is part of a larger class of problems that can be characterized as statistical acceptance sampling in a competitive environment. Sampling is used as an aid to decision-making by drawing inferences from limited data. These inferences depend on the sampling process and the structure of the population being sampled. If the sample is obtained in a competitive environment, the sample or the population or both may have been tampered with to accomplish some objective. Furthermore, the extent of tampering is not known.

The nuclear material accounting problem is an example of sampling in a competitive environment. This is so because a potential diverter may control partially the characteristics of the sample distribution. In the nuclear material accounting case he may control the mean of the MUF distribution. This could be achieved by stealing from an inventory where measurements are taken. The mean of the distribution would be related to the amount stolen. To contend effectively with this kind of non-stochastic problem, it is necessary that the decisionmaker take into account in his statistical acceptance sampling not only the uncertainty due to sampling errors, but also strategies available to a potential competitor for each specific situation. Classical statistical hypothesis testing is not appropriate in a competitive environment since such testing assumes only stochastic situations, i.e., all uncertainties are due to chance and there is no diversion. If statistical hypothesis testing is used, it is necessary to know the distribution of MUF when diversion takes place. A discussion of the limitations of classical statistical hypothesis testing for detecting diversion of nuclear material is presented in [1].

The approach taken by the Nuclear Material Accounting Study and first presented in [1] was based upon the theory of games. This theory represents an approach to decision-making under uncertainty in a competitive environment. The nuclear material accounting problem was formulated as a game between two players - the diverter and the defender. In particular, the MUF problem was analyzed by formulating the following two-person game:

Move 1. Diverter removes X grams of SNM.

Move 2. The defender, upon taking a sample inventory, measures that u grams of SNM are unaccounted for.

Move 3. The defender, knowing u, estimates that y grams of SNM have been diverted. Also, if the alarm threshold has not been pre-set at a fixed value a, the defender would determine whether to alarm or not. (If the alarm threshold has been pre-set, the decision to alarm would be determined simply by whether u is greater than a.)

A payoff function was established representing the decision utilities to the defender. This model was solved in terms of optimal strategies for both players and the value of the game (the value of the payoff function when both players play optimally.) Two cases were examined. In the first case the alarm threshold was pre-set at a fixed value. In the second case the alarm threshold was made a strategic variable for the defender. Both cases were applied to a generic facility representative of a small plutonium fabrication facility (Plant 1 of NUREG-0290, see [1]).

This report describes in detail the rationale for the selection of the specific payoff function, and its relationship to decisions faced by a decision-maker in a safeguards material accounting environment. The optimal strategies for both cases will be presented and the implications of the solution discussed. An appendix will present a sensitivity analysis of the parameters of the model and another appendix will present some simplified alternative alarm models.

Application of the game theoretic approach to establish MUF action limits and alarm thresholds can significantly improve the performance of material accounting for safeguards. This report will present the analysis and arguments to support the preceding statement.

2. The Criterion Function Approach to Analysis

In order to understand how operations research can aid decision-making, we must first understand decision-making. The process of individual decision-making can be characterized as either completely intuitive (based only upon the experience and judgment of the decision-maker) or a combination of intuition and analyses, the combination varying with the nature, information, and difficulty of the problem. There are two basic analytical approaches to aid a decision-maker. First is a simulation approach. This consists of creating a synthetic history of event-sequences, all that the analyst can legitimately evaluate within his computational limitations. The simulation analysis selects the best from the large number of possibilities under consideration. It is not appropriate for optimization; the optimum may not be contained in the set of event sequences under consideration. A more detailed discussion is presented in the text by Quade and Boucher [5].

The second type of analytical method is the criterion function approach. Based upon the objectives of the system about which a decision will be made, specific criteria are established related to the specific decision being made. These criteria are then translated into measures of effectiveness. Finally a mathematical model is selected or developed that applies a suitable criterion to the selected measures of effectiveness. Solution of the model will provide recommendations to the decision-maker by finding the optimum. The criterion function approach can optimize, and it provides an economy and breadth of solution not available by other techniques. Its limitations are the state-of-the-art of the mathematical techniques employed, the ability to model the complexity of the decision-maker's problem, and the availability of data to specify the model.

The selection of a suitable model with which to find an optimum for a specific criterion function is related to the type of information available to the decision-maker. It is generally recognized that the validity of a model's solution is related to the quality of the input data and the model's limiting assumptions. What is not so generally recognized is that the results are also sensitive to the type of model selected to analyze a given problem. Insofar as possible, the model should match the decision framework. There are two basic types of models, deterministic and probabilistic. In a deterministic model all the parameters are known and the solution has a specific optimum. Any departures from this optimum will only penalize the decision-maker insofar as the model represents the real world. The decision-maker completely controls the outcome.

There are two classes of probabilistic models. One treats problems of "decision-making under risk." This occurs when the outcome of the decision is sensitive not only to the decisions of the decision-maker, but also to chance events. The decision-maker knows the probability distribution, or the odds, for the occurrence of the chance events. He knows the risks when he makes a decision. The other class of probabilistic models is "decision-making under uncertainty." The decision-maker does not know the risk when he makes a decision. All situations involving competition belong to this class of models. The outcome of the decision is sensitive to competing decision-makers as well as chance events. Optimal decisions are based upon subjective perceptions of risk by each decision-maker. The nuclear material accounting safeguards problem when unauthorized deliberate diversion may be present belongs to this class of problems.

The model selected for analysis by the Nuclear Material Accounting Study was based upon the theory of games. The nuclear material accounting problem was formulated by representing the interests of the defender and the diverter as being in complete conflict. This is a reasonable formulation for the analysis of unauthorized deliberate diversion from domestic nuclear facilities. The minimax criterion merely indicates that there is an optimal decision policy (or strategy) for the defender even though he does not know what policy (or strategy) the diverter may employ or even the existence of a diverter. Furthermore, the defender by use of this policy can assure himself a certain payoff (in terms of expected value) no matter what the diverter does. A theory of games model represents the basic decision problem for safeguards management when deliberate diversion is possible.

It should be realized that the suitability of any model for aid in decision-making rests upon the accurate representation of the utilities influencing the decision-maker. Utilities are often subjective, particularly aversion to risk, and thus difficult to model faithfully. This process is the art of operations research and requires intimate insight into the problem if effective models are to be developed. However, this difficulty should not preclude serious efforts to develop suitable criterion functions. If the material accounting system is to be optimized, it will be necessary to find the appropriate criterion functions for analysis. The problem cannot be avoided.

The decision criterion is often implicit in the model rather than explicit. The classical statistical hypothesis testing approach is a criterion function model with implicit criteria. The implicit criteria are reflected in the measure of effectiveness such as the false alarm rate, which should be determined by cost-benefit consideration of the utilities of the decision-maker. On the other hand, the game theory approach has the criteria explicitly presented in the payoff function. Figure 1 presents a comparison of these two approaches.

3. A Game Theoretic Model Formulation

Let us consider a decision-maker's utilities for formulation of a game theoretic payoff function. Figure 2 presents the weighing of utilities and disutilities that appear to be representative of the material accounting safeguards problem. The decision-maker, or defender, may consist of more than one person, such as NRC and the plant management. In general, the defender consists of all participants who have the same objective with respect to the payoff. The defender's decisions are: given a MUF reading, whether to alarm or not and what preliminary estimate of unauthorized diversion to make, based upon the MUF reading. This preliminary diversion estimate will influence the defender in the resources he allocates for the post-alarm search. A final estimate of diversion should be based upon not only the results of the post-alarm search, but also relevant information from other safeguards systems (e.g., the physical security system and the material control system) as well as pertinent external intelligence information, police reports, etc. This study, however, is concerned only with the material accounting system. Consequently when we talk about the estimate

CRITERION FUNCTION APPROACH TO ANALYSIS

Statistical Hypothesis Testing Approach		Game Theory Approach
Detect Diversion	SYSTEM OBJECTIVES	Detect Diversion
Detect Theft within the Measurement State-of-the-Art without Major Disruption to the Industry	CRITERIA	Detect Theft and Allocate Resources for Recovery of Material with Minimum Penalty to the Defender and without Major Disruption to the Industry
False Alarm Rate, Probability of Detection	MEASURES OF EFFECTIVENESS	Penalty to the Defender
Normal Distribution for MUF. Dispersion Determined by Measurement Error Only. Mean of Distribution Equal to Zero or Alternatively, Some Specific Amount	CRITERION FUNCTION MODEL	Game Theory Payoff Function

Figure 1.

A DECISION-MAKER'S UTILITY BALANCE

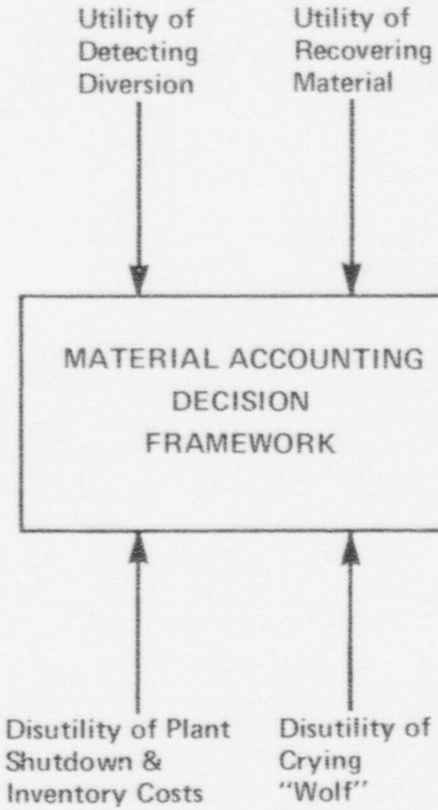

Figure 2.

of the amount diverted, we mean only the preliminary estimate based upon the MUF reading in and of itself.

Our discussion of diversion and the game theoretic model essentially assumes the existence of a diverter. In particular, the defender makes decisions to defend against actions by a diverter. Frequently the defender is uncertain about the existence of a diverter. If this uncertainty can be represented by a given probability distribution function, then the problem is a combination of a statistical problem and a game theoretic problem. Otherwise the problem is a game theoretic problem in which the defender assumes the existence of a diverter which is necessary if the defender is to protect himself against the actions of a possible diverter.

The utilities influencing the defender's decisions are relatively few for the essential material accounting decision. He has the utility of detecting diversion to assist in recovering material and to deter the would-be diverter, thereby preventing theft. This must be balanced against the disutility of material losses and inventory and post-alarm search costs. In addition, there is the disutility or penalty from making an error in the estimate of diversion. This penalty can reflect from desensitizing the system from a high false alarm rate, from adverse publicity concerning errors in the estimate, and numerous other factors. The optimal decision is thus seen to be a trade-off of these utilities.

The decision-maker does not know the outcome of the decision when he makes it since this is influenced by decisions of a possible diverter and also the chance errors in measurement and process systems.

There are many payoff functions that can be assumed with respect to nuclear material accounting actions and decisions. For example, the objective of the defender might be to minimize the loss of nuclear material, or to maximize the probability of detecting diversion, or to minimize the errors in detection, etc. By formulating a payoff in terms of costs and benefits obtained from efficient use of MUF data to detect diversion, we can include the above objectives. These costs and benefits to the defender may be summarized by the following five categories:

1) cost of material diverted
2) cost for search and recovery of diverted material
3) benefit obtained from recovery of diverted material
4) costs associated with search errors
5) cost of alarm, if alarm occurs.

The following payoff is considered to essentially embrace the major elements of the nuclear material accounting decision and will be used as a basis for this study.

The payoff function selected for this analysis is expressed as the penalty to the defender as follows:

Penalty to defender = [Inventory Cost + Search/Recovery Cost] + [Replacement Value of Material Lost] - [Utility of Material Recovered] + [Penalty for Error in the Estimate of Diversion]

This can be represented as follows:

$$M = \beta + cy + x - b \min (x, y) + e|y-x|$$

where:

M = payoff function representing penalty to defender

β = special clean-out inventory cost (if applicable)

x = amount deliberately diverted by diverter

y = estimate by the defender of the amount diverted

cy = recovery search cost

b min (x,y) = value to defender of recovery of the material diverted

$e|y-x|$ = error penalty from a wrong estimate by defender

The diverter may have a goal to divert sufficient material to constitute a credible threat. The defender desires to make decisions so as to minimize his penalties. He generally does not know the diverter's decisions or even the existence of a diverter, but he must take into account that a possible diverter may make decisions in such a fashion as to maximize the defender's penalties.

Let us assume that an amount of material k constitutes a credible threat to the defender. Furthermore assume that the facility under consideration has an amount of material vulnerable to diversion equal to or greater than k.

The defender can either pre-set a fixed value of the alarm threshold, $z=a$, or else he can select a value of the alarm z after the inventory is taken. From the inventory a MUF reading of u is obtained. In this model the actions the defender takes will be based upon the relationship of the MUF reading to the alarm threshold. If $u \geq a$, the defender will close the plant down for a special clean-out inventory, estimate an amount diverted y_2, and take search and recovery actions based upon the estimate y_2. If $u < a$, the defender will estimate an amount diverted y_1 and take more limited search and recovery actions based upon this estimate y_1. It is assumed that the amount diverted is bounded:

$$o \leq x \leq k \ .$$

The value k can be interpreted as the minimum of the plant inventory and the credible threat amount.

In general each component of the vector (z, y_1, y_2) is a function of u, the MUF reading, since each of the strategic variables z, y_1, y_2 are functions (of u). Further, we need consider only those functions which are monotonic non-decreasing in u, since we can show that for any x, non-monotonic functions are dominated by monotonic functions. Thus we need consider only the vector space (z, y_1, y_2) where each component is a choice of a number within a prescribed interval.

The defender desires to estimate y_1 and y_2 (as well as z when applicable) in such a fashion as to minimize M whereas a possible diverter may select x in such a fashion as to maximize M. Since the defender and diverter never cooperate, the problem may be treated as a zero-sum game.

The expected payoff for this game is:

$$M = [c_1 y_1 + x - b_1 \min(y_1, x) + e_1 |y_1 - x|] F(z,x)$$
$$+ [\beta + c_2 y_2 + x - b_2 \min(y_2, x) + e_2 |y_2 - x|] G(z,x)$$

The term F(z,x) represents the probability that the MUF reading u is below the alarm threshold z when an amount x has been diverted.

$$F(z,x) = P(u < z)$$

Also $$G(z,x) = P(u \geq z)$$

and $$G(z,x) = 1 - F(z,x).$$

The defender desires to select a set (z, y_1, y_2) in order to minimize the payoff M not knowing how much material x has been diverted nor the value of MUF, u, that will be obtained (although he does know the form of the distribution, F and G for u). The parameters of this model are c_1, c_2, β, b_1, b_2, e_1, e_2 and k which are site specific or unique to each facility under consideration.

4. Solution for the Fixed Alarm Threshold Model

The model described in the previous section was solved for the case z = a for any value of a and published in NUREG-0290 (A Study of Nuclear Material Accounting), June 1977. This represents the case of the fixed alarm threshold and corresponds to present practice. The model as applied to a generic plant will be briefly reviewed in this section.

The plant to be used for purposes of illustration is a small MOX fabrication plant. This generic facility is similar to an existing small plutonium fabrication facility, and was referred to as Plant 1 in NUREG-0290. The plant characteristics and model parameters are shown in Figure 3. A more complete description is presented in NUREG-0290 [1]. In general, even in the absence of diversion, the distribution of MUF is not normal for any real-world facility. Although measurement errors can be modeled with a normal distribution as characterized by LEMUF, there are other random errors not accounted for by the LEMUF values. These errors could be caused by such facility operations as unmeasured side streams, difficulties in measurement, human errors, adjustments, etc. In order to properly model

PLANT CHARACTERISTICS AND MODEL PARAMETERS

Plant 1 — Small Plutonium Fabrication Facility

Characteristic	Symbol	Value
LEMUF (Bimonthly)	2σ	0.6 Kg
Total Inventory		297 Kg
Throughput (Annual)		864 Kg
Replacement Value of SNM		$10,000 per Kg
Special Inventory Cost		$ 5,000 (dollar units)
Special Inventory Cost	β	1.67 (sigma units)
Variable Search Cost, Small MUF	c_1	.017
Variable Search Cost, Large MUF	c_2	.167
Utility of Recovery, Large MUF	b_2	100
Utility of Recovery, Small MUF	b_1	10
Error Coefficient	e	50
Safeguards Objective	k	10 (sigma units)
Safeguards Objective	k	3 (Kg units)

Figure 3.

Refer to Appendix A and Reference 1 for
Discussion of Parameter Derivations.

these random phenomena, it is necessary to understand the operations of each specific facility. The resulting MUF distribution will be site specific. Reference [3] presents results from modeling of a specific facility demonstrating the viability of the approach. For the generic facility under consideration in this report, it will be assumed that the distribution of MUF is normal as characterized by LEMUF. This is only for mathematical convenience. Any other distribution could be used for development of the game theoretic solution.

Under the assumption of a normal distribution for MUF, the following expressions hold:

$$F(z,x) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\Pi\sigma^2}} e^{-\frac{(u-x)^2}{2\sigma^2}} du$$

and $$G(z,x) = \int_{z}^{+\infty} \frac{1}{\sqrt{2\Pi\sigma^2}} e^{-\frac{(u-x)^2}{2\sigma^2}} du.$$

The minimax theorem was used to derive a solution for the game. The optimal strategies and game value for Plant 1 are presented in Figure 4, for the fixed alarm threshold case. The optimal alarm threshold, representing the minimum penalty to the defender, occurs at $a = 1.91\ \sigma$. Consider the case when the alarm is set at the optimum. If MUF is greater than a, the defender should estimate that $y_2 = 3.25\sigma = 975$ grams

OPTIMAL SOLUTION FOR PLANT 1 FIXED ALARM THRESHOLD CASE

Alarm Threshold	Defender of	Estimates Diversion	Amount Diverted	Value of Game	Prob. of Diverting X
a	Y_1	Y_2	X	V	P(X)
0.	.006	2.920	0.	74.217	.040
0.	.006	2.920	.064	74.217	.947
0.	.006	2.920	10.000	74.217	.013
1.000	.236	3.178	0.	35.499	.272
1.000	.236	3.178	.617	35.499	.638
1.000	.236	3.178	10.000	35.499	.091
1.911	.416	3.249	0.	24.823	.372
1.911	.416	3.249	1.104	24.823	.504
1.911	.416	3.249	10.000	24.823	.124
2.000	.433	3.249	0.	24.903	.378
2.000	.433	3.249	1.154	24.903	.496
2.000	.433	3.249	10.000	24.903	.126
3.000	.668	3.191	0.	33.573	.420
3.000	.668	3.191	1.776	33.574	.440
3.000	.668	3.191	10.000	33.573	.140
4.000	.986	3.086	0.	49.339	.437
4.000	.986	3.086	2.507	49.339	.417
4.000	.986	3.086	10.000	49.339	.146
5.000	1.366	3.389	0.	68.323	.373
5.000	1.366	3.389	3.337	68.323	.341
5.000	1.366	3.389	3.451	68.323	.285
6.000	1.779	4.267	0.	88.959	.374
6.000	1.779	4.267	4.227	88.959	.335
6.000	1.779	4.267	4.312	88.959	.292
8.000	2.632	6.089	0.	131.634	.374
8.000	2.632	6.089	6.063	131.634	.328
8.000	2.632	6.089	6.117	131.634	.298
10.000	3.506	7.962	0.	175.379	.374
10.000	3.506	7.962	7.943	175.379	.324
10.000	3.506	7.962	7.981	175.379	.302

Values of a, Y_1, Y_2, X and V are in σ Units (σ = 0.3 kgs.)

The Mixed Strategy of the Diverter Contains 3 Values of X.

Figure 4.

for the amount diverted and should search for this amount. When MUF is less than a, the defender should estimate that $y_1 = .43\sigma = 130$ grams and should conduct a more limited search for this amount. In this model if the defender interprets diversion based upon MUF in this manner he will protect himself in the best way against any diversion decision the diverter may make.

Whereas the defender has a best strategy, the diverter does not have a single strategy which is best. It is important for the diverter to withhold information about diversion. He accomplishes this by using a mixed strategy, which is a probability distribution of the amount he diverts.

The optimal decision for the diverter (in terms of penalizing the defender) when $a = 1.91\sigma$ is to divert nothing with probability .372, to divert 330 grams with probability .504, or to divert 3 kgs. with probability .124. Note that in this case the diverter favors taking a small amount relative to his objective of 3 kgs, but does so with a mixed strategy so that his moves will not be predictable by the defender. The expected amount diverted is 539 grams. Both diverter and defender optimal strategies are very sensitive to the alarm threshold. Note that in this game with these parameters, the defender should not estimate zero diversion when MUF is below the alarm threshold. To do so would be to expose himself to greater penalties than need be.

The distribution of MUF under the condition of optimal diversion can be expressed as:

$$P(MUF \leq u) = P_o\ F(u,o) + P_{x_2}\ F(u,x_2) + P_k\ F(u,k) \text{ where } 0 \leq x_2 \leq k$$

where (P_o, P_{x_2}, P_k) represent probabilities for the diverter's mixed strategies and $F(u,t)$ is a cumulative normal distribution with mean t. For Plant 1 under conditions of $a = 1.91\sigma$,

$$P(MUF \leq u) = .372\ F(u,o) + .504\ F(u,1.104) + .124\ F(u,10).$$

Note that this distribution is trimodal and is not normal. Let us calculate the probability that MUF will be less than a.

Then $u = 1.91\sigma$ and

$$P(MUF \leq a) = .372\ (.972) + .504\ (.790) + .124\ (.000) = .760$$

Thus under conditions of optimal diversion there is a .76 probability that MUF will be below the alarm and a .24 probability that MUF will be greater than the alarm. (There is a .63 probability that some diversion has taken place which compares with the .24 probability that the alarm will ring. The diversion will be detected by alarming only 38% of the time.) The null hypothesis, i.e., there is no diverter operating, would indicate a probability of .97 that MUF will be below the alarm. (This is the one-sided probability.) Thus the alarm rate will vary from .03 to .24 from best to worst case estimates.

These results are based upon the nominal parameter values given in Figure (3). Some of these parameters are based upon engineering cost estimates for the facility. The others represent subjective utilities to the decision-maker, particularly where societal values are involved. In order to determine the applicability of the game theoretic approach, it is necesary to examine the sensitivity of the results to variations in the values of the parameters. The approach to the sensitivity analysis is to examine the behavior of the optimum alarm threshold, the value of the game, and the defender's diversion strategies as the plant parameters are varied.

The nominal value of the parameters from Plant 1 were derived from the following considerations. Based upon engineering data for the facility, the standard deviation of the measurement accuracy distribution was determined to be σ = 0.3 kg. of special nuclear material (SNM). It was assumed that 3 kg. of SNM would constitute a credible threat and represent a reasonable safeguards objective. Therefore we took $k = 10\sigma$. The inventory cost model reported in [1] gave $5,000 for the cost of a special cleanout inventory. Using a "replacement" market value of $10,000 per kg. of SNM gave $\beta = 1.67\sigma$.

The remainder of the parameter values were based on these inventory costs. We assumed that the maximum cost of the variable search effort when MUF exceeds the alarm threshold would be equal to the fixed cost of the clean-out inventory $c_2k = \beta$. Moveover we assumed that when MUF was less than the alarm threshold the variable search would be performed at ten percent of the cost of a search effort for the same quantity of material above the alarm threshold. Thus $c_2 = 0.167$ and $c_1 = 0.017$.

Next, the utility of recovering the material was related to the disutility of plant shutdown and inventory. In this model the maximum utility of recovery is b_2k and the plant shutdown costs are β. We chose $b_2 = 100$ which gave a utility ratio of $b_2k/\beta = 600$ for plant 1. Again we took b_1 at ten percent of b_2. We decided that $e_1 = e_2 = 50$ by assuming that the penalty for making an error in the estimate of the amount diverted has a maximum value of one-half of the maximum utility of recovering material.

It is difficult, if not impossible, to make accurate estimates of all the parameters of this model. What is important is their consistency with each other and the sensitivity of the results to these estimates. However, the model may still be quite useful if results are reasonably insensitive to wide variations in parameter values. The sensitivity analysis explored this. Figure (5) shows the range of parameter values

RANGE OF PARAMETER VALUES FOR SENSITIVITY ANALYSIS

ITEM	PARAMETER SYMBOL	NOMINAL VALUE	RANGE STUDIED
Standard Deviation	σ	0.3 Kg	.03-3.0 Kg
Safeguards Objective	k	10σ	$1\text{-}100\sigma$
Inventory Cost	β	1.67σ	$0\text{-}10\sigma$
Search Cost Coefficient	c_1	.017	0-0.1
Search Cost Coefficient	c_2	.167	0-1.0
Recovery Coefficient	b_1	10	1.0-100
Recovery Coefficient	b_2	100	10-1000
Error Coefficient	e_1	50	5-500
Error Coefficient	e_2	50	5-500

Figure 5.

studied. We attempted to cover a range from an order of magnitude below to an order of magnitude above the nominal value.

Appendix [A] presents the details of the sensitivity analysis. These results will be summarized here for the determination of the optimal alarm threshold. The other variables are examined in the Appendix. Figure (6) shows the value of the game as a function of the alarm threshold. As can be seen, an optimum exists and is at a sharp point in the region of optimality. For Plant 1 $a^* = 1.91\sigma$ where a^* is the optimum fixed alarm threshold. Now we will consider sensitivity of this optimum value to variations in the parameters. Figure (7) shows that a linear relationship exists between a^* and the value of the safeguards objective k. This is not surprising since k represents the goal toward which system design is based. It has to be selected carefully and realistically with reference to what constitutes a credible threat.

The sensitivity analysis determined that a^* is insensitive to variations in the cost coefficients c_1, c_2 and inventory cost β. It is also insensitive to the value of the recovery coefficient b_1. Details are presented in Appendix A. However a^* is sensitive to the recovery coefficent b_2 for relatively small values of b_2. When b_2 is greater than 100, it is insensitive. This is illustrated in Figure (8).

VALUE OF GAME AS A FUNCTION OF ALARM

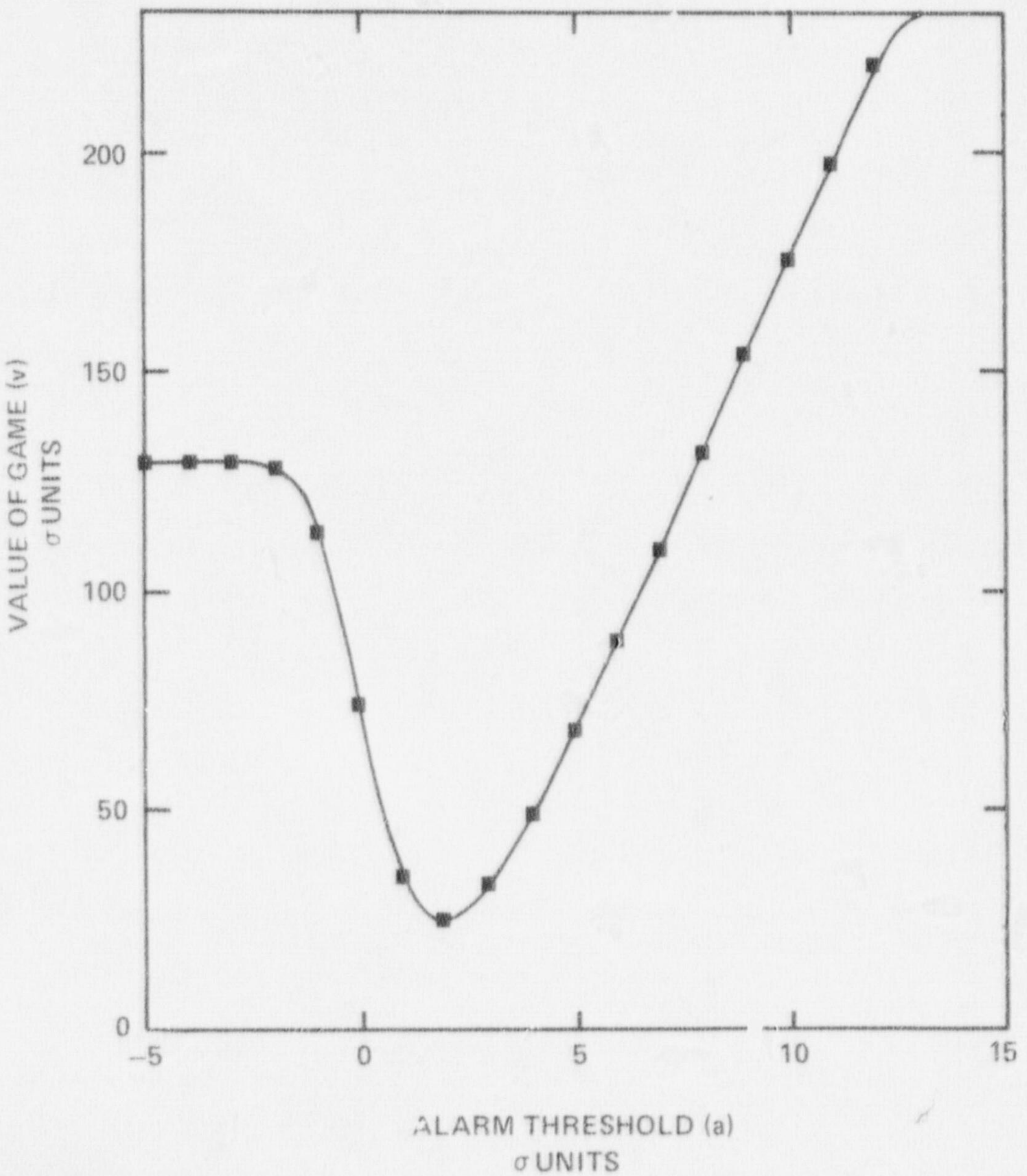

Figure 6.

RELATIONSHIP OF ALARM TO SAFEGUARDS OBJECTIVE

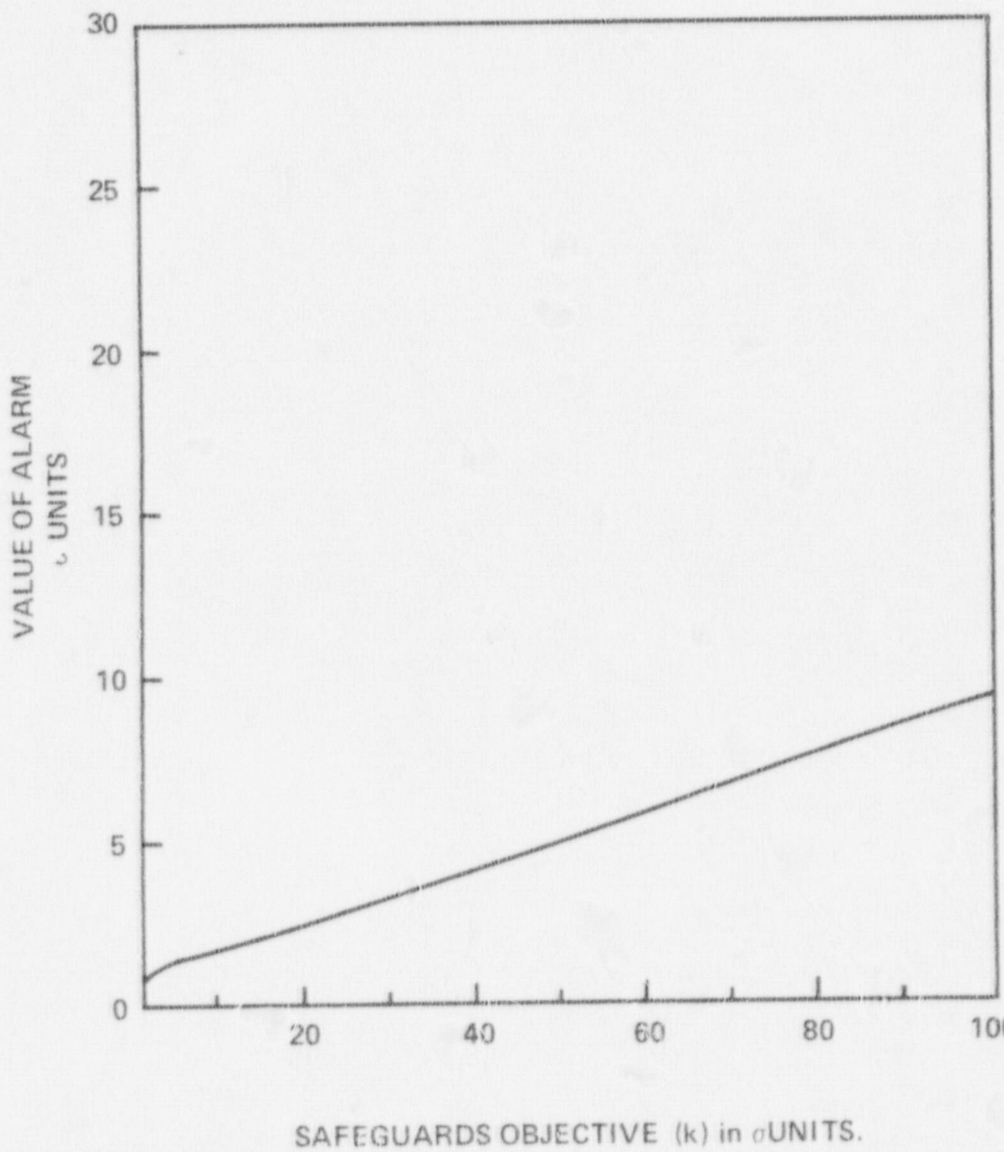

Figure 7.

RELATIONSHIP OF ALARM TO RECOVERY COEFFICIENT

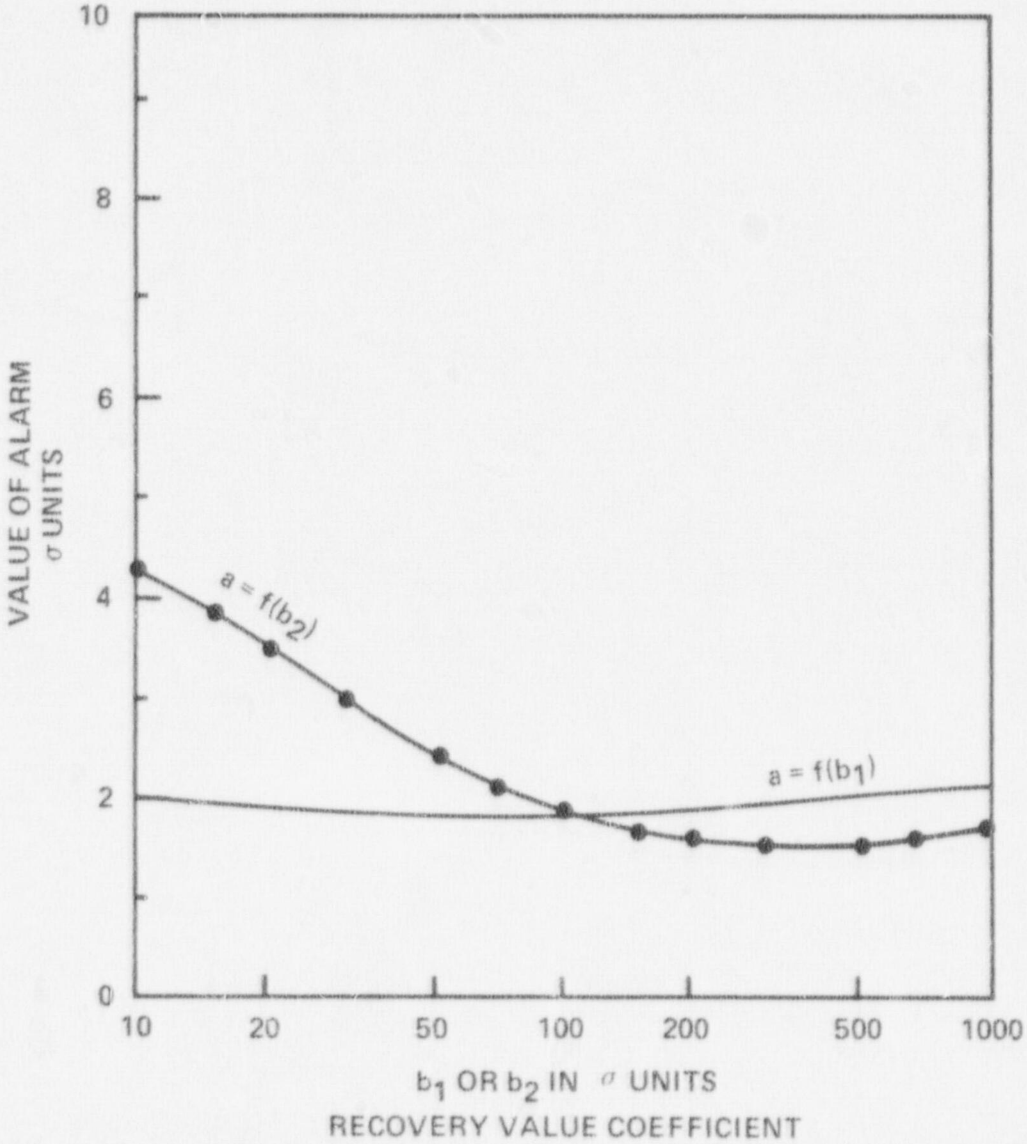

Figure 8.

The greatest sensitivity of a* was to the estimates of the error coefficients e_1 and e_2. Figure (9) presents the sensitivity to e_1. a* is a decreasing function of e_1. Figure (9) also presents the sensitivity to e_2. a* is an increasing function of e_2. Although the effects of e_1 and e_2 tend to cancel each other if the errors are in the same direction, e_2 has the most significant influence on a*. Figure (10) shows the effect on a* when $e_1 = e_2$. It becomes more significant for values of e greater than 50. Of all the parameters in the model, e_2 has the greatest influence on selection of an optimal alarm threshold. This represents the sensitivity to the decision-maker to making an error in the estimate of the amount diverted when MUF is large.

A review of the sensitivity of the optimal alarm to parameter values shows it to be reasonably insensitive. Thus, the result of the sensitivity analysis indicates that for Plant 1 it is feasible to establish fixed alarm thresholds based upon parameter estimation for the game theory model presented here. The greatest care must be exercised in estimates of e_2. Since the penalty in the error of the estimate of diversion should be based only on the magnitude of the error and not the value of MUF, it is reasonable to assume that $e_1 = e_2$. Thus, the effects of e_2 are reduced to a manageable dimension.

RELATIONSHIP OF ALARM TO ERROR COEFFICIENTS

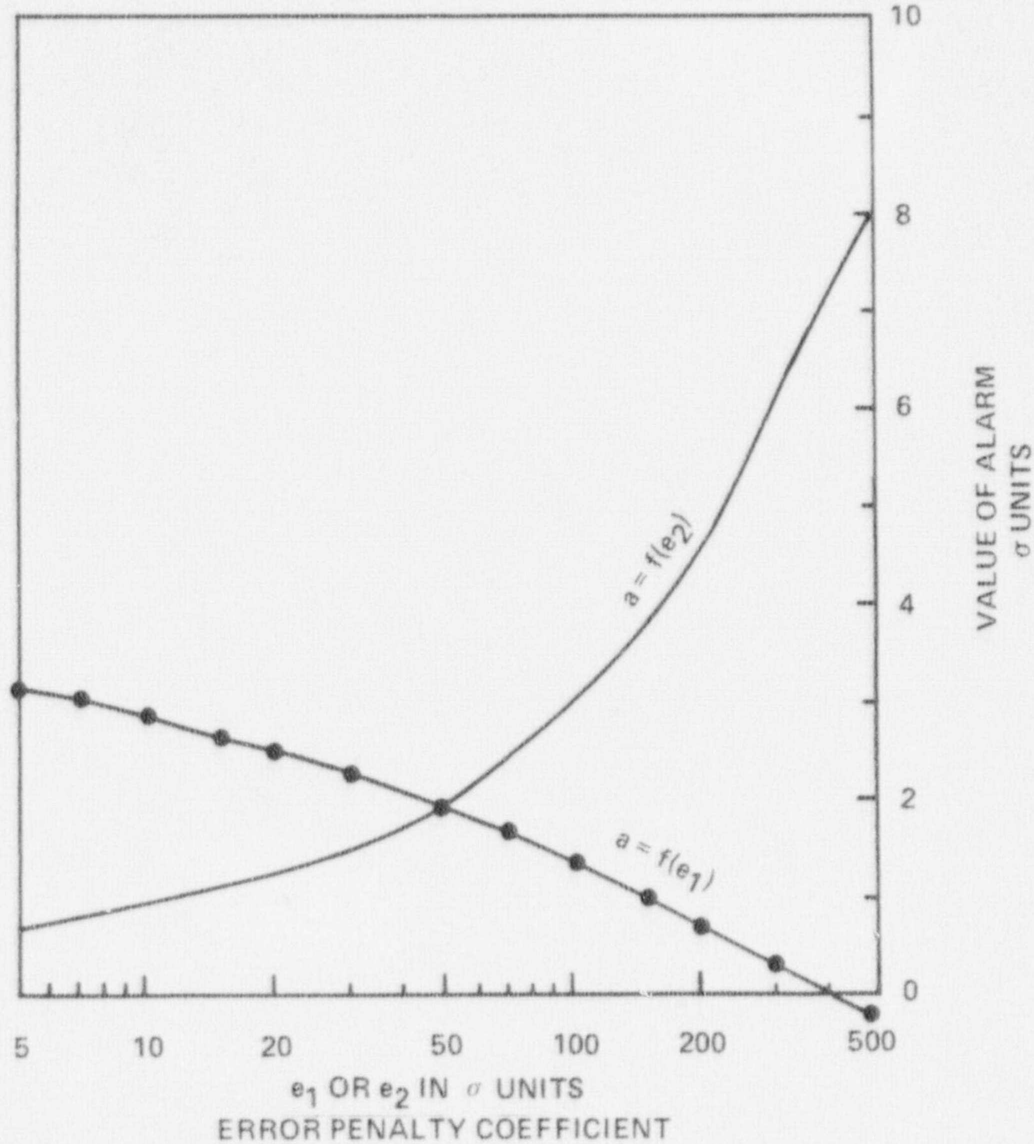

Figure 9.

RELATIONSHIP OF ALARM TO EQUAL ERROR COEFFICIENTS

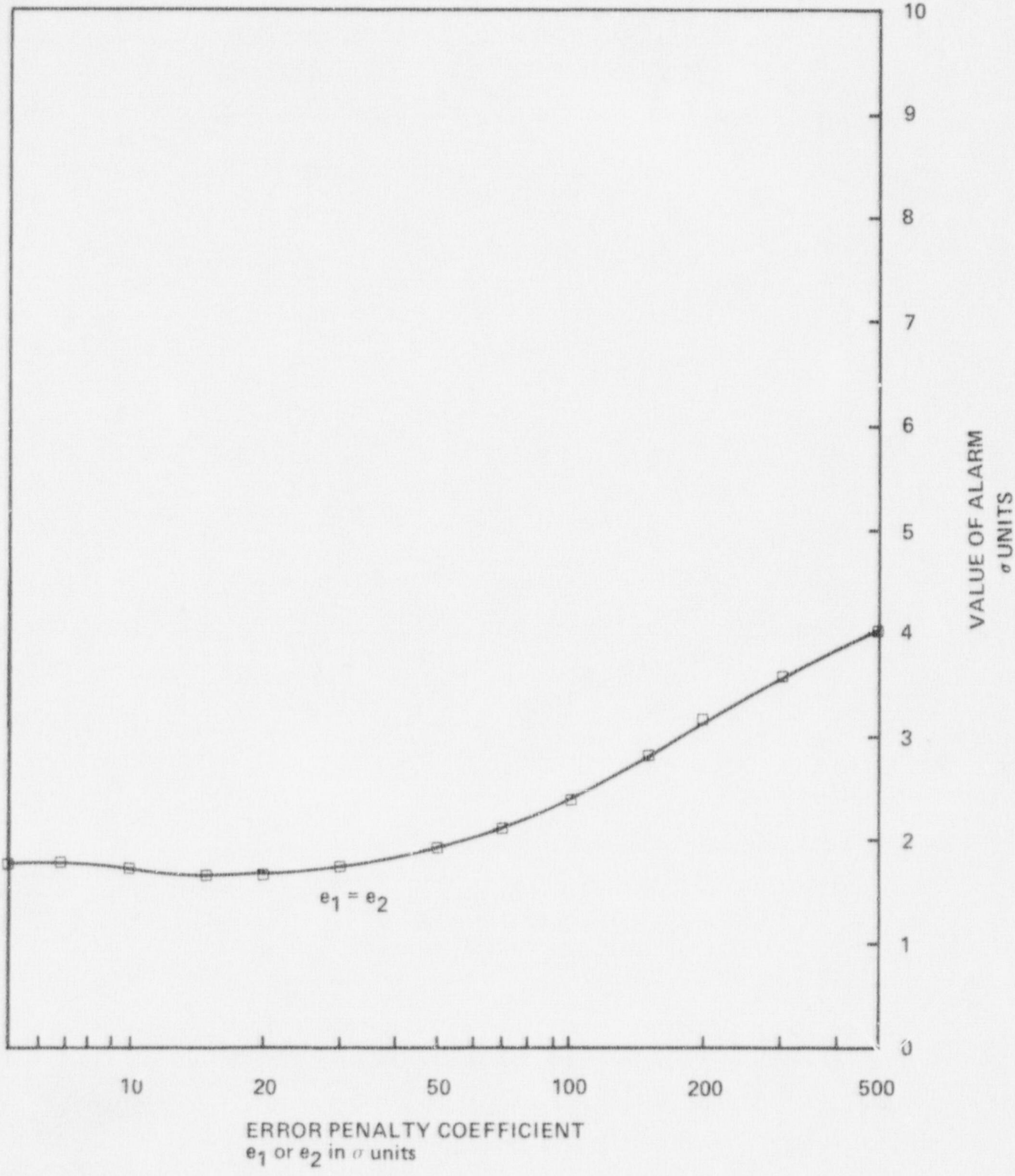

Figure 10.

5. Alarm Threshold as a Strategic Variable

The previous solution was based upon the assumption that the alarm threshold is predetermined and fixed at some value prior to the start of the game. Since the alarm threshold in this case is a parameter, its value is therefore known to the diverter as well as the defender. This information gives an undue advantage to the diverter. By making the alarm threshold a strategic variable of the defender rather than a parameter, we no longer restrict the defender to any particular value of the alarm threshold. This procedure withholds information about the alarm from the diverter at the time he makes a decision about the amount of material he will divert.

The game model will now be extended by introducing the alarm threshold as a strategic variable; one of the moves of the defender is the choice of an alarm threshold. The particular alarm threshold is known by the defender, but not by the diverter. Thus the strategic variables for the defender are the alarm threshold and the estimates of diversion for MUF values both above and below the alarm. This yields a three dimensional strategy space for the defender which will be analyzed for optimal strategies.

As will be recalled, the game in extensive form is described as follows:

Move 1. Diverter removes x grams, $0 \leq x \leq k$.

Move 2. Defender receives information that u grams are unaccounted for.

Move 3. Defender decides

i) Whether or not to alarm and take reinventory.

ii) How much effort to expend on search and recovery of diverted material based upon his estimate of diversion in each of the two cases of i).

In normalized form the above game may be described as follows:

1. Diverter picks a strategy x where $0 \leq x \leq k$.

2. Defender picks a strategy which is a three-component vector of functions $[z(u), y_1(u), y_2(u)]$ where $z(u)$ is the alarm threshold, $y_1(u)$ is the estimate of diversion when $u < z$ and $y_2(u)$ is the estimate of diversion when $u \geq z$.

In other words, if:

$u < z(u)$, then action based upon $y_1(u)$ is taken

$u \geq z(u)$, then action based upon $y_2(u)$ is taken.

These actions include search for and recovery of diverted material.

Now since u is a random variable, the payoff of the game will consist of two parts.

i) M_1 if $u < z(u)$

ii) M_2 if $u \geq z(u)$

Therefore the expected payoff M will be

$$M = M_1 \, P(u < z) + M_2 \, P(u \geq z)$$

and the complete payoff function in terms of the parameters is

$$M = [c_1 \, y_1 + x - b_1 \min (y_1, x) + e_1 |y_1 - x|] \, P(u < z)$$
$$+ [\beta + c_2 \, y_2 + x - b_2 \min (y_2, x) + e_2 |y_2 - x|] \, P(u \geq z)$$

Using the results of NUREG-0290 and the results of the one-dimensional game of Appendix [B], it is easy to show that

$$\max_{x} \; \min_{z,y_1,y_2} \; M(x,z,y_1,y_2) < \min_{z,y_1,y_2} \; \max_{x} \; M(x,z,y_1,y_2).$$

Hence this game requires mixed strategies for both players. Based upon the results in Appendix [C] it can be shown that the optimal mixed strategy for each player consists of a finite number of strategies rather than a density function.

If $F^*(x)$ and $G^*(z,y_1,y_2)$ represent the optimal mixed strategies of the diverter and defender respectively, then we need to solve the following optimizing equation:

$$\max_{x} \int M(x,z,y_1,y_2) \, dG^*(z,y_1,y_2) = \min_{z,y_1,y_2} \int M(x,z,y_1,y_2) \, dF^*(x)$$

Due to the complex form of the payoff function $M(x,z,y_1,y_2)$ which involves exponential and rational functions, a closed form solution is not possible.

Some approximating or iterative process is required to solve this mathematical problem. We shall use an iterative process designed specifically for game problems - the method of "fictitious play" or successive approximations. See Reference [2] for an explanation of the method.

Using this method of fictitious play, the three-dimensional game described above was solved. Figure (11) presents the value of the game for Plant 1 for the strategic alarm threshold case and compares it with alternative fixed alarm threshold policies. As can be seen from the figure, there is a substantial decrease in the defender's losses when he uses the alarm threshold as a strategic variable. The difference in value between the value of the game under an optimal fixed alarm and an optimal variable alarm represents the value to the defender for denying information to the diverter as to the location of the alarm with respect to the MUF reading. For Plant 1 it is considerable and is a 51% improvement in the payoff.

Figure (12) shows the optimal strategy for the defender. There are eleven pure strategy sets in the optimal mixed strategy solution for the defender. These strategies are approximate since we used a grid of points rather than a continuum. The alarm threshold range is from - 0.5 σ to 4.0 σ with a specific probability for selection of any specific alarm. Note that $y_1 = 0$ is optimal; the defender assumes that there is no diversion when MUF is below the alarm. However, the alarm level is selected by a mixture of several alarms. For any fixed alarm, $y_1 > 0$.

VALUE OF THE GAME FOR ALTERNATIVE ALARM POLICIES

	VALUE
ALARM THRESHOLD A STRATEGIC VARIABLE	12.2σ
FIXED ALARM THRESHOLD	
at Optimum (1.91σ)	24.82σ
at LEMUF (2σ)	24.90σ
at 2 LEMUF (4σ)	49.34σ
at 3 Kgs	175.40σ

Figure 11.

OPTIMAL STRATEGY FOR THE DEFENDER
PLANT 1 (σ = 0.3 Kgs)

Z	Y_1	Y_2	$P(Z, Y_1, Y_2)$
-0.5	0	0.5	.058
0	0	0.5	.188
+0.5	0	1.0	.113
1.0	0	1.0	.090
1.0	0	1.5	.090
1.5	0	1.5	.011
1.5	0	2.0	.083
2.0	0	2.5	.028
2.0	0	3.0	.096
3.5	0	10.0	.204
4.0	0	10.0	.039

NOTE: Z, Y_1, and Y_2 are in σ units.

Figure 12.

Figure (13) shows the optimal mixed strategy solution for the diverter. There are eight pure strategies in his optimal mix.

The expected value of y_2, $E(y_2)$, is equal to 5.14 σ = 1.54 kgs. Figure (14) shows the cumulative probability distribution for the selection of an alarm threshold based upon the value of MUF.

Notice the large deterrence effect derived from a variable alarm policy. The probability of no diversion, $P(x = 0)$, is .62. This compares with a probability of no diversion for the optimal fixed alarm threshold of .37. Thus the deterrence effect of a variable alarm threshold policy shows an improvement by a factor of 1.7. The expected amount of material diverted $E(x)$ = .437 σ or 131 grams. This compares to the case where $E(x)$ = 1.8 σ or 539 grams in the optimal fixed alarm case, an improvement by a factor of 4.1. Figure (15) shows a comparison of the results for alternative alarm threshold policies. The superiority of a mixed strategy policy for the defender is clearly evident.

Figure (16) presents the expected alarm rates for alternative alarm policies for the defender for two cases. In the first case a diverter playing optimally is assumed. There is not much significant change in the alarm rate. In the second case, there is no diverter present but the defender does not know this. The false alarm rate is higher for the variable alarm policy than for a fixed alarm policy.

OPTIMAL STRATEGY FOR THE DIVERTER
PLANT 1 (σ = 0.3 Kgs)

X	P(X)
0	.620
0.5	.241
1.0	.052
1.5	.043
2.0	.012
2.5	.008
3.5	.013
10.0	.011

NOTE: X is in σ units.

Figure 13.

PROBABILITY DISTRIBUTION FOR ALARM SELECTION

Plant 1 $\sigma = 0.3$ Kgs

MUF RANGE (σ units)	PROBABILITY OF ALARM
> 4.0	1.000
3.5-4.0	.961
2.0-3.5	.757
1.5-2.0	.633
1.0-1.5	.539
0.5-1.0	.359
0-0.5	.246
-0.5-0	.058
< -0.5	0.00

Figure 14.

COMPARISON OF RESULTS FOR ALTERNATIVE ALARM POLICIES

	EXPECTED AMOUNT OF MATERIAL DIVERTED	PROBABILITY OF NO DIVERSION
ALARM THRESHOLD A STRATEGIC VARIABLE	131 grams	0.62
FIXED ALARM		
at Optimum (1.91σ)	539 grams	0.37
at LEMUF (2σ)	550 grams	0.38
at 2 LEMUF (4σ)	750 grams	0.44

Figure 15.

EXPECTED ALARM RATES

	DIVERTER ACTING OPTIMALLY	FALSE ALARM
ALARM THRESHOLD AS STRATEGIC VARIABLE	.287	.088
FIXED ALARM THRESHOLD		
at Optimum (1.91σ)	.240	.028
at LEMUF (2σ)	.233	.023
at 2 LEMUF (4σ)	.174	$<$.001

Figure 16.

The effectiveness of the alarm policy is not determined by the false alarm rate as is often the case when a classical statistical approach is taken, but rather by the probability of the alarm going off when a diversion has occurred. This can be expressed quantitatively by the conditional probability $P(u > z | x > \bar{x})$ where $\bar{x}$ represents the expected amount of material diverted when the diverter uses his optimal strategy. This term represents the minimum probability of detecting a diversion of at least $\bar{x}$ because x has been chosen from the diverter's optimal distribution. Figure (17) presents this value for the alarm threshold as a strategic variable as well as a fixed optimum value for several different expected values of x. The probability of detecting a diversion of expected amount greater than 123 grams is .75 for the variable alarm case. This compares to .38 for 550 grams for the optimal fixed alarm case and .02 for 1495 grams for the case of the alarm at 3 kgs. This means that the values presented on Figure (17) are the assurance values for the material accounting system. The results indicate the advantages that may be realized by the game theoretic approach, particularly when the alarm threshold is treated as a strategic variable.

6. Summary and Conclusions

In this report we have presented the results of applying a game theoretic model to the interpretation of MUF data at a specific plant. We have shown in this case that there is a clear advantage in this approach over classical statistical hypothesis testing in terms of reducing the costs or penalty to

PROBABILITY OF ALARM WITH THREE DIFFERENT TYPES OF ALARM THRESHOLD

AMOUNT	ALARM THRESHOLD A STRATEGIC VARIABLE	ALARM FIXED AT OPTIMUM (1.91σ)	ALARM FIXED AT 3 Kgs
131 GMS.	.75	–	–
281 GMS.	.88	–	–
539 GMS.	–	.38	–
700 GMS.	–	.41	–
1495 GMS.	1.00	–	.02

Note: Table is Incomplete and Represents Only Calculations Made to Data.

Figure 17.

the defender. Withholding knowledge of the alarm threshold from the diverter, by treating it as a strategic variable, further decreased the defender's penalties. These results suggest that game theory may provide a better method for making decisions based on material accounting data.

It must be emphasized that this study is only a first step toward the possible use of game theory for material accounting purposes. We have examined in detail only one plant, and we have not completed the sensitivity analysis for the variable alarm threshold model. We should also apply the two models to other facilities.

Alternative formulations of the game also have to be explored. A first step would be to examine a variety of payoffs incorporating different utilities or functional forms. A realistic probability distribution for MUF, which could come from the simulation model described in Reference 4, should be used. The diverter may resort to other strategies to cover his diversion, e.g., tampering with the accounting records or with LEMUF. These possibilities should be investigated and means found for incorporating them into the model. Furthermore, the whole field of multi-move games remains to be examined. Before a game theoretic model can be implemented it must realistically incorporate the capabilities of the diverter as well as the utilities and behavior of the defender.

Other parts of the safeguards system can supply information on the existence and behavior of a diverter. Such information can be made use of by a game theory model in several ways. If these data include an _a priori_ estimate of the probability of there being a diverter, then using the results of the game theoretic analysis one can set up a statistical test by which the _a postiori_ probabilility of there being a optimal diverter can be calculated from the MUF value. A second approach is to restructure the game to incorporate this _a priori_ probability. Finally, the game can be enlarged to model the behavior of other parts of the overall safeguards systems, thereby optimizing this larger system.

Some of the results of strategic analysis are applicable to the existing system. As we have shown for a specific case, denying the diverter information about the alarm threshold is advantageous to the defender. The alarm threshold can be determined by NRC before the inventory period and kept secret, or the decision whether to alarm can be made after the MUF value is determined. The latter procedure is advantageous because it precludes the possibility of the diverter gaining access to classified data. However the legal and institutional ramifications of a variable alarm threshold have to be investigated. We believe that procedures necessary for implementing a variable alarm policy should be investitated.

Since our results to date suggest that game theory may provide an improvement in the method for interpreting and acting upon materials accounting data, we believe a vigorous program of research and development in this area would be benefit.

Fixing the alarm at predetermined amounts based on classical statistics can be shown to penalize the defender. The alarm should be the result of an analysis based upon modeling the competitive interaction that exists between a diverter and a defender. The theory of games is a logical methodology for this modeling.

In summary, alarm thresholds based upon classical statistical inference are not optimal in the sense defined in this study and in general penalize the defender. Furthermore, game theoretic derived optimal alarm thresholds are completely plant specific whereas those derived by the classical statistical approach are only partially plant specific. The results of this study suggest that steps to optimize alarms on a completely plant specific basis to achieve a common standard performance requirement throughout the industry should be evaluated.

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APPENDIX A

SENSITIVITY ANALYSIS

INTRODUCTION

During the first phase of this study a game theoretic model of nuclear material accounting systems was formulated. The model was applied to make a preliminary analysis of the material accounting systems at three generic plants. To perform the analysis it was necessary to evaluate the parameters of the model for each of the plants. Some of the parameter values could be chosen by engineering cost estimates, while for others we could only make rule-of-thumb estimates. The results of this preliminary analysis were published in NUREG-0290.

In order to determine the applicability of the game theoretic approach to evaluating material accounting systems, it is necesary to examine the sensitivity of the results to variations in the values of the parameters of the model. This is especially true in this case where some of the parameters represent societal values. In this report we present the results of a more detailed sensitivity analysis for one of the three plants, the small plutonium fuel fabrication facility.

We make the sensitivity analysis from the defender's point of view. We determine what the optimal strategies for him to follow are for different sets of plant parameters. Results are presented for the level at which the alarm threshold should be set, the defender's estimates of the amount of material diverted, and the maximum penalty to the defender when he uses the optimal strategy.

In applying the game theoretic model to a plant, all the plant parameters are fixed except the alarm threshold. The solutions are found for a series of alarm thresholds, and the alarm threshold which gives the lowest value of the game is determined. See Figure A-1. The purpose of the sensitivity analysis is to examine the behavior of this optimum alarm threshold a^*, the value of the payoff v^*, and the defender's strategic variables y_1^* and y_2^* as the plant parameters are varied.

1.0 Plant Parameters

In the game theoretic model the plant is characterized by ten parameters. Measurements of MUF are assumed to have normally distributed errors with mean x and a known standard deviation σ. This latter quantity is used as the unit in which we measure amounts of SNM. Thus k, a, x, y_1 and y_2 are all measured in units of σ.

The parameter k, which is defined as the amount of SNM vulnerable to diversion, has to be carefully interpreted. It does not represent the amount of material in the plant, only the amount the defender believes would constitute a credible threat. Thus k can be considered the safeguards objective. In this model if an amount greater than k were diverted, it would not increase the defender's losses.

Under current practice the alarm threshold is set as some multiple of the regulatory limit on LEMUF. The value chosen is a compromise between a high false alarm rate and not detecting a small diversion. In the game theoretic model the optimal alarm threshold a* is an outcome of the analysis. It therefore is a function of the other parameters.

The remaining parameters β, b_1, b_2, c_1, c_2, e_1 and e_2 are costs and cost coefficients in the payoff function. The cost of a special cleanout inventory β, which in this model is done if MUF exceeds the alarm threshold, can be determined using the inventory cost model developed during the first phase of this study. The cost coefficients c_1 and c_2 for the variable search effort also can be found by engineering estimates of labor and materials involved. The other coefficients have to be determined by less well established methods. The b_1 and b_2 coefficients contain such factors as the value to the defender (or to society in general) of recovering the material diverted and the probability of recovering it. The e_1 and e_2 coefficients, which represent penalties to the defender for wrongly estimating the amount diverted, also depend on societal values.

The values of these parameters for plant 1 that were determined in the first phase are shown in Figure 5 of the main report. They were derived from the following considerations. First, based on engineering data for the plant, the standard deviation of the measuring error distribution was determined to be σ = 0.3 kg of SNM. It was assumed that 3 kg of SNM would constitute a credible threat for the diverter; therefore, we took $k = 10\sigma$. The inventory cost model gave $5,000 for the cost of a special cleanout inventory at plant 1. Using a "market" price of $10,000 per kg of SNM gave $\beta = 1.67\sigma$.

The rest of the cost coefficients were based on the inventory costs. We assumed that the maximum cost of the variable search effort when MUF exceeds the alarm threshold would be equal to the fixed cost of the cleanout inventory $c_2k = \beta$. Moreover, we assumed that when MUF was less than the alarm threshold the variable search would be performed at ten percent of the cost of a search effort for the same quantity of material above the alarm threshold. Thus $c_2 = 0.167$ and $c_1 = 0.017$.

Next, the utility of recovering the material was related to the disutility of plant shutdown and inventory. In this model the maximum utility of recovery is b_2k and the plant shutdown costs are β. We chose $b_2 = 100$ which gave a utility ratio $b_2k/\beta = 600$ for plant 1. Again we take b_1 at ten percent of b_2. We decided that $e_1 = e_2 = 50$ by assuming that the penalty for making an error in the estimate of the amount diverted has a maximum value of one-half of the maximum utility of recovering the material.

It should be clear from this discussion that it is difficult, if not impossible, to make accurate estimates of all the parameters of this model. The best we can do is put plausible limits on the range of their values. In performing a sensitivity analysis we therefore attempted to vary the parameters by an order of magnitude in both directions from their nominal values. The range of parameter values studied are shown in Figure 5 of the main report.

2.0 RESULTS OF THE SENSITIVITY ANALYSIS

2.1 Nominal Values

Before presenting the results of the sensitivity analysis for plant 1, we will first discuss the results for the nominal values of the plant parameters. Figure A-1 shows the value of the game and the defender's strategic variables as a function of the alarm threshold. The penalty to the defender is large when the alarm threshold is much higher or lower than its optimal value $a^* = 1.911\sigma$ and $v^* = 24.8\sigma$. Note than even if $u < a^*$ the defender's optimal strategy is to estimate that there has been a small amount of SNM diverted and to make a search to recover any of this material. This is in contrast to the classical statistical hypothesis testing approach under which no diversion would be assumed if the alarm threshold has not been exceeded.

2.2 Sensitivity to Safeguards Objective

The parameter k represents the amount of SNM at a plant that is vulnerable to diversion measured in units of σ.

The results of varying k from 1 to 100σ are plotted in Figure A-2. These plots show a linear behavior with plant size except at the smallest values of k. The functional dependence can be expressed as:

$$a^* = 0.885 + 0.919\ k$$

$$y_1^* = -0.027 + 0.0445\ k$$

$$y_2^* = 0.059 + 0.3249\ k$$

$$v^* = 3.266 + 2.157\ k$$

For large k, these quantities are proportional to the safeguards objective. This analysis demonstrates that NRC must consider its safeguards objective in setting the alarm threshold for a plant. It is not sufficient to consider measurement errors only. A larger safeguards objective in physical units of material requires a proportionally larger alarm threshold and will result in larger losses to the defender.

2.3 Sensitivity to Cleanout Inventory and Recovery Search Costs

We now examine the sensitivity of our results to the costs for performing a cleanout inventory β, and to the cost coefficients for the recovery search, c_1 and c_2. These parameters can be fairly well evaluated using engineering cost estimates. In performing the sensitivity analysis, we varied these parameters from zero up to six times their nominal values. The results are shown in Table A-I.

Table A-I

Sensitivity to Cleanout Inventory and Recovery Search Costs

Parameter	a^*	y_1^*	y_2^*	v^*
Nominal Values	1.911	0.416	3.249	24.8
$\beta = 0$	1.901	0.410	3.240	24.6
$\beta = 10$	1.960	0.445	3.296	26.1
$c_1 = 0.0$	1.912	0.416	3.249	24.8
$c_1 = 0.1$	1.910	0.416	3.249	24.9
$c_2 = 0.0$	1.908	0.414	3.246	24.7
$c_2 = 1.0$	1.928	0.426	3.265	25.2

It is clear from these results that there is very little change in the four variables over this range of variation of the parameters. Since there is relatively little uncertainty in these parameters, we have not further investigated their sensitivity.

2.4 Sensitivity to the Value of Material Recovered

The parameters b_1 and b_2 measure the value to the defender of recovering any SNM that has been diverted. We first examine the case where one of these two parameters is held fixed at its nominal value while the other is allowed to vary up or down by a factor of ten. These results are displayed in Figures A-3 and A-4. Over the range shown both a^* and y_2^* show slight linear increases with b_1, whereas y_1^* shows a slight linear decrease. The value of the game also decreases slowly with increasing b_1; it changes by less than a factor of two when b_1 changes by two orders of magnitude.

The results show a much stronger dependence on b_2. All four quantities decrease rapidly with increasing b_2 for values of b_2 less than 100, the nominal value. Above $b_2 = 100$, y_1^*, y_2^* and v^* continue decreasing although at a slower rate. The optimal alarm threshold, however, goes through a minimum near $b_2 = 400$ and then increases very slowly. This behavior is due to the fact that the term with the b_2 coefficient has a negative sign in the payoff function. Therefore, as the value of the material recovered increases, the diverter removes less on the average. The defender can then decrease his alarm threshold and his estimates of the amount diverted, thus decreasing his penalties.

Since b_1 and b_2 both measure the value of the material recovered, we varied them simultaneously keeping the ratio $b_2/b_1 = 10$ fixed. These results, which are displayed in Figure A-5, are very similar to those where we varied only b_2. This again shows that variations in b_2 have larger effects than variations in b_1. For larger values of b_2 neither variable has much effect.

We have also varied b_1 and b_2 independently over their full range. Contour plots of the results are presented in Figures A-6 to A-9. For b_2 less than about 100, the results are nearly independent of b_1. For larger values of b_2 neither variable has much effect.

Our overall conclusion is that the model is very insensitive to variations in b_1 over the range studied. For larger b_2 the optimal alarm threshold is nearly independent of b_2 and the defender's strategic variables and penalties decrease slowly. Only for small values of b_2 do the results show much sensitivity. Because the value to society of recovering any diverted material is likely to be high, we believe these recovery value coefficients are well enough determined to allow NRC to set a meaningful alarm threshold.

2.5 <u>Sensitivity to Incorrect Estimates of the Amount Diverted</u>

We examined the sensitivity to the error penalty coefficients e_1 and e_2 in a similar way to that for the b parameters. The results for varying

e_1 while holding e_2 fixed at its nominal value are shown in Figure A-10; the corresponding plots varying e_2 for fixed e_1 are shown in Figure A-11. When e_1 is increased both y_1^* and y_2^* stay constant and then decrease slightly. The alarm threshold decreases as e_1 increases and becomes negative for e_1 larger than 400. Since this parameter increases the penalty to the defender below the alarm threshold, the defender prefers a low-alarm threshold when e_1 is large. We see exactly the opposite behavior when e_2 is increased; a^*, y_1^* and y_2^* all become larger. In both cases the value of the game increases as the parameters increase.

When we vary the two parameters simultaneously keeping $e_1 = e_2$, the behavior is more complex as shown in Figure A-12. Both the alarm threshold and y_1^* stay nearly constant for $e_1 = e_2$ up to 50; then they increase slowly. The value of y_2^* increases rapidly at first and then more slowly for larger values of the parameters. The penalty to the defender also increases and becomes nearly linear in e. The explanation for this behavior is as follows. As e_1 and e_2 become larger, the diverter attempts to remove a larger amount of material because the recovery value terms become relatively less important. The defender counters by increasing his estimate of the amount diverted, thus keeping $e|y-x|$ small. The optimal alarm threshold lies between y_1^* and y_2^* which balances the contributions to the payoff from the e_1 and e_2 terms.

These results indicate that the model is most sensitive to the error penalty coefficients and that the variations are more sensitive above threshold than below. In order to use this type of model to set the alarm threshold, these parameters have to be well determined, especially if they are large. Of all the parameters in this model, the largest effort must be put into evaluating e_2. Refer to Figures A13 - A16 for an illustration of this.

2.6 Other Two-Variable Correlations

We have looked at correlations when we vary the below-threshold parameters, b_1 and e_1, and also when we vary the above-threshold parameters, b_2 and e_2. Contour plots of the results for b_1 - e_1 are shown in Figures A-17 to A-20 and for b_2 - e_2 in Figures A-21 to A-24.

The results for a*, y_2^* and v* show very little sensitivity to the b_1 parameter as compared to the e_1 parameter. The below-threshold estimate of diversion y_1^* shows a more complex behavior. It is nearly independent of b_1 when e_1 is large. When e_1 is small, the dependence on b_1 becomes more important.

The results when we varied e_2 and b_2 show a strong negative correlation. All four quantities increase rapidly when b_2 decreases and e_2 increases. There is almost no change in y_1^*, y_2^* and v* when b_2 and e_2 increase simultaneously. The optimal alarm threshold does increase in this case because penalties are lower if the alarm threshold is exceeded less often. These plots reinforce our conclusion that the e_2 parameter is the one that must be most carefully evaluated. For safeguards purposes, the best situation is where b_2 is large and e_2 small.

Comparing the results when we vary $b_2 - e_2$ with the results of varying $b_1 - e_1$, we see that they are more dependent on the parameters that determine the payoff above alarm threshold than those below alarm threshold. This agrees with our intuitive feeling that we must be more careful when we think that there may be a large diversion.

3. CONCLUSIONS

Plant 1 is characterized by having a relatively small measuring error compared to the amount of material vulnerable to diversion. For this plant the optimal alarm threshold and the expected losses are proportional to k. The safeguards objective must be specified in order to set the alarm threshold for the plant. This is in contrast to the current practice of setting the alarm threshold based on measuring errors alone.

Since plant 1 is small, the special cleanout inventory costs are also small compared to the value of recovering diverted material. (The utility ratio $b_2k/\beta = 600$ is large.) Consequently the alarm threshold and losses are insensitive to cleanout inventory and recovery search costs The parameters β, c_1 and c_2 can be determined by considerations outside this model, for example, by how much they contribute to the amount of material recovered.

Because the value of recovering material that has been diverted is high relative to the inventory and recovery search costs, the value of the alarm threshold is nearly independent of the b_1 and b_2 parameters for this plant. As long as the b_2 parameter is known to be large, it does not have to be well determined. The error penalty coefficients are the parameters in the model for which the most effort must be expended to determine accurately. This is especially true for the e_2 parameter when it is large.

In general it appears that the model shows greater sensitivity to the parameters that determine the defender's losses when MUF is above the alarm threshold than those when MUF is below threshold. This is in line with the intuitive notion that we need more accurate knowledge when we think a significant amount of material has been diverted.

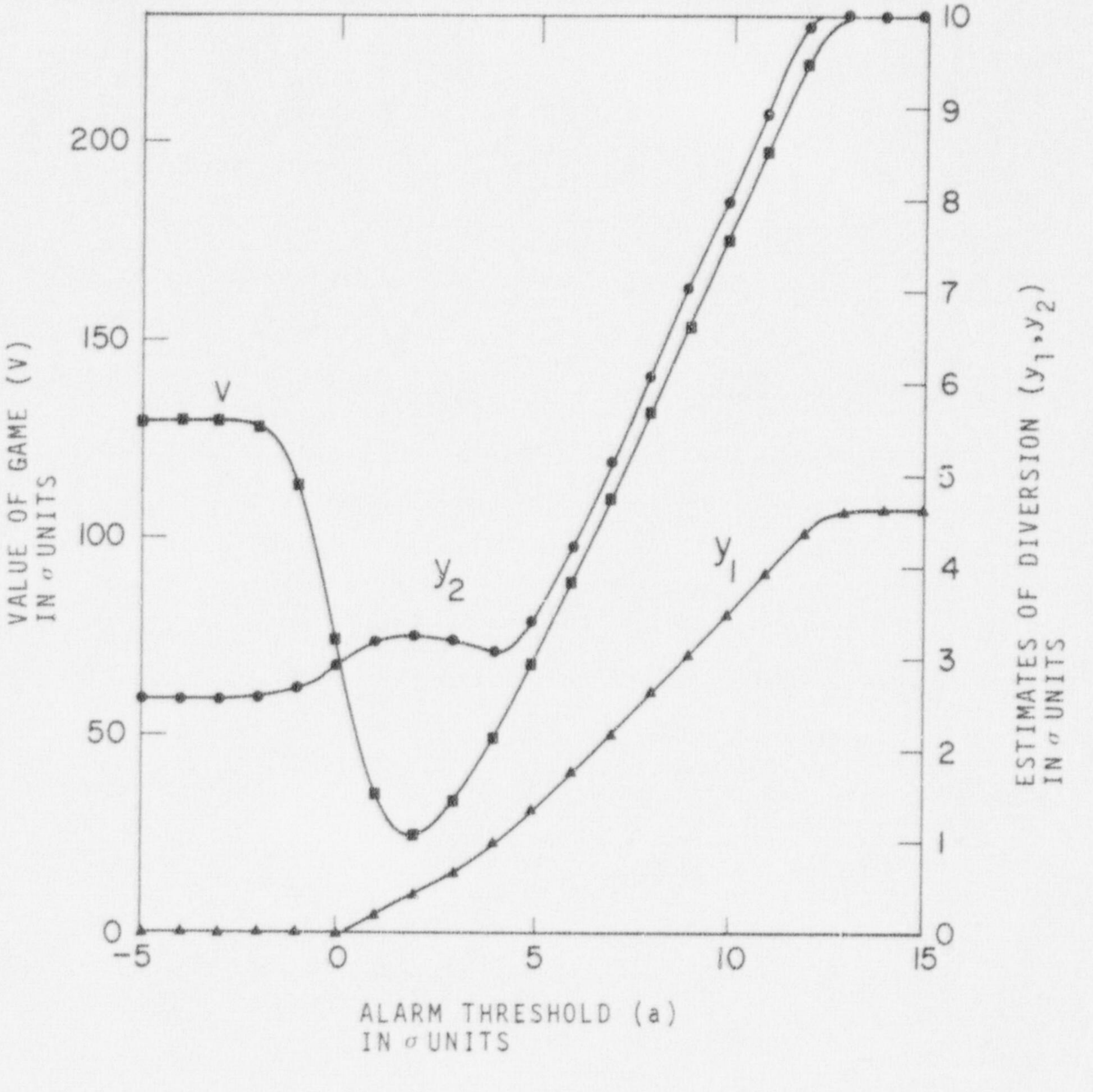

Figure A1. Value of Game and Estimates of Diversion as a Function of Alarm Threshold

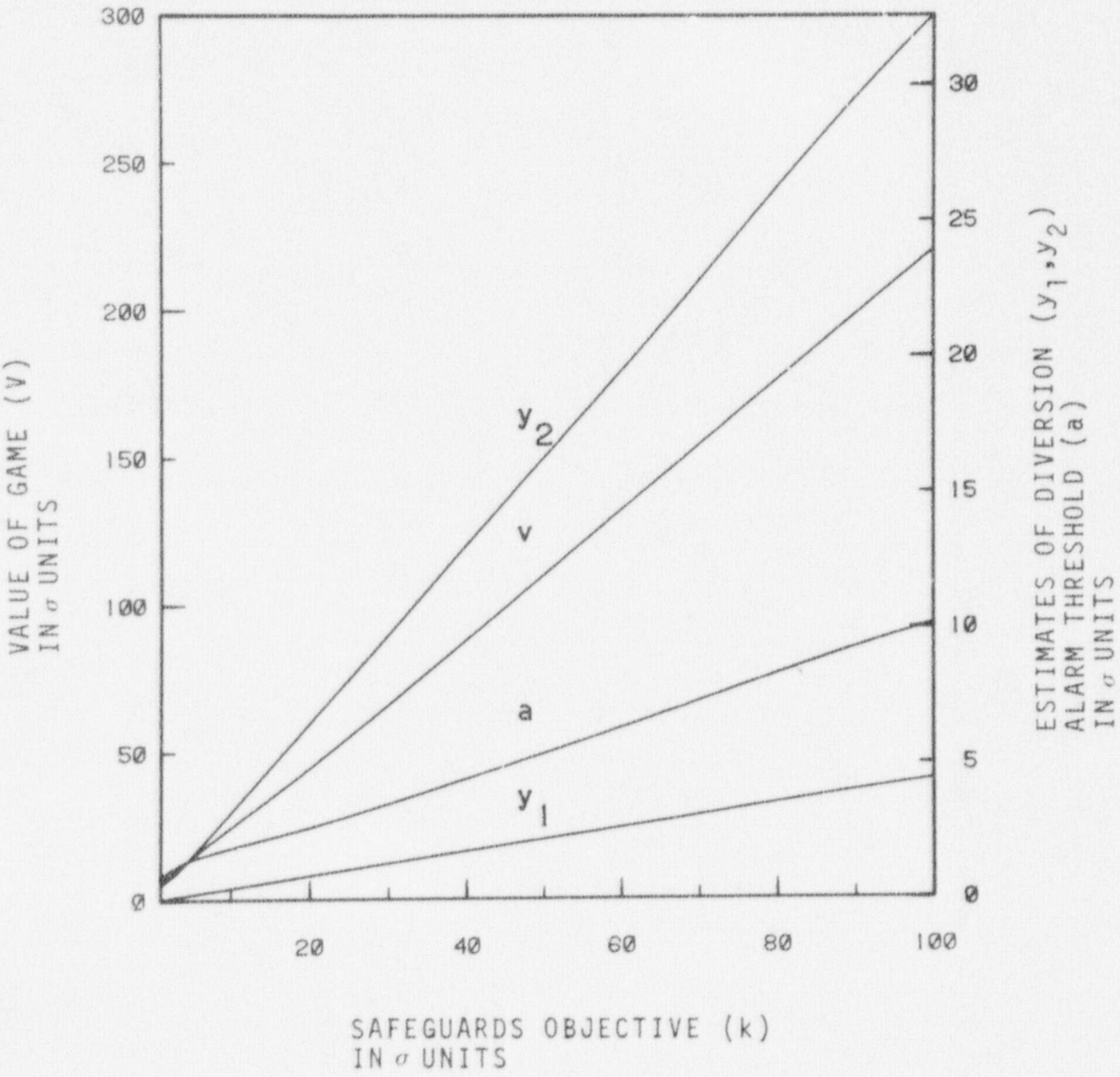

Figure A2. Results of Varying the Amount of Material Vulnerable to Diversion

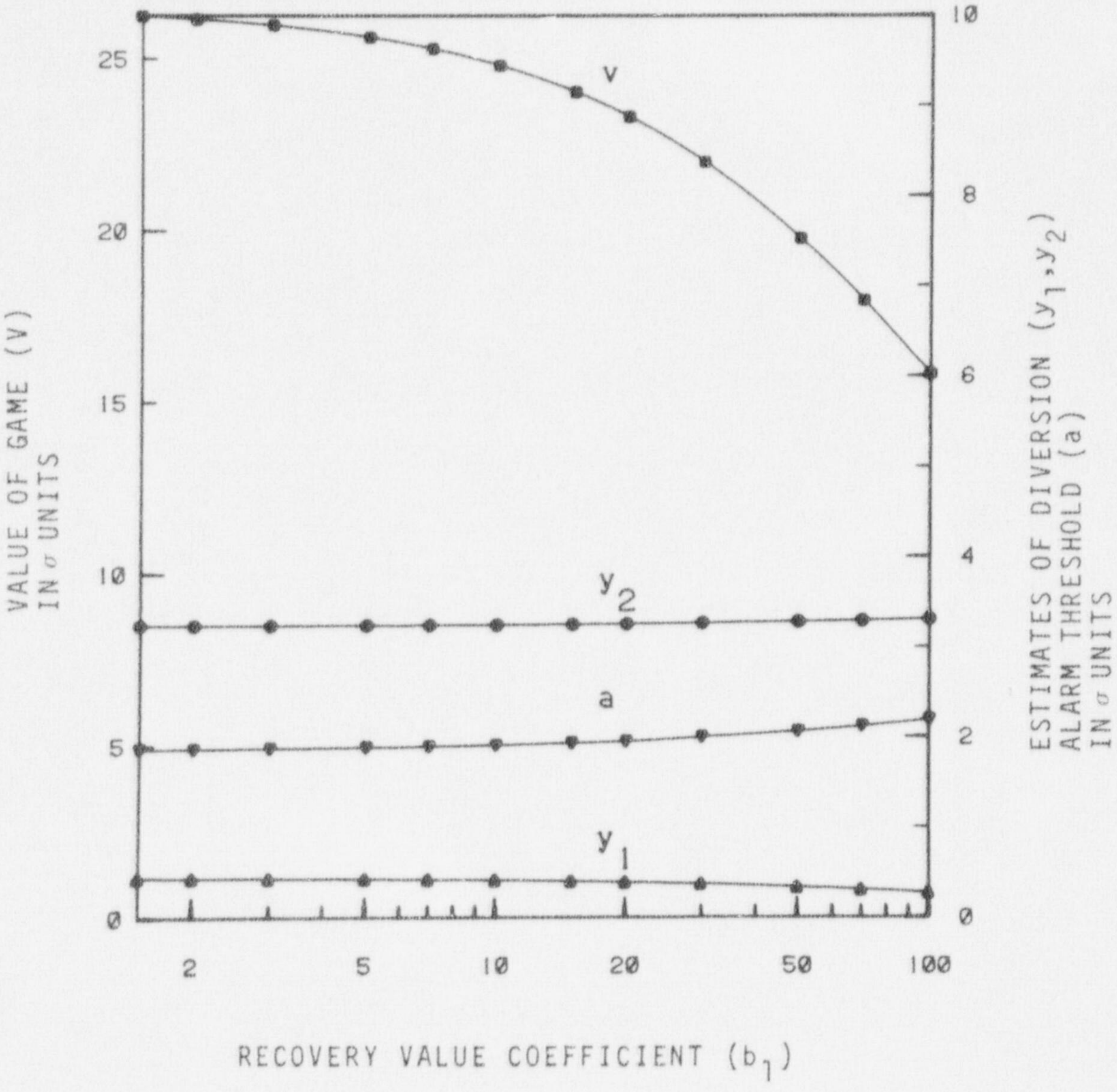

Figure A3. Result of Varying the Below-Threshold Recovery Value Coefficient

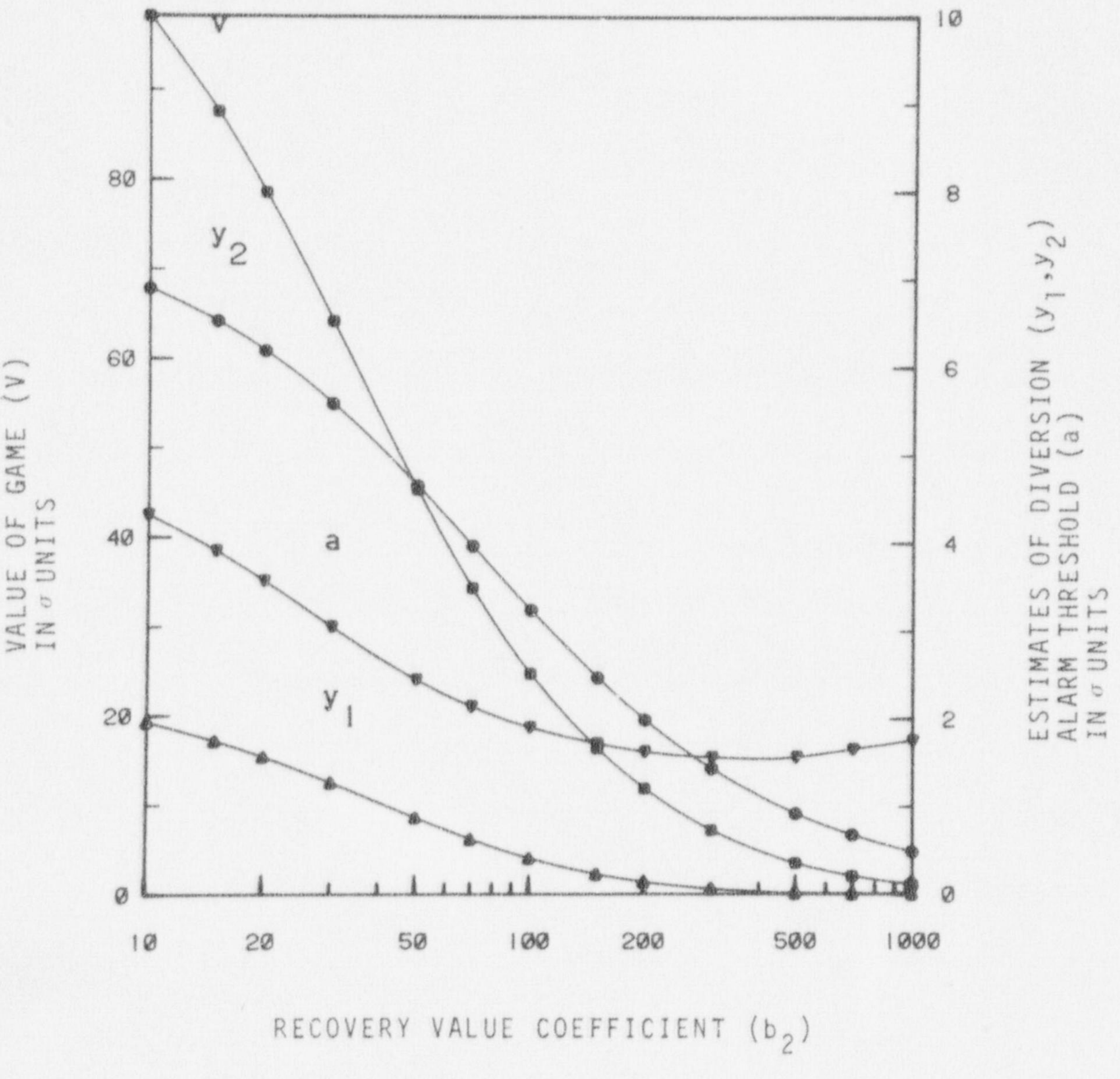

Figure A4. Results of Varying the Above-Threshold Recovery Value Coefficient

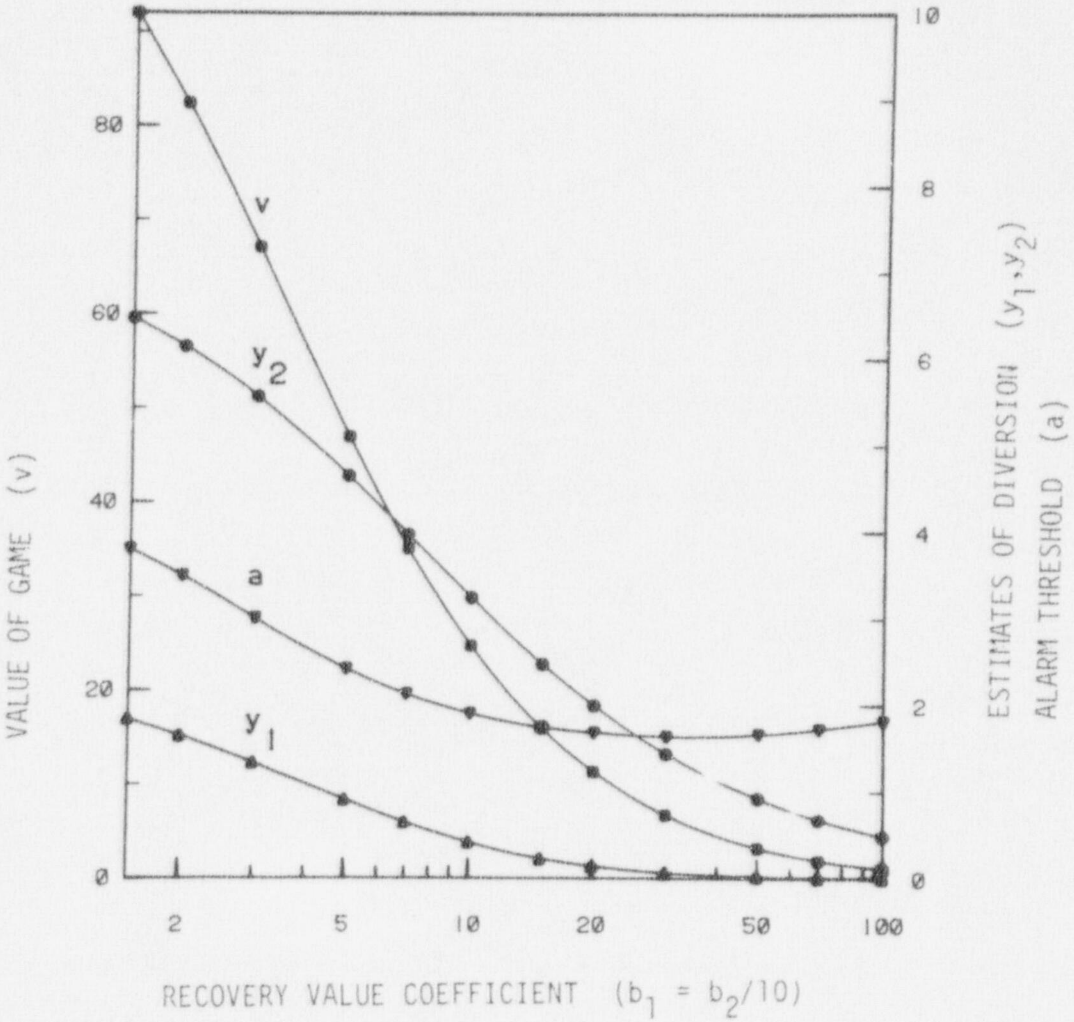

Figure A5. Results of Simultaneously Varying Both Recovery Value Coefficients

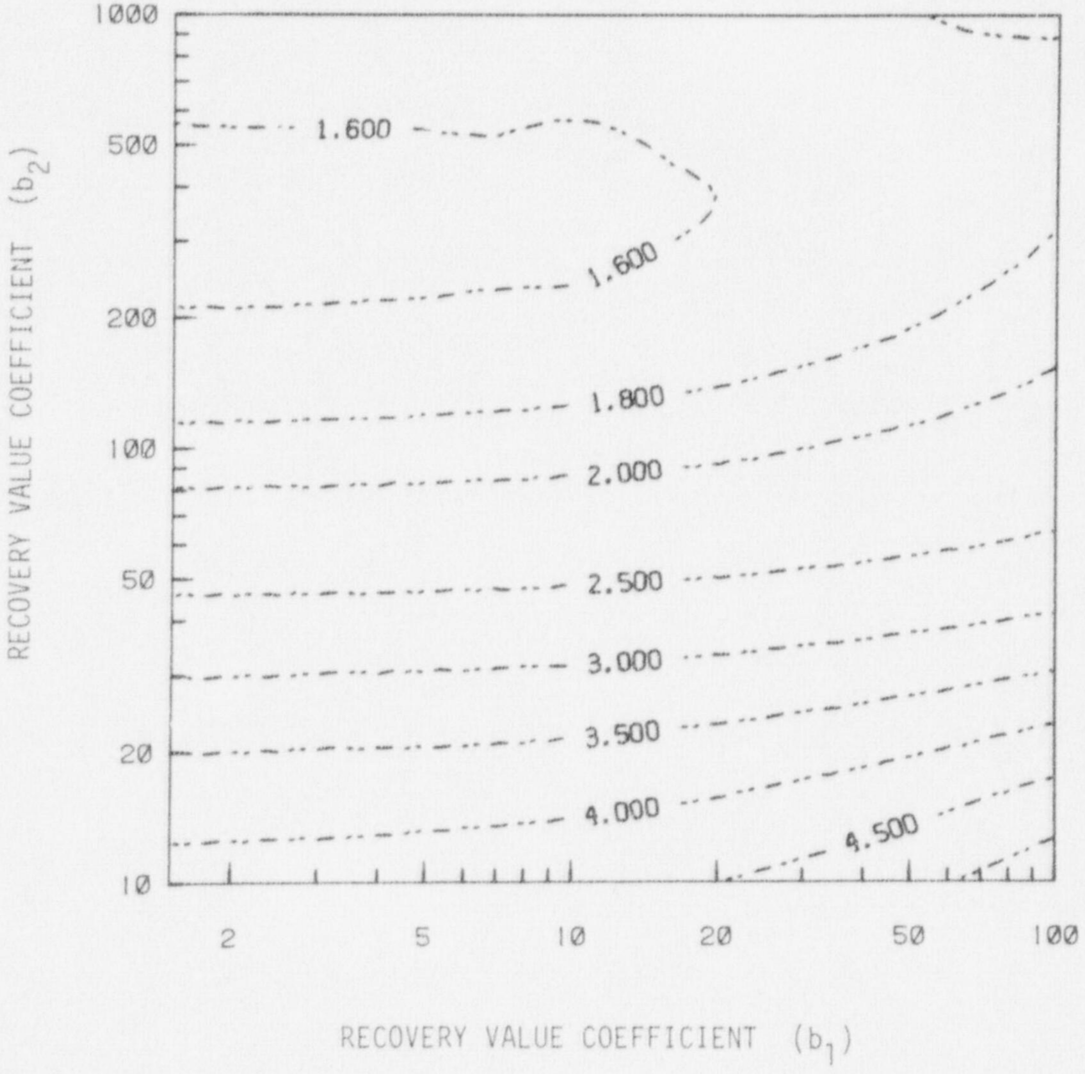

Figure A6. Optimal Alarm Threshold for Different Recovery Value Coefficients

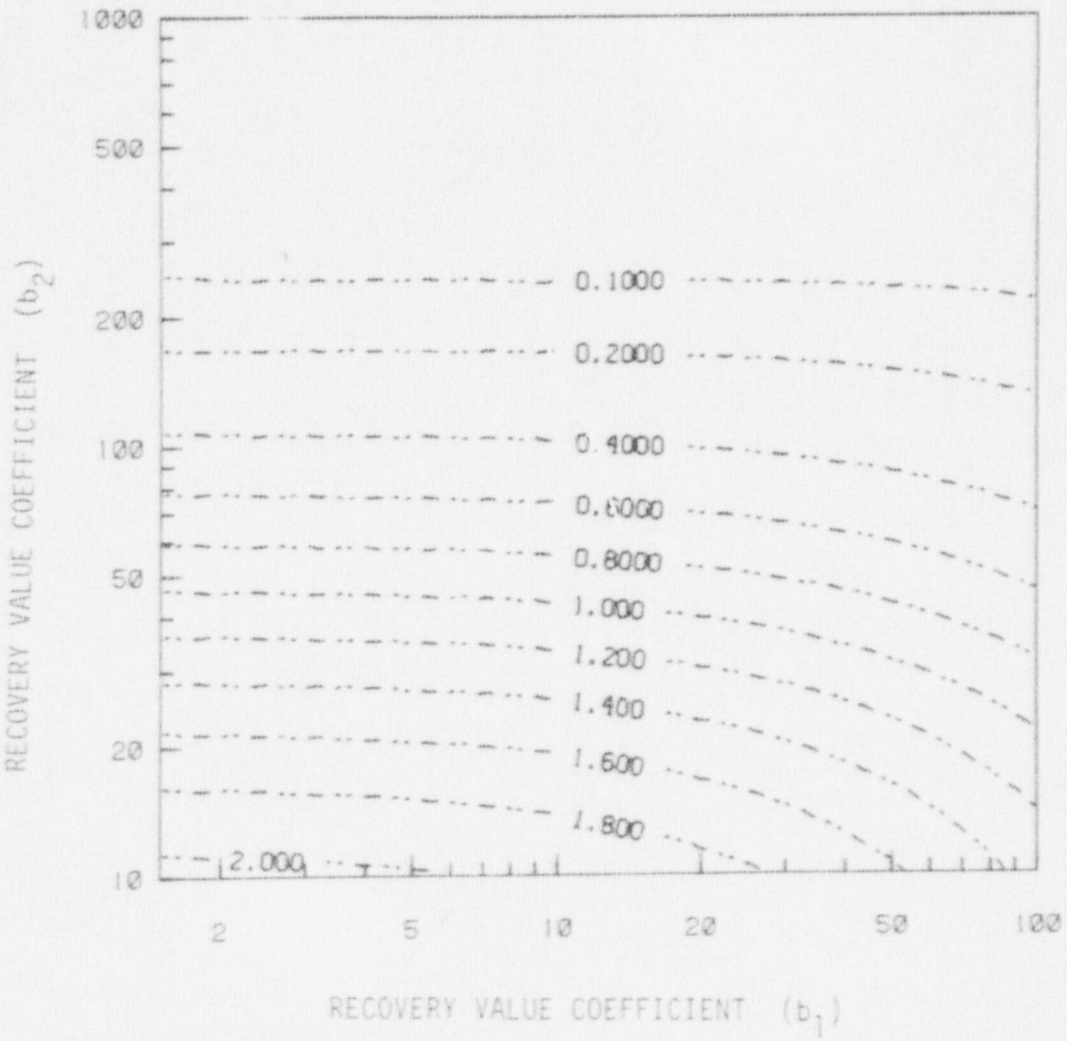

Figure A7. Estimated Diversion Below-Threshold for Different Recovery Value Coefficients

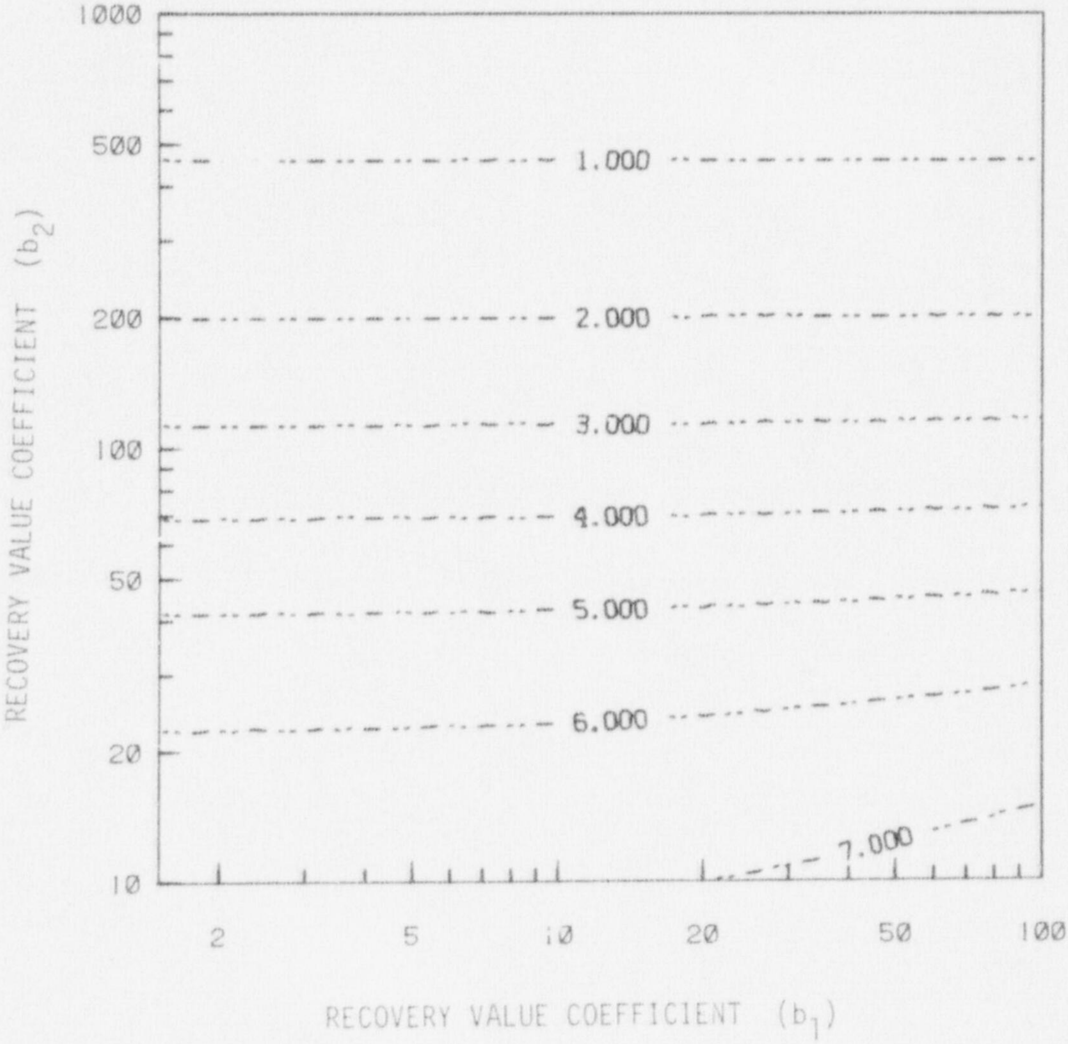

Figure A8. Estimated Diversion Above-Threshold for Different Recovery Value Coefficients

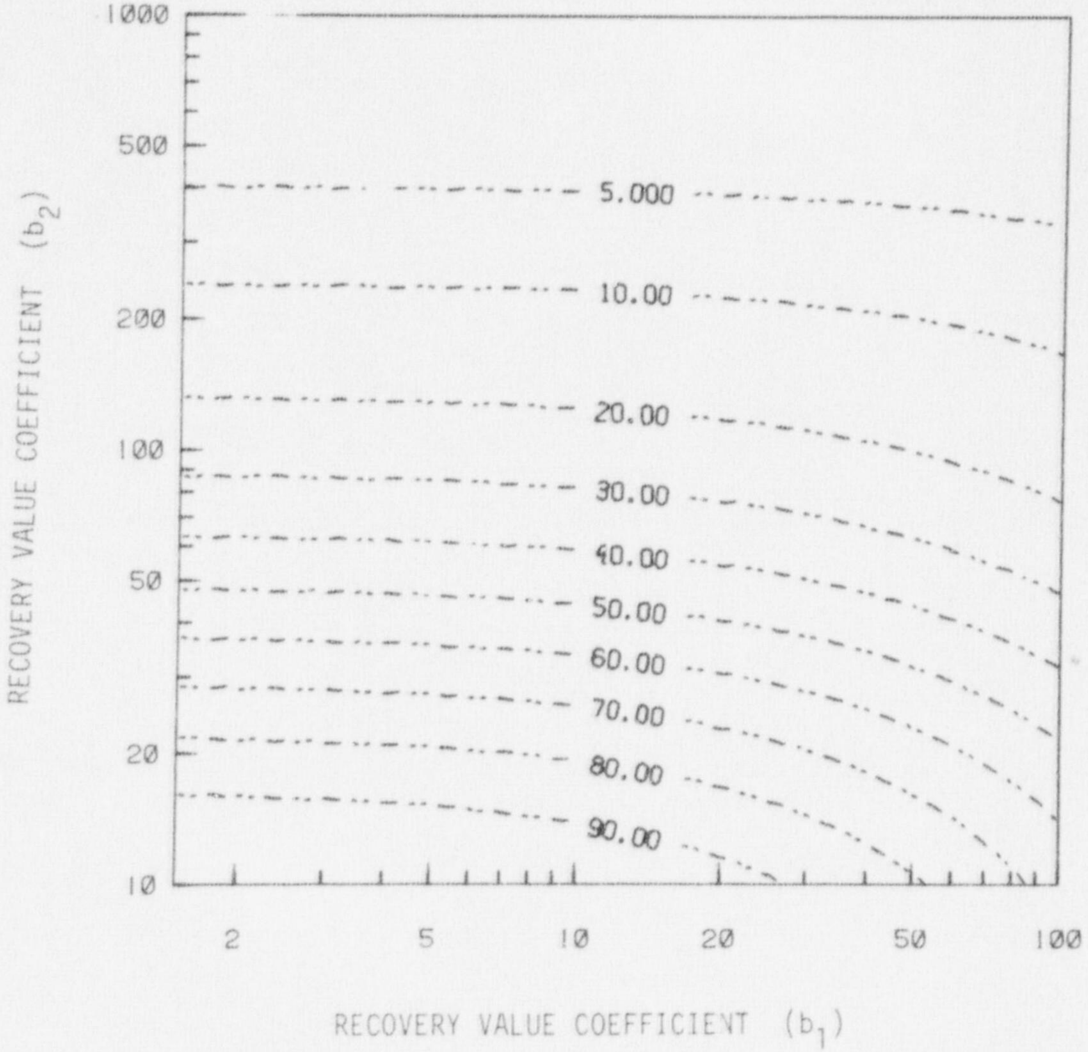

Figure A9. Expected Losses to the Defender for Different Recovery Value Coefficients

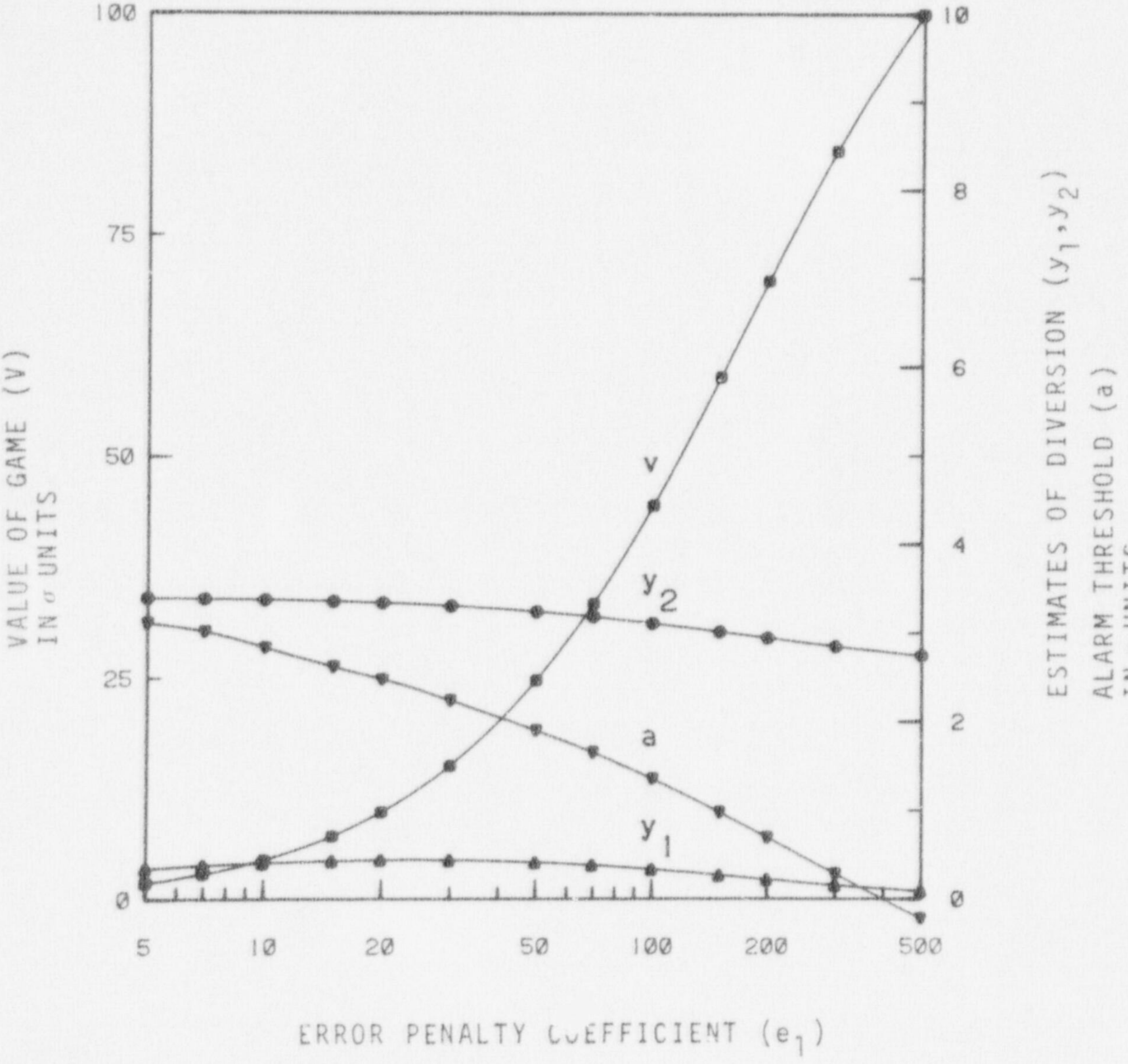

Figure A10. Results of Varying the Below-Threshold Error Penalty Coefficient

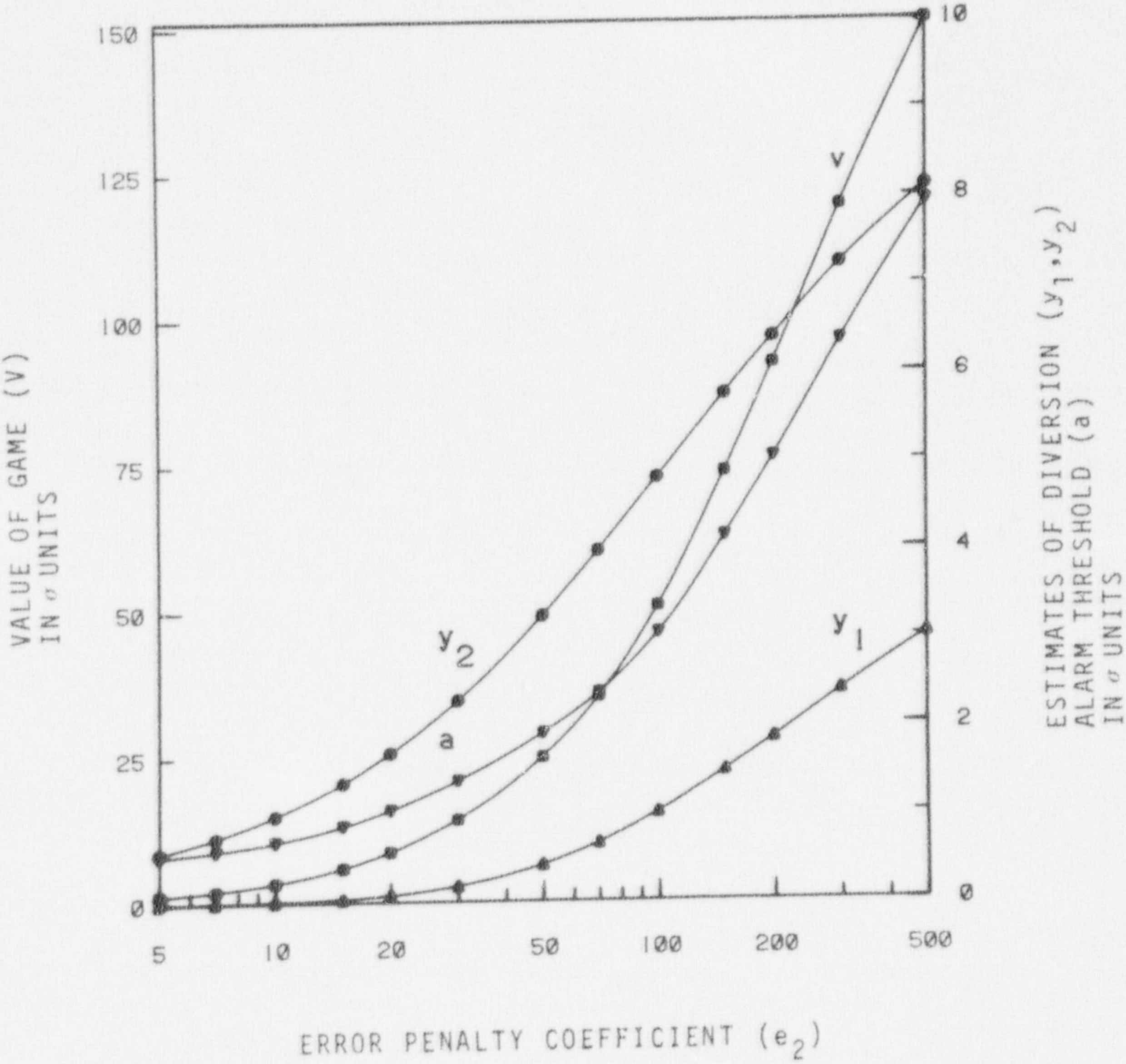

Figure A11. Results of Varying the Above-Threshold Error Penalty Coefficient

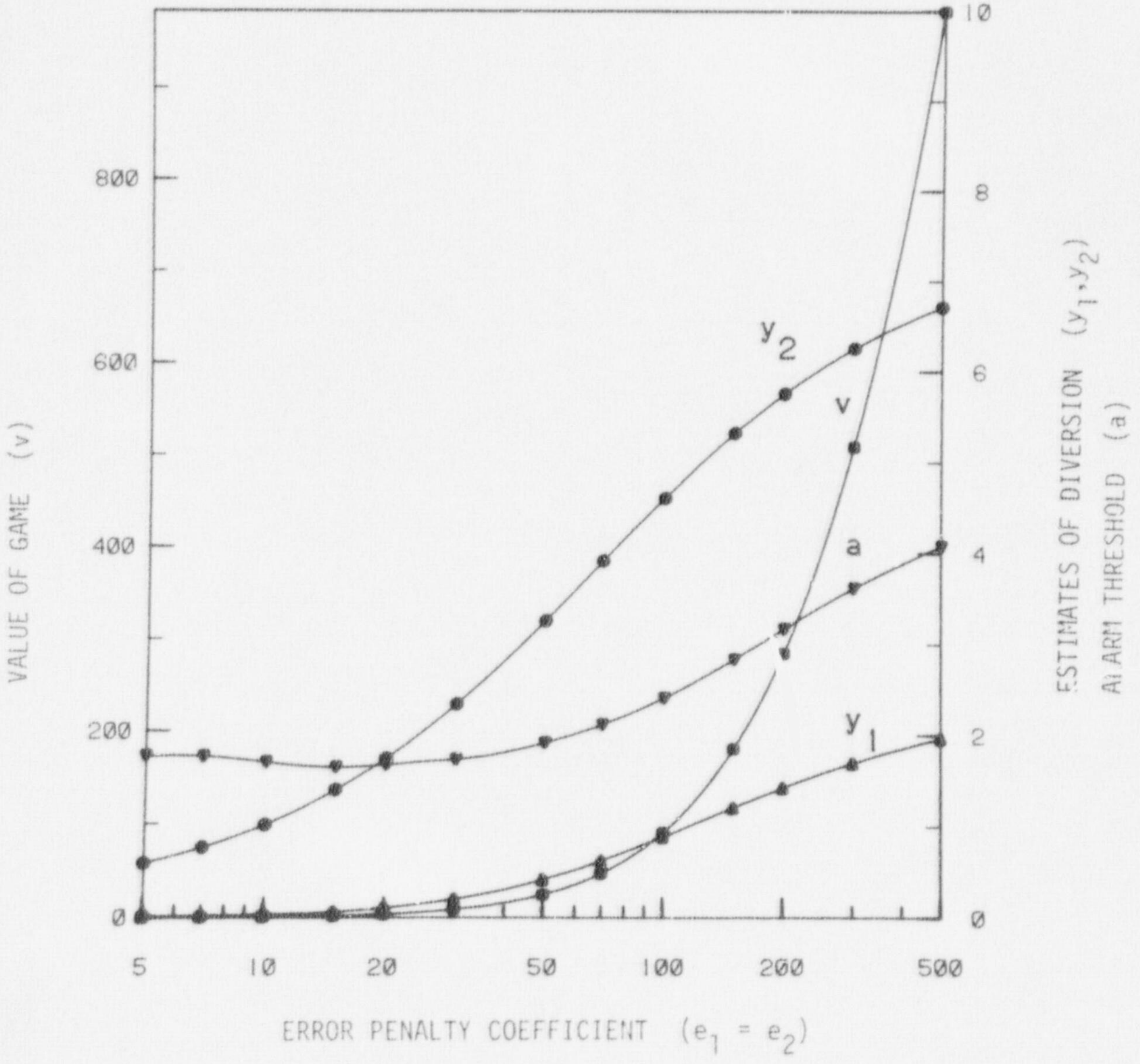

Figure A12. Results of Simultaneously Varying Both Error Penalty Coefficients

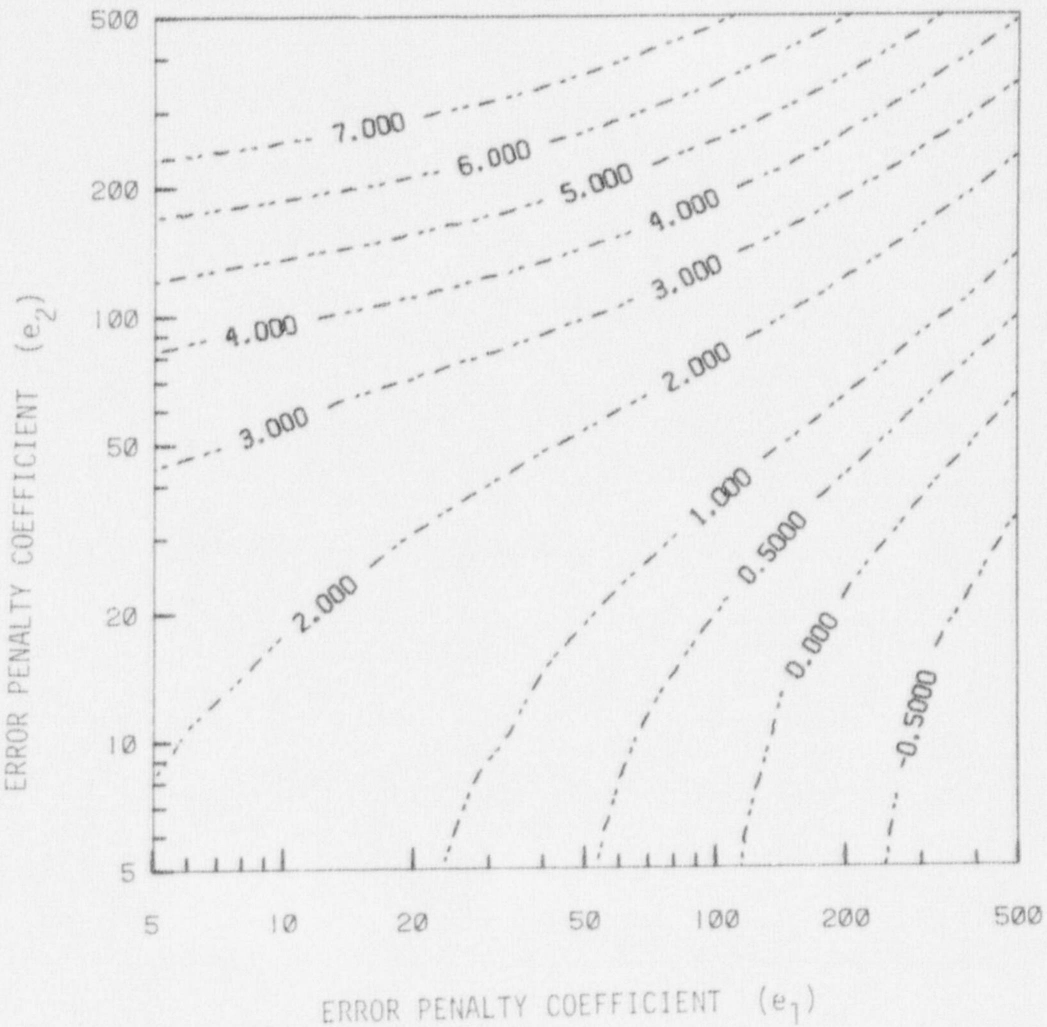

Figure A13. Optimal Alarm Threshold for Different Error Penalty Coefficients

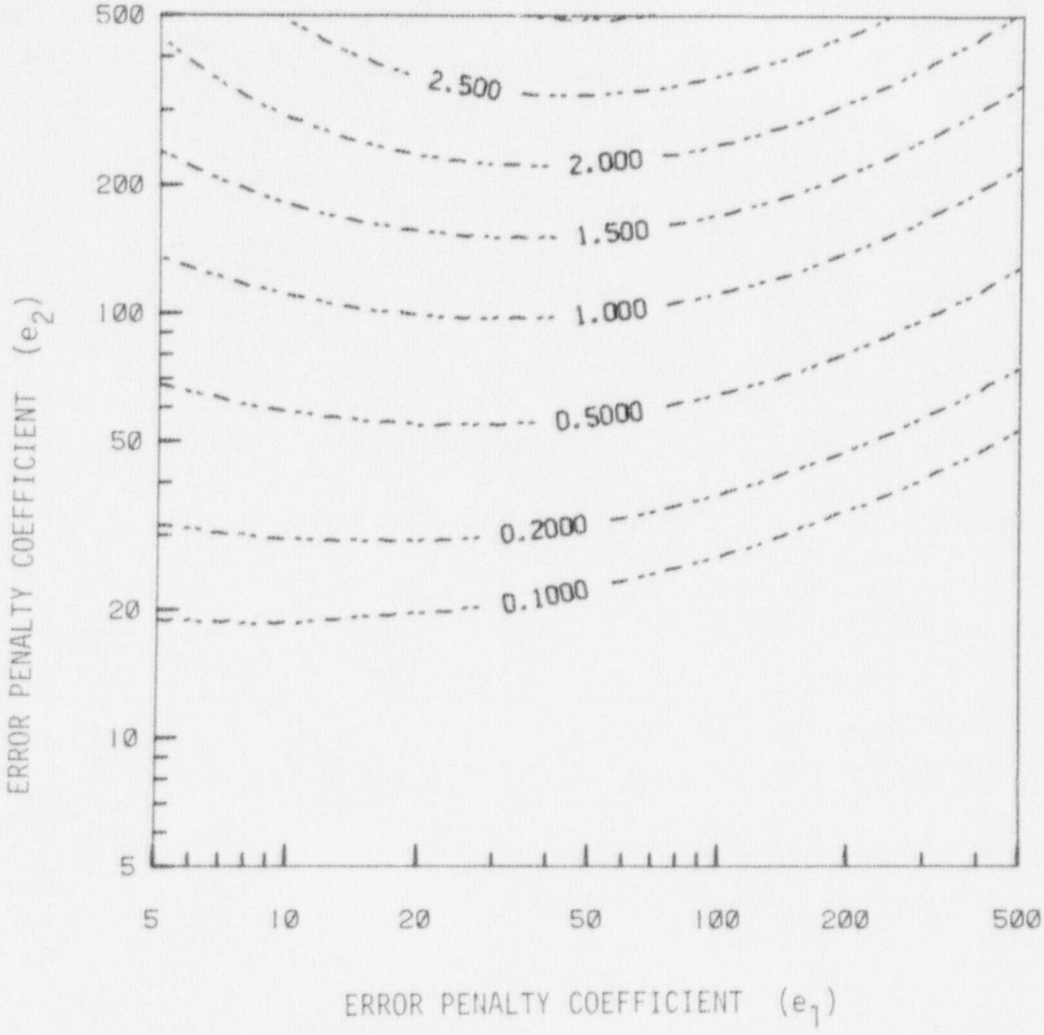

Figure A14. Estimated Diversion Below Threshold for Different Error Penalty Coefficients

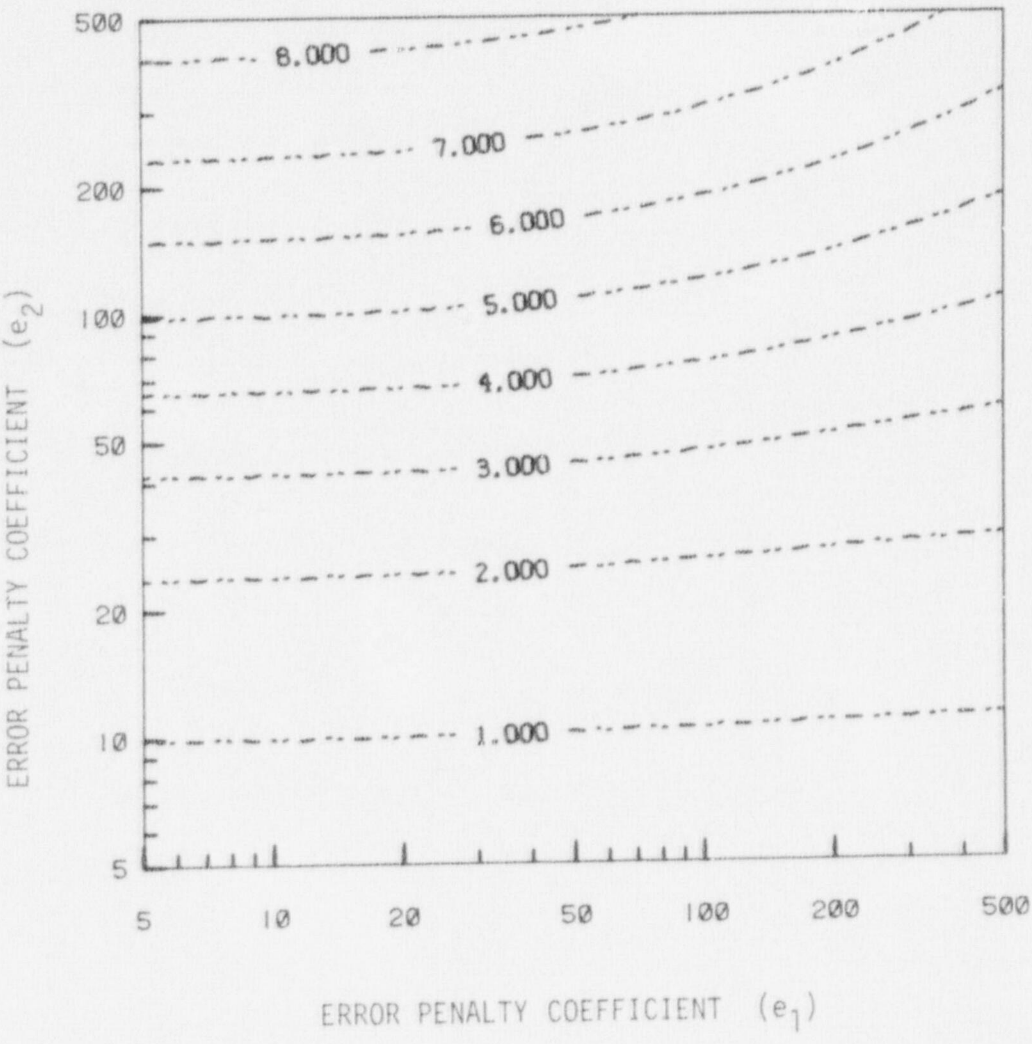

Figure A15. Estimated Diversion Above Threshold for Different Error Penalty Coefficients

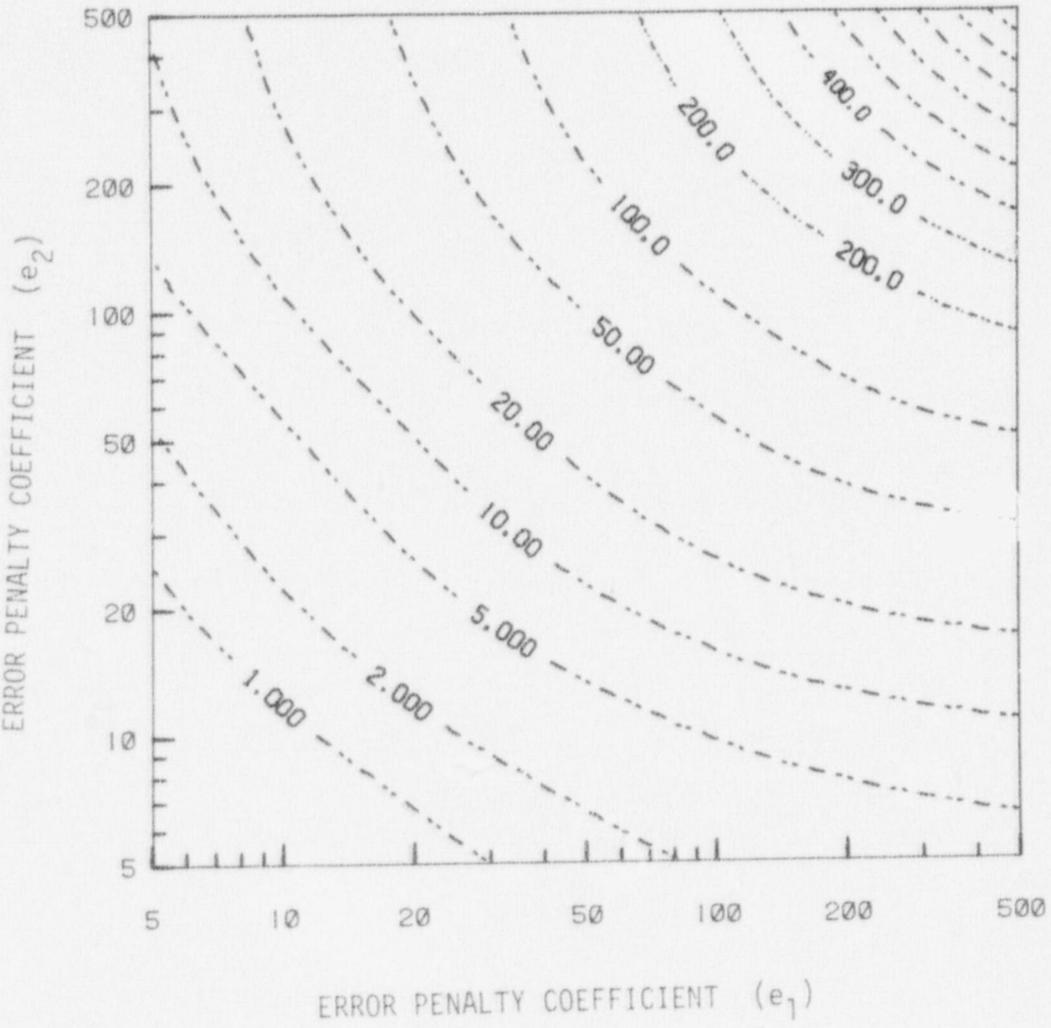

Figure A16. Expected Losses to the Defender for Different Error Penalty Coefficients

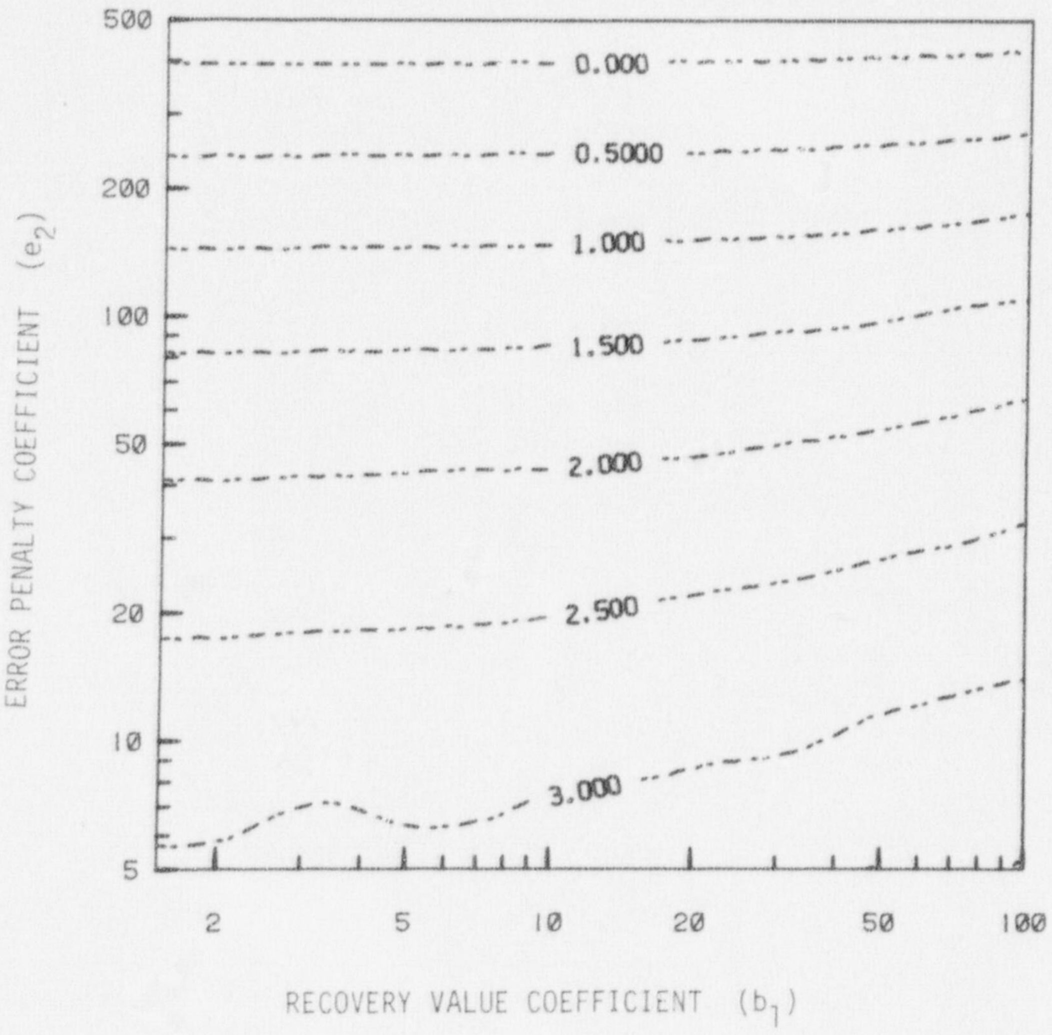

Figure A17. Optimal Alarm Threshold for Different Values of b_1 and e_1

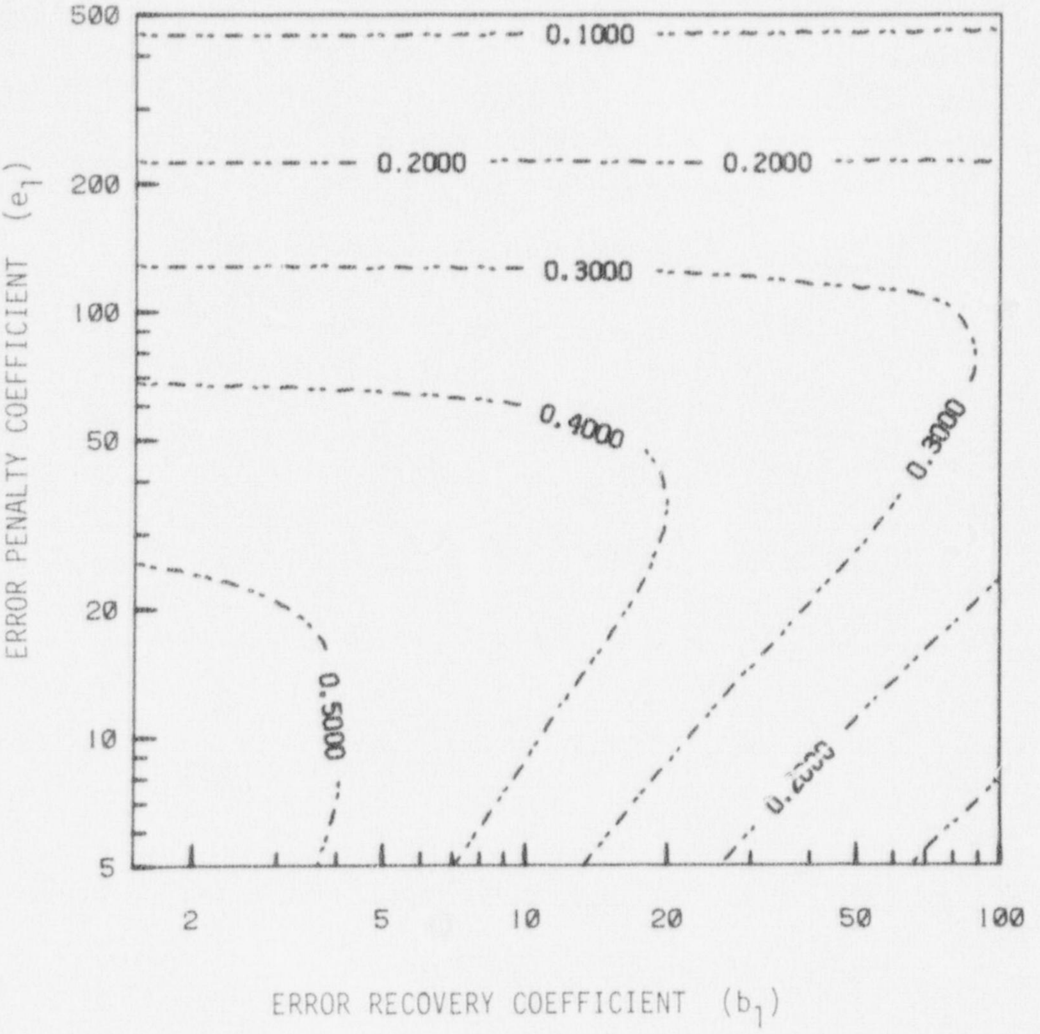

Figure A18. Estimated Diversion Below Threshold for Different Values of b_1 and e_1

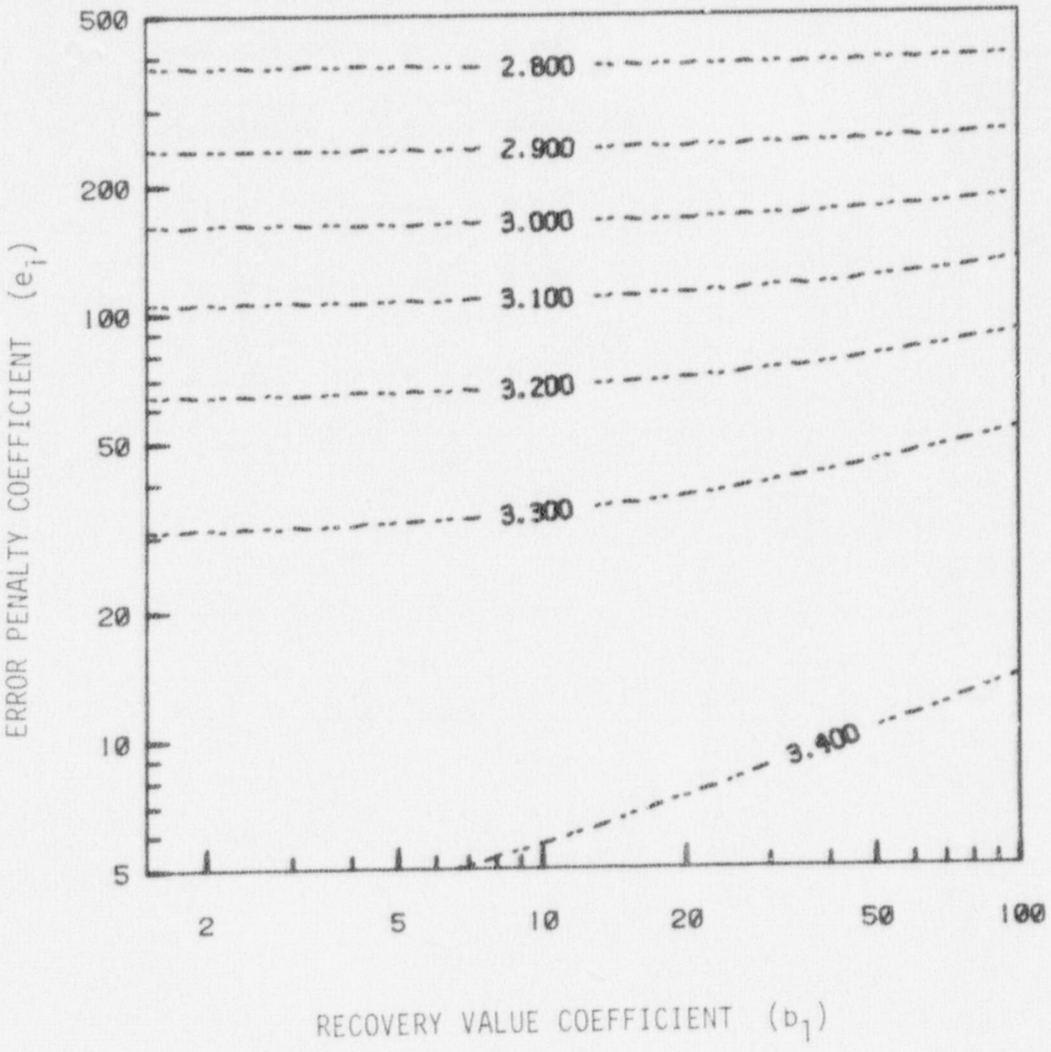

Figure A19. Estimated Diversion Above Threshold for Different Values of b_1 and e_1

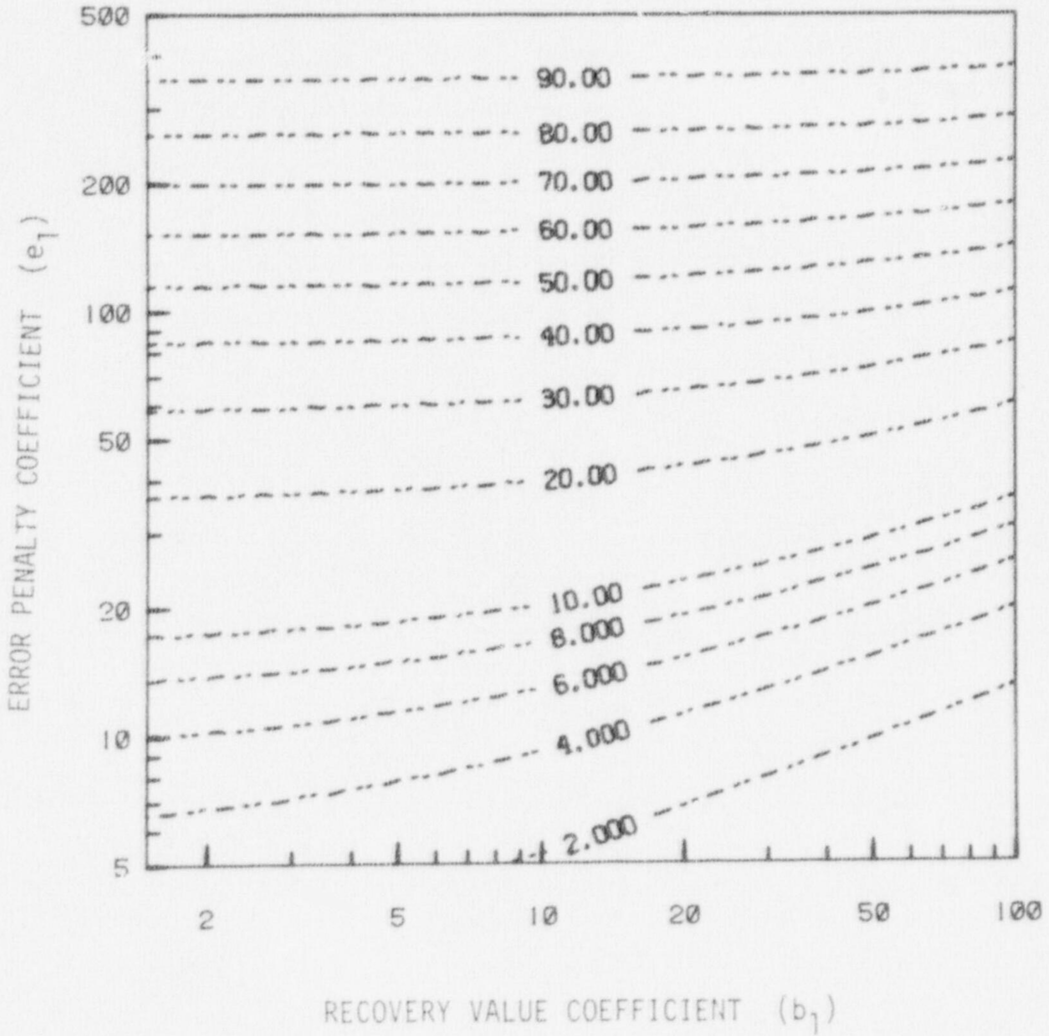

Figure A20. Expected Losses to the Defender for Different Values of e_1 and b_1

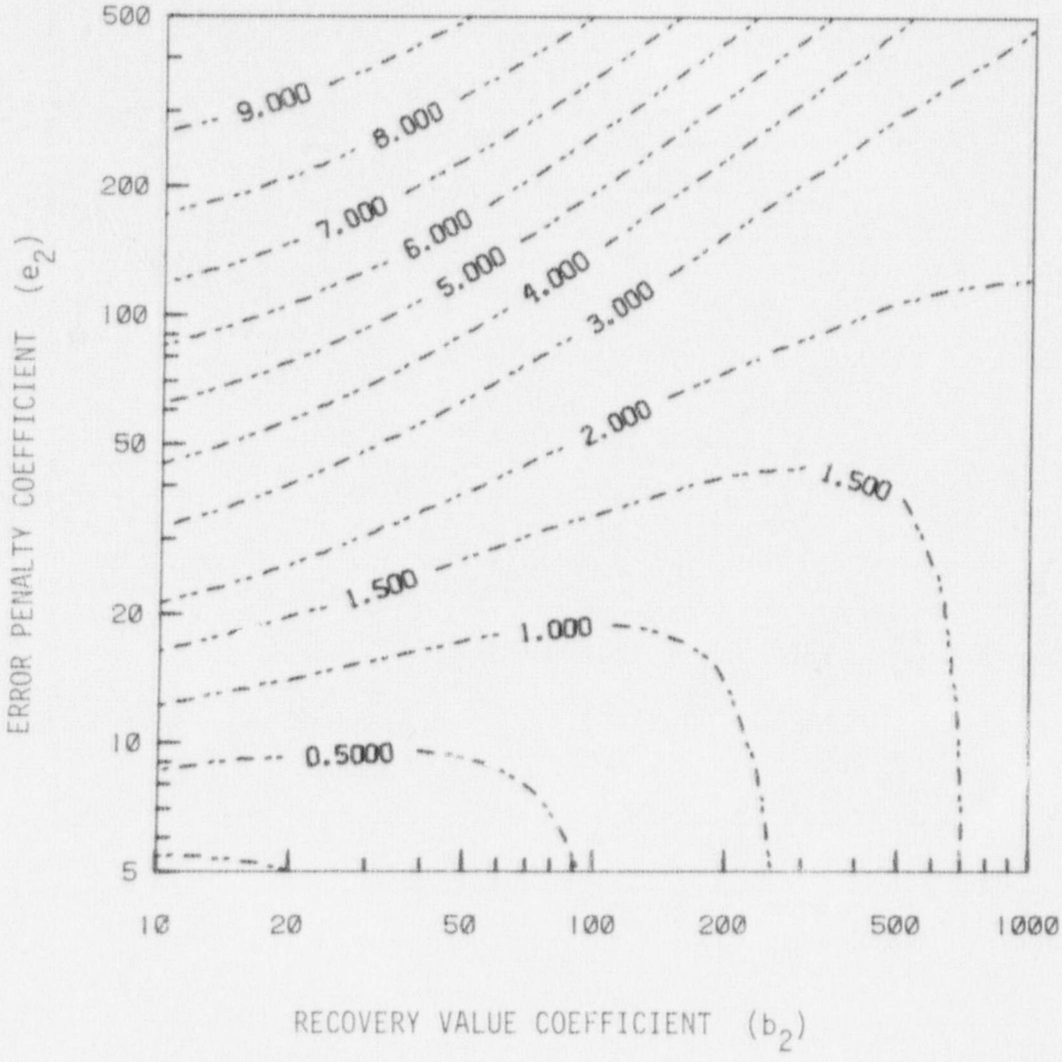

Figure A21. Optimal Alarm Threshold for Different Values of b_2 and e_2

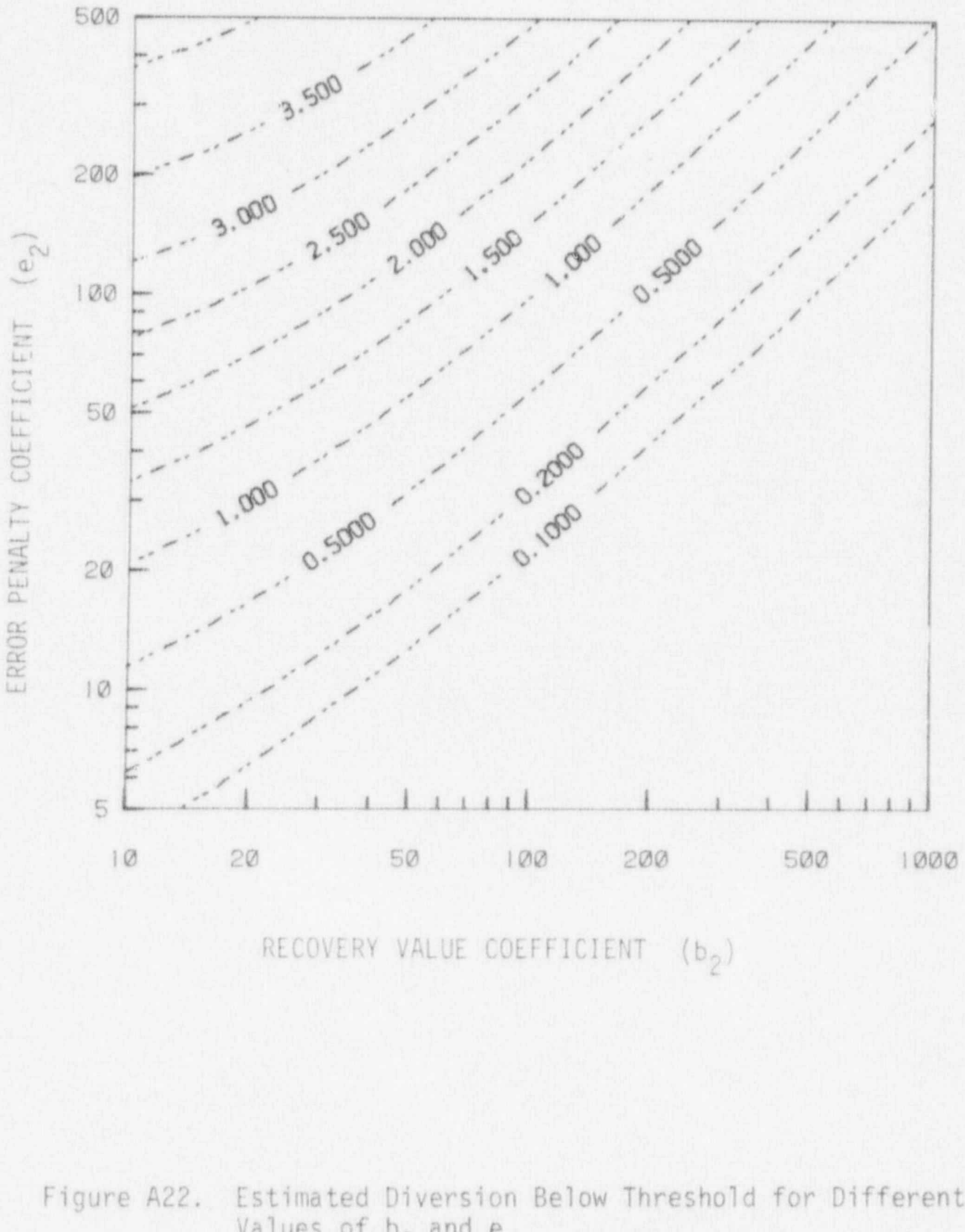

Figure A22. Estimated Diversion Below Threshold for Different Values of b_2 and e_2

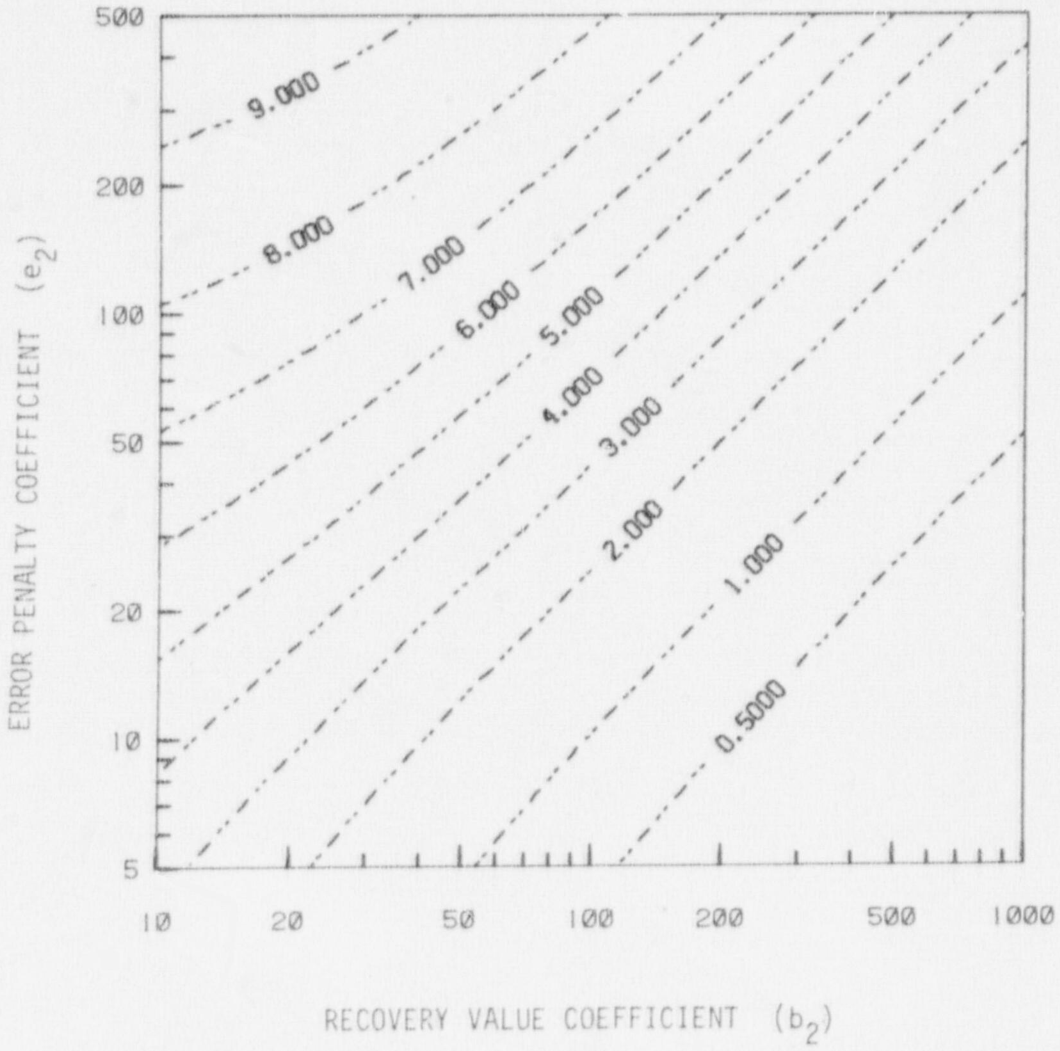

Figure A23. Estimated Diversion Above Threshold for Different Values of b_2 and e_2

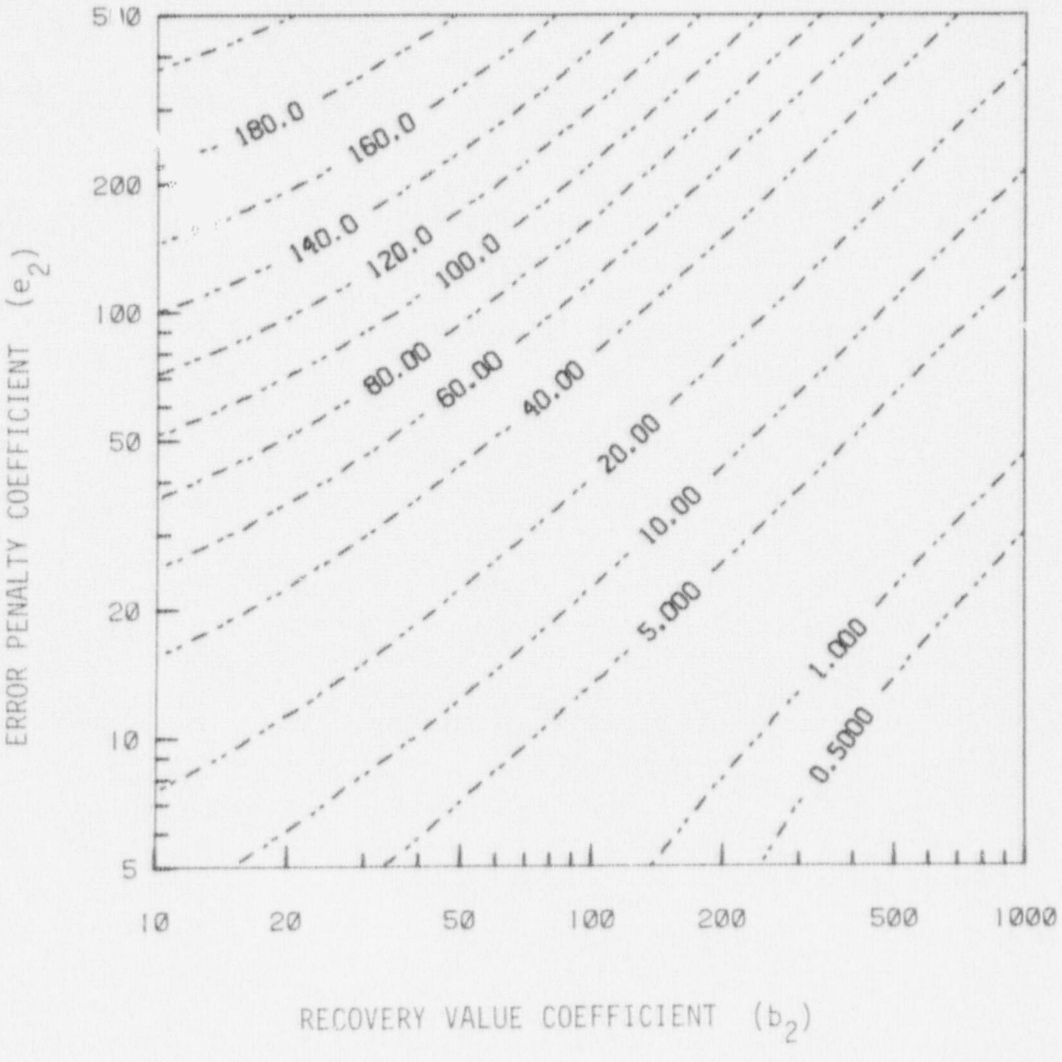

Figure A24. Expected Losses to the Defender for Different Values of b_2 and e_2

APPENDIX B

SOME ALARM THRESHOLD MODELS

1. INTRODUCTION

In this appendix we extend the game model of NUREG-0290 by introducing the alarm threshold as a strategic variable. We shall assume that one of the moves of the defender is the choice of an alarm threshold whereas in NUREG-0290 the alarm threshold was fixed. The particular alarm threshold chosen is known by the defender, but not by the diverter. We shall formulate some games in which the alarm threshold is the only strategic variable of the defender. This simplification yields a one-dimensional strategy space for the defender which can be analyzed for optimality in closed form. This was not possible when the defender's strategy space was three-dimensional.

2. THE GENERAL ALARM THRESHOLD GAME MODEL

The general strategic and competitive situation that exists in nuclear material accounting and what we wish to model may be described as follows:

The nuclear material accounting system reports that u grams of SNM is unaccounted for, or the imbalance of inventory, u, is due to measuring errors and possible unauthorized removal of SNM. Based on this value of u, a decision is to be made or some action is to be taken with respect to this material unaccounted for. From the set of possible alternatives or actions available, the problem is to choose the optimal action--optimal in the sense that it satisfies some objective or criterion of the decision-maker.

This situation may be modeled by formulating a two-person game. We shall first describe the game in extensive form--in terms of the moves of the game. From this extensive form we shall derive the normalized form of the game, which will be used for the analysis.

The game may be reduced to the following three-move game:

Move 1. Diverter removes x grams of SNM

Move 2. Defender observes that u grams of SNM are unaccounted for

Move 3. Defender takes some action α.

The first move of the game is a strategic decision by the Diverter. It is a choice x where $0 \leq x \leq k$. Move 2 is a chance move, a choice $u(x)$ by Nature. Move 3 is a strategic decision $\alpha(u)$ by the Defender.

Having described the game in extensive form, one may now formulate the game in normal form. This is necessary for analytic purposes. This is accomplished by defining a strategy for each player. A strategy for the Diverter is a number x, where $0 \leq x \leq k$. A strategy for the Defender is a function, since he has information about u. Thus a strategy for the Defender is a function α where

$$\alpha_0(u) \leq \alpha(u) \leq \alpha_1(u)$$

From the above it is clear that the Diverter has a one-dimensional strategy space, namely the line 0 to k. However, the strategy space of the Defender may be any dimension. We shall analyze the game where α is one-dimensional. This is the situation where the alarm threshold is the only strategic variable.

3. PAYOFF

In order to complete the description of the game, we need to describe the payoff to the Defender for each strategic choice of the Diverter and Defender. Suppose the Defender wishes to minimize his losses; then we need to describe how these losses depend on x and $\alpha(u)$--i.e., we need to give the function $M(x,\alpha(u))$.

In general, it is also necessary to describe the payoff to the Diverter--$N(x, \alpha(u))$. However, if we assume that with respect to these payoffs, M and N, the Diverter and Defender are non-cooperative, then this is equivalent to a constant-sum game and we need to consider only one payoff, M. This rules out the possibility of collusion and cooperation between the Diverter and Defender.

An optimal strategy of the Defender in such a situation protects the Defender against anything the Diverter may do and even against any payoff the Diverter may set for himself. It also includes the case that no Diverter exists.

4. ALARM THRESHOLD GAME WITH OPTIMAL PURE STRATEGIES

Let us analyze the one-dimensional alarm threshold game, when the alarm threshold is the only strategic variable of the Defender. This game is described as follows:

Diverter diverts x where $0 \leq x \leq k$

Defender, knowing u, chooses alarm threshold $\alpha(u)$ where $-\infty \leq \alpha(u) \leq \infty$.

Since $\alpha(u)$ is the alarm threshold, this implies that

i) Defender takes action 1 if $u \leq \alpha(u)$

ii) Defender takes action 2 if $u > \alpha(u)$

Further, since $-\infty \leq \alpha(u) \leq \infty$ this is equivalent to the Defender picking a strategy z where $-\infty \leq z \leq \infty$ with the condition that

i) Defender takes action 1 if $u \leq z$

ii) Defender takes action 2 if $u > z$

Having formulated the strategy spaces, we now need to formulate the payoff associated with these strategies. The payoff to the Defender is measured by his losses or costs as follows:

1) If Defender takes action 1, a minimal effort, he loses the diverted material, x, or $M_1 = x$.

2) If Defender takes action 2, for instance by taking inventory at a cost β, then with probability b he recovers the diverted material. His losses and costs are

$$M_2 = \beta + x - bx = \beta + (1-b)x.$$

Now M_1 and M_2 are conditional payoffs, conditional upon MUF being below or above, respectively, the alarm threshold. Letting $P(u \leq z) = F(z,x)$ and $P(u > z) = G(z,x)$, then the expected payoff to the Defender or his expected losses are given by

$$M(x,z) = xF(z,x) + [\beta+(1-b)x]G(z,x) \tag{1}$$

In order to solve this game, we shall first obtain upper and lower bounds to the game value. First, let us derive the lower game value, $\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z)$. Since

$$G(z,x) = 1 - F(z,x)$$

we can rewrite the payoff as

$$M(x,z) = \beta + (1-b)x + (bx-\beta)F(z,x) \tag{2}$$

For any x, we have

$$\underset{z}{\text{Min}}\ M(x,z) = \begin{cases} \beta + (1-b)x + (bx-\beta)\underset{z}{\text{Min}}F(z,x), & \text{if } x \geq \frac{\beta}{b} \\ \beta + (1-b)x + (bx-\beta)\underset{z}{\text{Max}}F(z,x), & \text{if } x \leq \frac{\beta}{b} \end{cases}$$

Now let us assume that F(z,x) is a normal distribution function with mean x, then

$$\underset{z}{\text{Min}}\ F(z,x) = 0,\ \underset{z}{\text{Max}}\ F(z,x) = 1,$$

and

$$\underset{z}{\text{Min}}\ M(x,z) = \begin{cases} \beta + (1-b)x, & \text{if} \quad x \geq \frac{\beta}{b} \\ x, & \text{if} \quad x \leq \frac{\beta}{b} \end{cases}$$

Now maximizing with respect to x we have

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \text{Max} \left[\underset{bx \geq \beta}{\text{Max}}[\beta + (1-b)x] \quad , \quad \underset{bx \leq \beta}{\text{Max}}\ x \right] \tag{3}$$

Suppose $\frac{\beta}{b} \leq k$, then from (3) we get

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \text{Max}\ [\beta + (1-b)k, \frac{\beta}{b}]$$

Further, we have that

$$\beta + (1-b)k \geq \beta + (1-b)\frac{\beta}{b} = \frac{\beta}{b}$$

Hence we have

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \beta + (1-b)k$$

and the maximum is assumed at x=k.

Now suppose that $\frac{\beta}{b} \geq k$, then from (3) we get

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \text{Max}\ [\phi, k] = k$$

where ϕ is the null function (over a null set). The maximum is assumed at x=k.

We have thus shown that

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \begin{cases} \beta + (1-b)k, & \text{if } \frac{\beta}{b} \leq k \\ k, & \text{if } \frac{\beta}{b} \geq k \end{cases}$$

In each case the maximum is assumed at x=k.

Now let us compute $\underset{z}{\text{Min}}\ \underset{x}{\text{Max}}\ M(x,z)$ where

$$M(x,z) = xF(z,x) + [\beta + (1-b)x]G(z,x)$$

We have for any fixed z

$$\underset{x}{\text{Max}}\ M(x,z) = kF(z,k) + [\beta + (1-b)k]G(z,k)$$
$$= \beta + (1-b)k + (bk-\beta)F(z,k).$$

Now minimizing with respect to z, we have

$$\underset{z}{\text{Min}}\ \underset{x}{\text{Max}}\ M(x,z) = \begin{cases} \beta + (1-b)k, & \text{if } \frac{\beta}{b} \le k \\ k, & \text{if } \frac{\beta}{b} \ge k \end{cases} \qquad (5)$$

where the minimum is assumed at $z = -\infty$ and $z = +\infty$, respectively.

Comparing (4) and (5) we see they are the same. We have proven that this game has a saddle-point at $(k,-\infty)$ or (k,∞) depending on the parameters β, b, k.

The solution of this game can be summarized as follows:

i) The Diverter should always divert the maximum amount k.

ii) The Defender has an optimal pure strategy, $-\infty$ or $+\infty$, depending on the relative costs β and bk. If $\beta \ge bk$ then the Defender sets the alarm threshold at $-\infty$. If $\beta \le bk$, then the Defender sets the alarm threshold at $+\infty$.

iii) The game value is

$$v = \begin{cases} \beta + (1-b)k, & \text{if } \beta \le kb \\ k, & \text{if } \beta \ge kb \end{cases}$$

5. ALARM THRESHOLD GAME WITH OPTIMAL MIXED STRATEGIES

We now keep the strategy space the same as in the previous game but modify the payoff as follows: If an inventory is taken at a cost β, then with probability r the Defender will recover the diverted material. The payoff in this game becomes

$$M(x,z) = xF(z,x) + (\beta-rx)G(z,x) \tag{6}$$

This game model, as compared to the previous one, does not include a penalty to the defender for loss of diverted material when an inventory is taken.

We shall show that this game does not have a saddle-point, and hence mixed strategies will be required. We shall also obtain upper and lower bounds of the game.

First, we rewrite the payoff as follows:

$$M(x,z) = (\beta-rx) + [(1+r)x-\beta]F(z,x) \tag{7}$$

We have for any x

$$\operatorname*{Min}_z M(x,z) = \begin{cases} (\beta-rx) + [(1+r)x-\beta]\operatorname*{Min}_z F(z,x), & \text{if } x \geq \frac{\beta}{1+r} \\ (\beta-rx) + [(1+r)x-\beta]\operatorname*{Max}_z F(z,x), & \text{if } x \leq \frac{\beta}{1+r} \end{cases}$$

$$= \begin{cases} \beta - rx, & \text{if } x \geq \frac{\beta}{1+r} \\ x, & \text{if } x \leq \frac{\beta}{1+r} \end{cases}$$

Now maximizing with respect to x we have

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \text{Max}\left[\underset{x \geq \frac{\beta}{1+r}}{\text{Max}}(\beta - rx), \quad \underset{x \leq \frac{\beta}{1+r}}{\text{Max}}\ x\right]$$

We have to consider two cases. Suppose

$\frac{\beta}{1+r} \leq k$, then $\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \text{Max}\left[\frac{\beta}{1+r}, \frac{\beta}{1+r}\right] = \frac{\beta}{1+r}$

and the maximum is assumed at $x = \frac{\beta}{1+r}$.

Now suppose $\frac{\beta}{1+r} \geq k$, then

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \text{Max}[\phi, k] = k$$

where ϕ is a null function (over a null set). The maximum is assumed at $x = k$.

We have thus proven that

$$\underset{x}{\text{Max}}\ \underset{z}{\text{Min}}\ M(x,z) = \min\left(\frac{\beta}{1+r}, k\right). \qquad (8)$$

and the maximum is assumed at $x = \min\left(\frac{\beta}{1+r}, k\right)$.

Now let us compute the upper bound of the game or $\underset{z}{\text{Min}}\ \underset{x}{\text{Max}}\ M(x,z)$. For any z, we can show that, by computing $dM(x,z)/dx$, that the function $M(x,z)$ has at most two critical values and hence at most one maximum point which is also a critical point. Let this critical maximum be designated by $x_m(z)$, then $\underset{x}{\text{Max}}\ M(x,z)$ will be assumed at one of the following three values

$$x = 0,\ x_m(z),\ k$$

depending on z. Further, small values of z are associated with $x = 0$, large values yield $x = k$ and intermediate values of z yield x_m, where $0 < x_m < k$ and $M'(x_m,z) = 0$.

We have from the preceding argument

$$\min_z \max_x M(x,z) = \min \left[\min_{-\infty \le z \le z_1} M(0,z), \quad \min_{z_1 \le z \le z_2} M(x_m,z), \quad \min_{z_2 \le z \le \infty} M(k,z) \right] \tag{9}$$

$$= \min_{z_1 \le z \le z_2} M(x_m,z)$$

Clearly,

$$\min_{z_1 \le z \le z_2} M(x_m,z) \neq \min\left(\frac{\beta}{1+r}, k\right) .$$

We have shown that

$$\min_z \max_x M(x,z) \neq \max_x \min_z M(x,z) \tag{10}$$

and hence this game requires mixed strategies for both players.

6. SOLUTION OF ALARM THRESHOLD GAME--OPTIMAL MIXED STRATEGIES

In Section 5 we showed that the one-dimensional game of alarm threshold as a strategic variable requires mixed strategies for both players. Since the payoff function is continuous in the strategic variables, there exists a solution which requires mixing over a finite number of strategies for each player, rather than a density function. Thus the game is fundamentally a finite game in the sense that only a finite number of strategies are selected from the continuum of strategies. Thus the solution problem is reduced to finding a finite set of x's and a finite set of z's such that

$$\underset{x}{\text{Max}} \int_{z=-\infty}^{+\infty} M(x,z)dG^*(z) = \underset{z}{\text{Min}} \int_{0}^{k} M(x,z)dF^*(x)$$

where $F^*(x)$ and $G^*(z)$ are the optimal mixed strategies of the diverter and defender, respectively. Both $F^*(x)$ and $G^*(z)$ are step-functions with jumps at the critical points of the integral functions.

Analyzing these integral functions, we can prove that for any z,

$$\int M(x,z)dG^*(z)$$

can assume a critical maximum, at most twice. Similarly for any x,

$$\int M(x,z)dF^*(x)$$

can assume a minimum for at most three different values of z. This implies that

$$F^*(x) = \alpha I_{x_1}(x) + (1-\alpha)I_{x_2}(x)$$

and

$$G^*(z) = \bar{\alpha} I_{z_1}(z) + \beta I_{z_2}(z) + \gamma I_{z_3}(z)$$

where

$$\bar{\alpha} + \beta + \gamma = 1.$$

In order to obtain a solution of the game, it is necessary to solve the following equations in closed form

$$\underset{x}{\text{Max}} \int_{-\infty}^{\infty} M(x,z)dG^*(z) = v$$

$$\underset{z}{\text{Min}} \int_{0}^{k} M(x,z)dF^*(x) = v$$

where v is the value of the game. Because of the form of the payoff function, $M(x,z)$, which involves exponential functions as well as algebraic functions, this is impossible. Hence only solutions for particular values of the parameters can be obtained.

We shall do this for three sets of σ, β, k parameters and five values of the recovery parameters. In order to reduce the computing time to a minimum, we shall use the method of "fictitious play" to solve the game, rather than the classical methods of optimizing a function. Both methods are iteration processes requiring many iterations. However, the technique of "fictitious play" can perform high numbers of iterations at negligible costs.

Table 1 presents game values for each of the three plants and for five probabilities of recovery. We also present, for comparison, game values when the alarm threshold is not a strategic variable but strictly determined at LEMUF.

The table shows the improvement in game value--i.e., the defender's losses are reduced--by using a mixed strategy of three alarm thresholds.

Table 2 shows the optimal mixed strategy for the defender for each of the three plants and for each of the five recovery rates.

Table 1

Game Values--Losses of Defender
One-Dimensional Alarm Threshold Game

Probability of Recovery	Plant #1 $\sigma=0.3$ $\beta=1.61$ $k=10$ Alarm Threshold		Plant #2 $\sigma=1.62$ $\beta=6.17$ $k=10$ Alarm Threshold		Plant #3 $\sigma=15.9$ $\beta=3.14$ $k=10$ Alarm Threshold	
r	LEMUF	Strategic	LEMUF	Strategic	LEMUF	Strategic
0.1	1.83	1.51	5.78	5.60	2.92	2.84
0.3	1.61	1.26	5.15	4.72	2.50	2.40
0.5	1.44	1.09	4.59	4.08	2.22	2.07
0.7	1.29	0.97	4.16	3.60	1.99	1.81
0.9	1.21	0.84	3.73	3.20	1.82	1.61

Table 2

Optimal Strategy for Defender
$G^*(z) = \alpha I_{z_1} + \beta I_{z_2} + \gamma I_{z_3}$

r	Plant #1		Plant #2		Plant #3	
	z	P(z)	z	P(z)	z	P(z)
	0.2	0.16	4.2	0.27	1.0	0.38
0.1	0.6	0.68	4.6	0.71	1.4	0.59
	1.0	0.16	5.0	0.02	1.8	0.03
	0.2	0.39	3.4	0.02	1.0	0.24
0.3	0.6	0.59	3.8	0.53	1.4	0.72
	1.0	0.02	4.2	0.45	1.8	0.04
	0.6	0.18	3.4	0.02	1.4	0.02
0.5	1.0	0.78	3.8	0.62	1.8	0.58
	1.4	0.04	4.2	0.36	2.2	0.40
	-0.2	0.11	3.4	0.40	1.0	0.06
0.7	0.2	0.58	3.8	0.58	1.4	0.83
	0.6	0.31	4.2	0.02	1.8	0.11
	0.2	0.16	3.0	0.53	1.4	0.11
0.9	0.6	0.79	3.4	0.46	1.8	0.83
	1.0	0.05	3.8	0.01	2.2	0.06

z represents alarm threshold value

P(z) represents probability of Defender selecting that value of z.

It should be pointed out that playing $G^*(z)$ by the defender is not the same as playing $z^* = E(z)$ because the defender must use a mixed strategy and $E(z)$ is a pure strategy.

Also note that for Plant #1, at $r = 0.7$, a negative $z(0.2)$ is optimal.

Table 3 shows the diverter's optimal strategy which is a mixture of two strategies for each plant and each recovery rate. We also show the expected diversion, $E(x)$.

Table 3

Optimal Strategy for Diverter
$F^*(x) = \alpha I_{x_1} + \beta I_{x_2}$

r	Plant #1			Plant #2			Plant #3		
	x	P(x)	E(x)	x	P(x)	E(x)	x	P(x)	E(x)
0.1	1.5	0.93	1.54	5.5	0.65	5.67	2.5	0.16	2.92
	2.0	0.07		6.0	0.35		3.0	0.84	
0.3	1.0	0.33	1.33	4.5	0.42	4.78	2.0	0.12	2.44
	1.5	0.67		5.0	0.58		2.5	0.88	
0.5	1.0	0.73	1.13	4.0	0.75	4.13	2.0	0.80	2.10
	1.5	0.27		4.5	0.25		2.5	0.20	
0.7	0.5	0.02	0.99	3.5	0.73	3.63	1.5	0.25	1.87
	1.0	0.98		4.0	0.27		2.0	0.75	
0.9	0.5	0.21	0.89	3.0	0.50	3.25	1.5	0.69	1.65
	1.0	0.79		3.5	0.50		2.0	0.31	

x represents amount diverted.

p(x) represents probability of diverter selecting that value of x.

E(x) represents expected value of x.

APPENDIX C

MATHEMATICAL PROOF FOR FORM OF DIVERTER OPTIMAL STRATEGY

We will present a semi-constructive proof that the diverter optimal strategy, $F^*(x)$, is a step function with a finite number of jumps. Define as follows the generalized moments associated with each distribution function $F(x)$ of the diverter. First let us set:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z} e^{-\frac{(u-x)^2}{2}} du = f(x)$$

Now define the following hypercurve, parametrically in x, where $0 \leq x \leq k$:

$r_1 = f(x)$

$r_2 = xf(x)$

$r_3 = \min(y_1,x)\ f(x)$

$r_4 = |y_1-x|f(x)$

$r_5 = 1-f(x)$

$r_6 = x[1-f(x)]$

$r_7 = \min(y_2,x)[1-f(x)]$

$r_8 = |y_2-x|[1-f(x)]$.

These functions are obtained from the payoff $M(x,y)$.

As x varies between 0 and k, the above curve is traced out in 8-dimensional space (at most). Now form the convex hull of this curve. This convex hull, H, will be at most an 8-dimensional convex volume.

Let F(x) be any distribution function (density or step function). We can compute the following 8 moments:

$$\rho_i = \int_0^K r_i(x)\, dF(x) \qquad 1 \leq i \leq 8$$

for that F, or we can associate a $\rho_F = (\rho 1, \rho 2, \ldots \rho 8)$, a point in 8-dimensional space. Letting F vary over all possible distribution functions, we obtain an 8-dimensional volume R which is closed, bounded, and convex. Further R = H, the convex hull of the previously defined curve.

Thus selecting a distribution function F(x) is equivalent to selecting a point ρ in H = R, an 8-dimensional convex set. In this convex set each vertex corresponds to a distribution function with a single step at x where $0 \leq x \leq k$, or $F(x) = I_x(x)$. Now from Fenchel's Theorerm on convex sets--every point of an n-dimensional convex set can be represented by a convex linear combination of (n+1) vertices of the convex set--it follows that there exists an F*(x) such that it consists of at most 9 steps.

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