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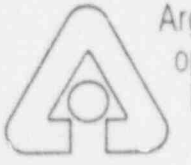
Time- and Volume-Averaged Conservation Equations for Multiphase Flow Using Mass-Weighted Velocity and Internal Energy

by W. T. Sha, B. T. Chao,
and S. L. Soo



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by

W. T. Sha, B. T. Chao,*
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Materials and Components Technology Division

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ABSTRACT

Conservation equations of mass, momentum, and energy for multiphase flow, formulated on the basis of local volume averaging followed by time-averaging for turbulent flows, are presented. They are differential equations of transport with area integrals associated with interfacial transport. Because the spatial averaging theorems used in the analysis are subject to certain length scale restrictions, the resulting equations are best suited for dispersed systems.

The local instantaneous variable is decomposed as a linear combination of its local intrinsic volume average and a spatial deviation. Use of the mass-weighted, volume-averaged velocity and internal energy simplified certain relationships between the volume average of products and the product of volume averages. Recognition of the fact that the spatial deviation component takes on positive and negative values within the averaging volume makes further simplifications feasible. Inasmuch as information is always lost as a result of averaging, be it volume-averaging or time-averaging or both, the lost information must somehow be replaced before the equations can be solved. This is commonly done by the development of appropriate constitutive relations, which, however, is not treated in this report.

The difficulties of making direct comparisons of the volume-averaged and time-averaged conservation equations for multiphase flow are discussed. Nevertheless, an attempt was made to compare the time-averaged equations of Ishii and the energy equation used in the TRAC code with the present set of rigorously derived equations after considerable simplifications. Apparent agreement is found in all cases, although some differences remain.

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CONTENTS

NOMENCLATURE

A — Area; A_k is the total interfacial area associated with phase k inside the local averaging volume v

c_v — Specific heat at constant volume

d — Characteristic length of a dispersed phase

D — Diffusivity

E — Total energy per unit mass, $E_k = u_k + (1/2)\,\underline{U}_k \cdot \underline{U}_k$

$^{ti}\langle \underline{E}_k^T \rangle$ is the time- and volume-averaged turbulent total energy flux vector defined by equation (5.6.4)

$^{i}\langle \underline{E}_k^D \rangle$ is the volume-averaged dispersive total energy flux vector defined by equation (5.6.13)

$^{ti}\langle \underline{\tilde{E}}_k^T \rangle$ is the time- and volume-averaged turbulent, dispersive total energy flux vector defined by equation (5.6.17)

$\underline{f}$ — Field force per unit mass

$\underline{F}_s$ — Interfacial force per unit interfacial area

g — Gravitational acceleration

H_{kf} — Mean curvature of interface between phases k and f

$\underline{\underline{I}}$ — Unitary tensor

J_E — Internal heat source per unit volume

$\underline{J}_q$ — Heat flux vector

K — Kinetic energy per unit mass, $K_k = \frac{1}{2}\,\underline{U}_k \cdot \underline{U}_k$

$^{ti}\langle \underline{K}_k^T \rangle$ is the time- and volume-averaged turbulent kinetic energy flux vector defined by equation (5.6.9)

$^{i}\langle \underline{K}_k^D \rangle$ is the volume-averaged dispersive kinetic energy flux vector defined by equation (5.6.15)

$^{ti}\langle \underline{\tilde{K}}_k^T \rangle$ is the time- and volume-averaged turbulent, dispersive kinetic energy flux vector defined by equation (5.6.19)

ℓ — Characteristic length of local averaging volume v

L Characteristic length of physical system

$\dot{m}$ Interfacial mass flux; $\dot{m}_k$ is defined by equation (3.5)

$\underline{M}$ Interfacial momentum source per unit volume; ${}^{t}\langle \underline{M}_k \rangle$ is defined by equation (5.5.16)

$\underline{n}$ Unit outward normal vector of interface as illustrated in Fig. 2

N Number of bubbles per unit volume

P Static pressure

$\dot{Q}$ Interfacial heat transfer rate per unit volume; ${}^{t}\langle \dot{Q} \rangle$ is defined by equation (5.6.27)

t Time

T Temperature; also averaging time interval

u Internal energy per unit mass

${}^{ti}\langle \underline{u}_k^T \rangle$ is the time- and volume-averaged turbulent internal energy flux vector defined by equation (5.7.4)

${}^{i}\langle \underline{u}_k^D \rangle$ is the volume-averaged dispersive internal energy flux vector defined by equation (5.7.5)

${}^{ti}\langle \underline{u}_k^T \rangle$ is the time- and volume-averaged turbulent, dispersive internal energy flux vector defined by equation (5.7.6)

$\underline{U}$ Velocity; U is velocity in one-dimensional flow

v Local averaging volume

$\underline{W}_s$ Interface velocity

z Elevation

α Local volume fraction

Γ Interfacial mass source per unit volume, equation (4.2); ${}^{t}\langle \Gamma_k \rangle$ is defined by equation (5.4.4)

ε Perturbation parameter as defined

$\underline{\xi}$ Interfacial total energy source per unit volume; ${}^{t}\langle \underline{\xi}_k \rangle$ is defined by equation (5.6.35)

$\mathcal{J}$ Interfacial internal energy source per unit volume; $^{t}\langle \mathcal{J}_k \rangle$ is defined by equation (5.7.18)

κ Thermal conductivity; κ^T is turbulent conductivity

λ Bulk viscosity

Λ Integral length scale

μ Dynamic viscosity

ρ Density

σ Interfacial tension

τ Characteristic time

$\underline{\underline{I}}$ Viscous stress

$^{ti}\langle \underline{\underline{I}}_k^T \rangle$ is the time- and volume-averaged Reynolds stress tensor defined by equation (5.5.2)

$^{i}\langle \underline{\underline{I}}_k^D \rangle$ is the volume-averaged dispersive stress tensor defined by equation (5.5.3)

$^{ti}\langle \underline{\underline{I}}_k^T \rangle$ is the time- and volume-averaged turbulent, dispersive stress tensor defined by equation (5.5.4)

ϕ Dissipation function; $\phi_k = \underline{\underline{I}}_k : \nabla \underline{U}_k$

Φ_{Ek} Scalar total energy function defined by equation (5.6.2)

Φ_{Pk} Scalar pressure work function defined by equation (5.7.8)

Φ_{uk} Scalar internal energy function defined by equation (5.7.2)

$\Phi_{\tau k}$ Scalar viscous dissipation function defined by equation (5.7.11)

Ψ Intensive property

$\underline{\Psi}_{mk}$ Vector mass flux function defined by equation (5.4.3)

$\underline{\Psi}_{Pk}$ Vector pressure work function defined by equation (5.6.21)

$\underline{\Psi}_{\tau k}$ Vector viscous stress work function defined by equation (5.6.23)

Superscripts

~	Local spatial deviation
'	High-frequency fluctuation
D	Dispersive transport
T	Transport associated with high-frequency fluctuation

Subscripts

c	Conjugate
f	Phase f
HF	High frequency
i	Interface
k	Phase k
LF	Low frequency
m	Mass; also mixture

Symbols

$(\underline{\ })$	Vector
$(\underline{\underline{\ }})$	Tensor second order
$\langle\ \rangle$	Local volume average
$^{i}\langle\ \rangle$	Intrinsic volume average
$^{i}\langle\ \rangle^{*}$	Mass-weighted intrinsic volume average
$^{t}\langle\ \rangle$	Time average
O( )	Order of magnitude

Operators

∇	Gradient; ∇_s denotes surface gradient operator
$\nabla\cdot$	Divergence

$\mathbb{V}$ (•) Dyad

$[\mathbb{V}$ (•)$]_c$ Conjugate of dyad

Acronyms

(HTI) Interfacial heat transfer integral defined by equation (5.7.15)

(IETI) Interfacial internal energy transfer integral defined by equation (5.7.17)

(MTI) Interfacial mass transfer integral defined by equation (5.4.5)

(MMTI) Interfacial momentum transfer integral defined by equation (5.5.14)

(PTI) Interfacial pressure transfer integral defined by equation (5.5.11)

(PWI) Interfacial pressure work integral defined by equation (5.6.30)

$(PWI)^{(u)}$ Interfacial pressure work integral associated with internal energy production, defined by equation (5.7.9)

(TETI) Interfacial total energy transfer integral defined by equation (5.6.33)

(VDI) Interfacial viscous dissipation integral defined by equation (5.7.12)

(VSTI) Interfacial viscous stress transfer integral defined by equation (5.5.12)

(VWI) Interfacial viscous stress work integral defined by equation (5.6.31)

All equations referred to in the above list are written for phase k. For phase f, it is necessary only to change subscript k to f for the entries that appear in the equations.

TIME- AND VOLUME-AVERAGED CONSERVATION EQUATIONS FOR MULTIPHASE FLOW USING MASS-WEIGHTED VELOCITY AND INTERNAL ENERGY

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EXECUTIVE SUMMARY

Multiphase flows consist of interacting phases that are dispersed randomly in space and in time. Although the intraphase conservation equations for mass, momentum, and energy, and their initial and boundary conditions, can in principle be written, the cost of detailed fluid flow and heat transfer analysis is often prohibitive, if not impossible. In most engineering applications, all that is required is to capture the essential features of the system and to express the flow and temperature field in terms of local, global quantities while sacrificing some of the details. The present study is an attempt to achieve this goal by applying time averaging after local volume averaging.

Local volume averaging of conservation equations of mass, momentum, and energy for a multiphase system yields equations in terms of local-volume-averaged products of density, velocity, energy, stresses, and field forces together with interfacial transfer integrals. These averaging relations are subject to the following length scale restrictions:

$$d \ll \ell \ll L,$$

where d is a characteristic length of the pores of dispersed phases, ℓ is a characteristic length of the averaging volume, and L is a characteristic length of the physical system.

Solutions of local-volume-averaged conservation equations call for expressing these local volume-averaged products in terms of products of averages. In nonturbulent flows, this may be achieved by expressing the "point" variable as the sum of its intrinsic volume average and a spatial deviation. In turbulent flows, the same can be achieved via subsequent time averaging over a duration T such that

$$\tau_{HF} \ll T \ll \tau_{LF},$$

where τ_{HF} is a characteristic time of high-frequency fluctation and τ_{LF} is a characteristic time of low-frequency fluctuation. In this case an instantaneous "point" variable ψ_k of phase k is decomposed into a low-frequency component ψ_{kLF} and a high-frequency component ψ_k' as in Reynolds analysis of turbulent flow. The low-frequency component consists of the sum of the local intrinsic volume average ${}^{i}\langle\psi_k\rangle_{LF}$ and its local spatial deviation $\tilde{\psi}_k$. Time

averaging then reduces the volume-averaged products to products of averages plus terms representing eddy and dispersive diffusivities of mass, Reynolds and dispersive stresses, and eddy and dispersive conductivities of heat, etc. These terms arise from both high-frequency fluctuations and local spatial deviations. In either case, turbulent or nonturbulent, the procedure leads to differential equations of conservation with area integrals representing interfacial transport. Inasmuch as information is always lost as a result of averaging, be it volume-averaging or time-averaging or both, the lost information must somehow be replaced before the equations can be solved. This is commonly done by the development of appropriate constitutive relations which, however, is not treated in this report.

A point of departure from an earlier analysis on the same subject (Ref. 12, NUREG/CR-3989, ANL-84-66) is that mass-weighted fluid velocity and internal energy are used. This results in some simplification and facilitates interpretation of certain terms in the averaged equations. Several errors in Ref. 12 were also unearthed and corrected. Furthermore, the interfacial balance equations for mass, momentum, and total energy are examined in more detail. It was shown that the so-called jump conditions are embodied in the differential-integral equations of conservation. At the present stage of development, the same cannot be demonstrated for the internal energy equation owing to the presence of the two terms: $\langle P_k \nabla \cdot \underline{U}_k \rangle$ and $\langle \underline{\underline{\tau}}_k : \nabla \underline{U}_k \rangle$. This points to the need for developing volume-averaging theorems for the product of P_k and $\nabla \cdot \underline{U}_k$ and for the scalar product of $\underline{\underline{\tau}}_k$ and $\nabla \underline{U}_k$.

The difficulties of making direct comparisons of volume-averaged and time-averaged conservation equations for multiphase flow are discussed. Nevertheless, an attempt was made to compare the time-averaged equations of Ishii and the energy equation used in the TRAC code with the present set of rigorously derived equations after considerable simplifications. Apparent agreement is found in all cases, although some differences remain.

1. INTRODUCTION

Multiphase flows consist of interacting phases that are dispersed randomly in space and in time. Although the intraphase conservation equations for mass, momentum, and energy and their associated initial and boundary conditions can in principle be written, along with their interfacial conservation relations, they are far too complicated to permit detailed solutions. In fact, they are usually not needed in engineering applications. An alternative is to describe the essential dynamics and thermodynamics of such a system in terms of locally averaged quantities. This can be achieved by applying some form of averaging process, such as time averaging [1-3], space averaging [2-10], statistical averaging [2,3], or space/time and time/space averaging [2,3]. The present work begins with local volume averaging to be followed by time averaging [11,12].

Local volume averaging has been successfully used in analyzing viscous flow through porous media [4,6,7,13]. Volume averaging leads naturally to volume fraction of phases while a priori time averaging yields their fractional residence time. The thermodynamic properties of a mixture, such as density and specific heat, are cumulative with volume fraction but not with fractional residence time, which becomes identical to volume fraction only in the special case of one-dimensional, uniform flow. The several disadvantages associated with time averaging as the basis of analysis for multiphase flows have been pointed out by Reynolds [14] in his review of Ishii's book [1] and were also discussed in Ref. 11.

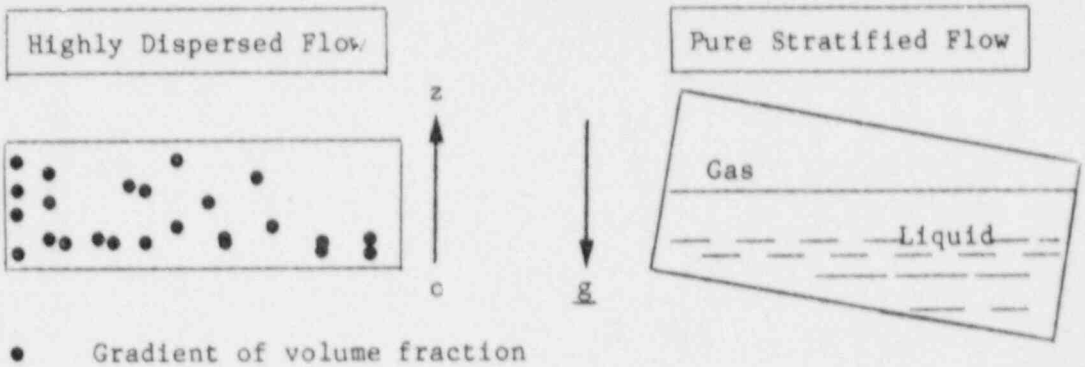
The configuration of phases plays a major role in determining the dynamics of multiphase flows and the concomitant heat and mass transport processes when they occur. This is illustrated in Fig. 1 for the two extreme cases of highly dispersed flow and ideally stratified flow, which, by definition, has a plane interface. The figure is largely self-explanatory. Given the defining relations (1.1) for the mixture density ρ_m and (1.2) for the mixture velocity U_m, it is easy to show that $\rho_m U_m^2$ and $\sum_k \alpha_k \rho_k U_k^2$ are not the same. It is also easy to demonstrate that if the Bernoulli relationship for an ideal mixture in highly dispersed flow is written as equation (1.5), then that for the individual phase must be given by equation (1.6). For the ideally stratified flow, the Bernoulli relationship for the individual phase is given by equation (1.7). It follows that for the mixture it is given by equation (1.8). Clearly, the form of the Bernoulli equation depends on the configuration of the phases. The Bernoulli equation for other systems, such as bubbly flow, annular wavy flow with dispersed liquid, intermittent flow, stratified wavy flow, etc., are far more complex, representing cases intermediate between the highly dispersed flow and ideally stratified flow.

2. AVERAGING RELATIONS

The flow system under consideration is depicted in Fig. 2. The local averaging volume v considered in this paper is invariant in both space and time, and its orientation relative to the inertial frame of reference is fixed. The region consists of a fluid mixture with dynamic phases k, f, and g. Without loss of generality, attention is focused on phase k, which is chosen arbitrarily. Phase k has a variable volume v_k with **total** interfacial area A_k inside v. For convenience of discussion, all phases that form an interface with phase k will be collectively represented by f. Thus, $A_k = A_{kf}$ and A_{kf} implies $A_{kf} + A_{kg}$. Here, A_{kf} and A_{kg} are respectively the total interfacial area between phases k/f and k/g inside v. The unit normal vector $\underline{n}_k$ of A_k is always drawn outwardly and away from phase k, regardless of whether it is associated with A_{kf} or A_{kg}, as illustrated in Fig. 2. The local velocity of phase k is $\underline{U}_k$ and that of the interface between phases k and f is $\underline{W}_s$. Obviously, $v = \sum_k v_k$ and the volume fraction of phase k is

$$\alpha_k = v_k/v, \tag{2.1}$$

which is a dependent variable. In general, it varies with time and location.



- Gradient of volume fraction

  $\nabla\alpha \propto \nabla$(conc.) — $\nabla\alpha$ is of no significance

- Finite diffusivity, D
- Diffusion velocity, $D\nabla\alpha$

  } Concept of diffusivity is irrelevant

- Wave propagation
  Existence of speed of sound in the mixture — No common speed of sound
  Common characteristics — Individual characteristics of phases
  Transfer of inertia force across interface — Plane interface; no inertia force transferred across interface

- Bernoulli relationships for steady, incompressible, inviscid, one-dimensional flow

$$\rho_m = \sum_k \alpha_k \rho_k, \qquad k = 1, 2, \cdots \tag{1.1}$$

where $\alpha_k\rho_k$ is the density of phase k based on mixture volume.

$$\rho_m U_m = \sum_k \alpha_k \rho_k U_k \tag{1.2}$$

Clearly, $$\boxed{\rho_m U_m^2 \neq \sum_k \alpha_k \rho_k U_k^2} \tag{1.3}$$

$$P_m = \sum_k \alpha_k P_k \tag{1.4}$$

Ideal Mixture

$$(1/2)\rho_m U_m^2 + P_m + \rho_m g z = \text{Constant} \tag{1.5}$$

Individual Phase

$$(1/2)\rho_k U_k^2 + P_k + \rho_k g z = \text{Constant} \tag{1.7}$$

Individual Phase

$$(1/2)\rho_k U_k^2 - (1/2)\rho_k (U_k - U_m)^2 + P_k + \rho_k g z = \text{Constant} \tag{1.6}$$

Mixture

$$(1/2)\rho_m U_m^2 + (1/2)\sum_k \alpha_k \rho_k (U_k - U_m)^2 + P_m + \rho_m g z = \text{Constant} \tag{1.8}$$

Fig. 1. Significance of phase configurations in multiphase flows

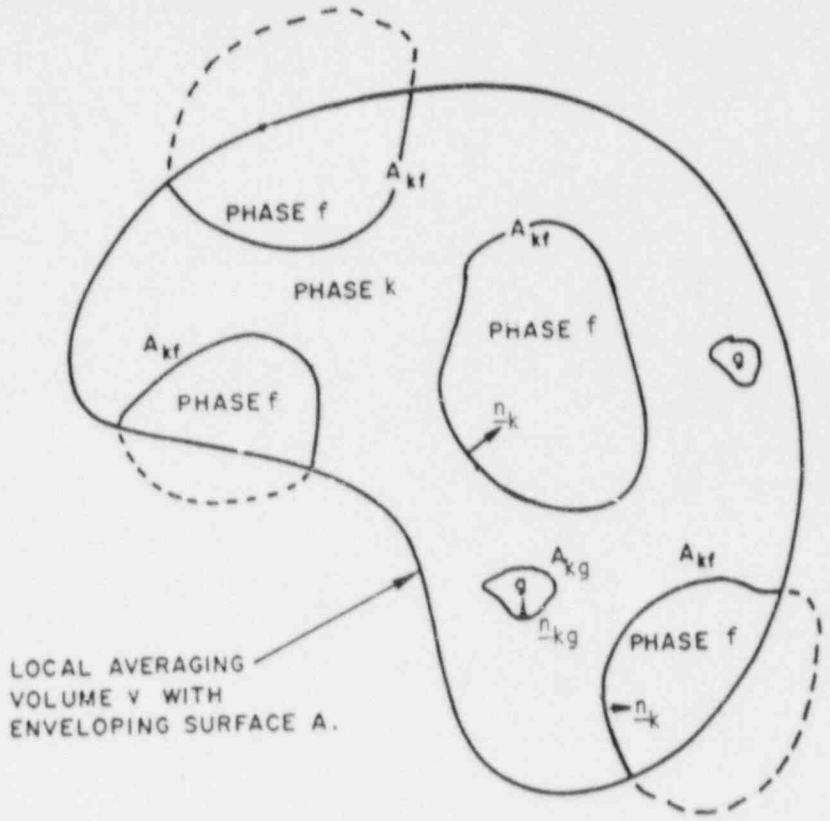


Fig. 2. Multiphase flow system

2.1 Local and Intrinsic Volume Average

For any intensive property ψ_k associated with phase k of the fluid mixture--scalar, vector, or tensor--the local volume average of ψ_k is defined by

$$\langle \psi_k \rangle = \frac{1}{V} \int_{V_k} \psi_k \, dv = \alpha_k \frac{1}{V_k} \int_{V_k} \psi_k \, dv \, , \tag{2.1.1}$$

which has also been called the phase average. The intrinsic volume average of ψ_k is defined by

$$^{i}\langle \psi_k \rangle = \frac{1}{V_k} \int_{V_k} \psi_k \, dv \, . \tag{2.1.2}$$

Clearly, the two averages are related according to

$$\langle \psi_k \rangle = \alpha_k \, ^{i}\langle \psi_k \rangle \, . \tag{2.1.3}$$

In order that the volume average $\langle\psi_k\rangle$ or $^i\langle\psi_k\rangle$ be physically meaningful and amenable to the usual mathematical operations, certain length scale restrictions need to be imposed. They were first recognized by Whitaker [7], namely,

$$\boxed{d \ll \ell \ll L} \quad , \tag{2.1.4}$$

where d is a characteristic length of the dispersed phase, ℓ is a characteristic length of v, and L is that of the physical system. Therefore, the averaging volume cannot be made arbitrarily small. We note that both $\langle\psi_k\rangle$ and $^i\langle\psi_k\rangle$ exist everywhere in space, not just in the region occupied by phase k. To be sure, the basic equations and discussions presented in this report are pertinent to dispersed systems.

It is easy to demonstrate that

$$\langle\langle\psi_k\rangle\rangle = \langle\psi_k\rangle, \quad ^i\left\langle ^i\langle\psi_k\rangle\right\rangle = {}^i\langle\psi_k\rangle, \quad \left\langle ^i\langle\psi_k\rangle\right\rangle = a_k\, {}^i\langle\psi_k\rangle = \langle\psi_k\rangle$$

$$\text{and} \quad ^i\langle\langle\psi_k\rangle\rangle = {}^i\left\langle a_k\, {}^i\langle\psi_k\rangle\right\rangle = \langle\psi_k\rangle \ . \tag{2.1.5}$$

If we set $\psi_k = 1$ in equations (2.1.2) and (2.1.3), we obtain $^i\langle 1\rangle = 1$ and $\langle 1\rangle = a_k$.

When density is not uniform, it is physically meaningful to speak of mass-weighted, volume-averaged velocities and internal energies. These mass-weighted averages will be denoted by an asterisk. Accordingly, we define

$$^i\langle \underline{U}_k\rangle^* = \int_{V_k} \rho_k \underline{U}_k \, dv \Big/ \int_{V_k} \rho_k \, dv \ , \tag{2.1.6a}$$

or

$$^i\langle \underline{U}_k\rangle^* = \langle \rho_k \underline{U}_k\rangle / \langle \rho_k\rangle = {}^i\langle \rho_k \underline{U}_k\rangle / {}^i\langle \rho_k\rangle \ . \tag{2.1.6b}$$

Likewise,

$$^i\langle u_k\rangle^* = \langle \rho_k u_k\rangle / \langle \rho_k\rangle = {}^i\langle \rho_k u_k\rangle / {}^i\langle \rho_k\rangle \ . \tag{2.1.7}$$

Equivalent relationships are

$$\langle \rho_k \underline{U}_k \rangle = \langle \rho_k \rangle \, {}^{i}\langle \underline{U}_k \rangle^{*} = \alpha_k \, {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}_k \rangle^{*} \tag{2.1.8}$$

and a similar expression for internal energy. When ρ_k = constant, ${}^{i}\langle \underline{U}_k \rangle^{*} = {}^{i}\langle \underline{U}_k \rangle$ and equation (2.1.8) leads to $\langle \underline{U}_k \rangle = \alpha_k \, {}^{i}\langle \underline{U}_k \rangle$ as it must. We shall soon see that when the mass-weighted, volume-averaged velocity and internal energy are used, the time- and volume-averaged conservation equations are simplified.

One may easily show that

$$i\left\langle {}^{i}\langle \underline{U}_k \rangle^{*} \right\rangle = {}^{i}\langle \underline{U}_k \rangle^{*}, \quad i\left\langle {}^{i}\langle \underline{U}_k \rangle^{*} \right\rangle^{*} = {}^{i}\langle \underline{U}_k \rangle^{*} , \tag{2.1.9}$$

and similar relationships can be shown for the scalar u_k. Also,

$$i\left\langle {}^{i}\langle \underline{U}_k \rangle^{*} \, {}^{i}\langle u_k \rangle^{*} \right\rangle = {}^{i}\langle \underline{U}_k \rangle^{*} \, {}^{i}\langle u_k \rangle^{*} ,$$

$$i\left\langle {}^{i}\langle \underline{U}_k \rangle^{*} \, {}^{i}\langle u_k \rangle^{*} \right\rangle^{*} = {}^{i}\langle \underline{U}_k \rangle^{*} \, {}^{i}\langle u_k \rangle^{*} . \tag{2.1.10}$$

2.2 Local Volume Averaging Theorems

The local volume averages of the spatial and time derivatives of a fluid property ψ_k, which may be a scalar, vector, or tensor, have been given by Whitaker [4,7], Slattery [6], Anderson and Jackson [5], Gray and Lee [10], and others. They relate the average of the derivatives to the corresponding derivatives of the average and an interfacial area integral according to the following formulas:

$$\langle \nabla \psi_k \rangle = \nabla \langle \psi_k \rangle + v^{-1} \int_{A_k} \psi_k \, \underline{n}_k \, dA , \tag{2.2.1}$$

$$\langle \nabla \cdot \underline{\psi}_k \rangle = \nabla \cdot \langle \underline{\psi}_k \rangle + v^{-1} \int_{A_k} \underline{\psi}_k \cdot \underline{n}_k \, dA , \tag{2.2.2}$$

and

$$\left\langle \frac{\partial \psi_k}{\partial t} \right\rangle = \frac{\partial \langle \psi_k \rangle}{\partial t} - v^{-1} \int_{A_k} \psi_k \, \underline{W}_s \cdot \underline{n}_k \, dA . \tag{2.2.3}$$

As noted previously, A_k denotes the sum of **all** interfacial areas associated with phase k inside the local averaging volume v. A recent review of the various approaches to derive the spatial averaging theorem can be found in [15].

Upon setting $\psi_k = 1$ in equations (2.2.1) and (2.2.3), one obtains, respectively,

$$\nabla \alpha_k = - v^{-1} \int_{A_k} \underline{n}_k \, dA \tag{2.2.4}$$

and

$$\frac{\partial \alpha_k}{\partial t} = v^{-1} \int_{A_k} \underline{W}_s \cdot \underline{n}_k \, dA \, . \tag{2.2.5}$$

For those elements of phase k that are completely inside v, $\int_{A_k} \underline{n}_k \, dA = 0$. Thus, equation (2.2.4) implies that the gradient of the volume fraction of phase k depends only on those elements that are cut through by the bounding surface A of v. A corollary of equation (2.2.4) is

$$\sum_k \int_{A_k} \underline{n}_k \, dA = 0 \, , \tag{2.2.6}$$

since $\sum_k \alpha_k = 1$.

The physical meaning of equation (2.2.5) can be seen by considering phase k to be evaporating vapor bubbles in water. The right side of the equation gives the rate of increase of the vapor volume per unit volume of the mixture, which is obviously $\partial \alpha_k / \partial t$.

To conclude this section, we reiterate that the volume averaging theorems given by equations (2.2.1), (2.2.2), and (2.2.3) are subject to the length scale restrictions of equation (2.1.4).

3. PHASIC CONSERVATION EQUATIONS AND INTERFACE BALANCE EQUATIONS

The equations of conservation for single-phase flow are well known; they are the foundations of continuum mechanics. Although a "single" phase commonly refers to one **physical** phase, such as vapor, liquid, or solid, it may include nonreactive mixtures, such as room air, an aqueous solution of glycerine, or composite polymers. The identification of a multiphase system is best made in terms of the dynamic behavior of its component phases [16],

despite the fact that they may be of the same material. Examples are air bubbles of widely different sizes in a gas-liquid flow or solid particles of identical size, shape, and density that carry widely different electric charges in a gas-solid suspension.

For a phase k, the equations of continuity, momentum, total energy, and internal energy are, respectively,

$$(\partial \rho_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k) = 0 \ , \tag{3.1}$$

$$(\partial \rho_k \underline{U}_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k \underline{U}_k) = - \nabla P_k + \nabla \cdot \underline{\underline{\tau}}_k + \rho_k \underline{f} \ , \tag{3.2}$$

$$\frac{\partial \rho_k E_k}{\partial t} + \nabla \cdot (\rho_k \underline{U}_k E_k) = - \nabla \cdot \underline{U}_k P_k + \nabla \cdot (\underline{U}_k \cdot \underline{\underline{\tau}}_k) + \rho_k \underline{U}_k \cdot \underline{f} - \nabla \cdot \underline{J}_{qk} + J_{Ek} \ , \tag{3.3}$$

where the total energy $E_k = u_k + \frac{1}{2} \underline{U}_k \cdot \underline{U}_k$, and the internal energy equation is

$$\frac{\partial \rho_k u_k}{\partial t} + \nabla \cdot (\rho_k \underline{U}_k u_k) = - P_k \nabla \cdot \underline{U}_k - \nabla \cdot \underline{J}_{qk} + J_{Ek} + \underline{\underline{\tau}}_k : \nabla \underline{U}_k \ . \tag{3.4}$$

The double dot in the last term of equation (3.4) denotes the scalar product of two second-order tensors. In the literature, $\underline{\underline{\tau}}_k : \nabla \underline{U}_k$ is commonly denoted by ϕ_k, the dissipation rate per unit volume of phase k. The energy equation may also be given in terms of enthalpy. It is given in [12] and will not be repeated here.

The mass, momentum, and internal energy balance equations for phase interfaces have been given by Scriven [17], Standart [18], Slattery [19], Delhaye [20], Ishii [1], Deemer and Slattery [21], Bouré and Delhaye [22], and more recently, Kataoka [23]. These balance equations are also known as interfacial jump conditions in the literature. A simple case is an interface with negligible thickness and mass. For the convenience of presentation, we introduce an interfacial mass flux defined by

$$\dot{m}_k = \rho_k (\underline{U}_k - \underline{W}_s) \cdot \underline{n}_k \ . \tag{3.5}$$

The interfacial mass balance equation is then

$$\sum_{k,f} \dot{m}_k = 0 \tag{3.6}$$

where the subscripts k and f refer to two adjacent fluids separated by the interface whose velocity is $\underline{W}_s$. In equation (3.5), ρ_k and $\underline{U}_k$ denote the density and velocity of phase k as the interface is approached. That is, $\rho_k = \rho_{ki}$, $\underline{U}_k = \underline{U}_{ki}$, and $\underline{n}_k = \underline{n}_{ki}$, the unit vector normal to the interface and away from phase k. Clearly $\underline{n}_{ki} = -\underline{n}_{fi}$. Thus, in equation (3.5) or (3.6), $\dot{m}_k$ implies $\dot{m}_{ki}$. For the sake of simplifying writing, the subscript i will be dropped. Equation (3.6) states that at the interface the mass flux from phase k and from phase f must add to zero since the interface is without mass.

The interfacial balance equation for linear momentum is

$$\sum_{k,f} \left(\dot{m}_k \underline{U}_k + P_k \underline{n}_k - \underline{\underline{I}}_k \cdot \underline{n}_k \right) + \underline{F}_s = 0 \ , \tag{3.7}$$

where $\underline{F}_s$ is the incerfacial force per unit interfacial area. It may be expressed in terms of interfacial tension σ according to

$$\underline{F}_s = \nabla_s \sigma - 2H_k \sigma \, \underline{n}_k \ , \tag{3.8}$$

in which ∇_s is the surface gradient operator and H_k is the mean curvature, being positive when the associated radius vector is pointing outward from phase k. The existence of the surface gradient of σ could be the result of nonuniform temperature or nonuniform chemical composition or both.

The interfacial balance equation for the total energy is

$$\sum_{k,f} \left[\dot{m}_k E_k + \underline{J}_{qk} \cdot \underline{n}_k + \left(P_k \underline{U}_k - \underline{\underline{I}}_k \cdot \underline{U}_k \right) \cdot \underline{n}_k \right]$$

$$= - \underline{F}_s \cdot \underline{W}_s \ . \tag{3.9}$$

It is of interest to note that the interfacial balance equation for the internal energy is

$$\sum_{k,f} \left(\dot{m}_k u_k + \underline{J}_{qk} \cdot \underline{n}_k \right) + \sum_{k,f} \left[\frac{1}{2} \dot{m}_k \left(\underline{U}_k - \underline{W}_s \right) \cdot \left(\underline{U}_k - \underline{W}_s \right) \right.$$

$$\left. + \left(P_k \underline{n}_k - \underline{\underline{I}}_k \cdot \underline{n}_k \right) \cdot \left(\underline{U}_k - \underline{W}_s \right) \right] = 0 \ , \tag{3.10}$$

which is valid regardless of whether $\underline{F}_s$ exists or not. Equation (3.10) is derived by forming the dot product of equation (3.7) and $\underline{W}_s$, followed by sub-

tracting the result from equation (3.9) and making use of the relation given by equation (3.6). Equation (3.10) was previously given by Bouré and Delhaye [22]. An unexpected feature of equation (3.10) is that the balance of internal energy and heat flux at the interface requires the inclusion of the kinetic energy $\frac{1}{2}\dot{m}_k\left(\underline{U}_k - \underline{W}_s\right)^2$ and the work done by the pressure and viscous stresses at the relative velocity $\left(\underline{U}_k - \underline{W}_s\right)$.

In principle, the coupled phasic equations might be solved for given initial and boundary conditions together with the interfacial balance relations. Because the configuration and location of the fluid-fluid interfaces are not known a priori, their detailed solutions are next to impossible. When the length scale over which the point variables undergo significant changes is small compared with that over which the knowledge of these variables is of practical interest, information on their volume averages is all that is needed. To preserve the identity of the dynamic phases, local volume averaging is performed first; this is done in the following chapter. Time averaging of the volume-averaged equations is presented in Chapter 5.

4. LOCAL VOLUME-AVERAGED CONSERVATION EQUATIONS AND INTERFACE BALANCE EQUATIONS

An application of the local volume averaging theorems [equations (2.2.1) to (2.2.3)] to the phasic conservation equations given in Chapter 3 leads to the following set of local volume-averaged conservation equations for multiphase flow. Since the derivation is quite straightforward, only the results are listed here.

- Mass Conservation Equation

$$\frac{\partial}{\partial t}\alpha_k{}^i\langle\rho_k\rangle + \nabla\cdot\alpha_k{}^i\langle\rho_k\underline{U}_k\rangle = -v^{-1}\int_{A_k}\rho_k\left(\underline{U}_k - \underline{W}_s\right)\cdot\underline{n}_k\,dA\,. \tag{4.1}$$

The integral on the right side of equation (4.1) denotes the rate of interfacial mass generation of phase k per unit volume of v. Denoting it by Γ_k, we have

$$\Gamma_k = -v^{-1}\int_{A_k}\rho_k\left(\underline{U}_k - \underline{W}_s\right)\cdot\underline{n}_k\,dA = -v^{-1}\int_{A_k}\dot{m}_k\,dA\,. \tag{4.2}$$

We reiterate that for adjacent fluids k and f separated by a common interface, $\underline{n}_k = -\underline{n}_f$.

For N_k bubbles per unit volume, each of mass m_k and evaporating (or condensing) at the rate dm_k/dt,

$$\Gamma_k = N_k\left(dm_k/dt\right), \tag{4.3}$$

which is determined by the average relative velocity between the bubble and its surrounding liquid, the system pressure and temperature, and the enthalpy of evaporation or condensation. It is pertinent to note that the interfacial mass source is not necessarily limited to phase change. It could also be the result of chemical reaction at the interface.

- Linear Momentum Conservation Equation

$$\frac{\partial}{\partial t}\alpha_k\,{}^{i}\langle\rho_k\underline{U}_k\rangle + \nabla\cdot\alpha_k\,{}^{i}\langle\rho_k\underline{U}_k\underline{U}_k\rangle = -\nabla\alpha_k\,{}^{i}\langle P_k\rangle + \nabla\cdot\alpha_k\,{}^{i}\langle\underline{\underline{\tau}}_k\rangle + \alpha_k\,{}^{i}\langle\rho_k\rangle\,\underline{f}$$

$$+ v^{-1}\int_{A_k}\left(-P_k\underline{\underline{I}}_k + \underline{\underline{\tau}}_k\right)\cdot\underline{n}_k\,dA$$

$$- v^{-1}\int_{A_k}\rho_k\underline{U}_k\left(\underline{U}_k - \underline{W}_s\right)\cdot\underline{n}_k\,dA\,, \tag{4.4}$$

in which the field force per unit mass $\underline{f}$ is taken to be constant. The first integral on the right side of equation (4.4) accounts for the pressure and viscous forces acting on the interface in a unit volume of the fluid mixture. The second integral relates to the momentum transport at the interface. Both are directly proportional to the so-called interfacial area concentration, i.e., total interfacial area associated with phase k per unit volume of the mixture.

- Total Energy Conservation Equation

$$\frac{\partial}{\partial t}\alpha_k\,{}^{i}\langle\rho_k E_k\rangle + \nabla\cdot\alpha_k\,{}^{i}\langle\rho_k\underline{U}_k E_k\rangle = -\nabla\cdot\alpha_k\,{}^{i}\langle\underline{U}_k P_k\rangle + \nabla\cdot\alpha_k\,{}^{i}\langle\underline{U}_k\cdot\underline{\underline{\tau}}_k\rangle$$

$$- \nabla\cdot\alpha_k\,{}^{i}\langle\underline{J}_{qk}\rangle + \alpha_k\left({}^{i}\langle J_{Ek}\rangle + {}^{i}\langle\rho_k\underline{U}_k\rangle\cdot\underline{f}\right) + \dot{Q}_k$$

$$+ v^{-1}\int_{A_k}\left(-P_k\underline{U}_k + \underline{\underline{\tau}}_k\cdot\underline{U}_k\right)\cdot\underline{n}_k\,dA - v^{-1}\int_{A_k}\rho_k E_k\left(\underline{U}_k - \underline{W}_s\right)\cdot\underline{n}_k\,dA\,, \tag{4.5}$$

where $\dot{Q}_k$ denotes the rate of interfacial heat transfer to phase k per unit volume of the fluid mixture. It is given by

$$\dot{Q}_k = -v^{-1}\int_{A_k}\underline{J}_{qk}\cdot\underline{n}_k\,dA\,. \tag{4.6}$$

- Internal Energy Conservation Equation

$$\frac{\partial}{\partial t}\alpha_k {}^{i}\langle\rho_k u_k\rangle + \nabla\cdot\alpha_k {}^{i}\langle\rho_k \underline{U}_k u_k\rangle = -\alpha_k {}^{i}\langle P_k \nabla\cdot\underline{U}_k\rangle$$

$$-\nabla\cdot\alpha_k {}^{i}\langle\underline{J}_{qk}\rangle' + \alpha_k\left({}^{i}\langle J_{Ek}\rangle + {}^{i}\langle\phi_k\rangle\right) + \dot{Q}_k$$

$$-v^{-1}\int_{A_k}\rho_k u_k(\underline{U}_k - \underline{W}_s)\cdot\underline{n}_k\, dA \qquad (4.7)$$

in which $\phi_k = \underline{\underline{\tau}}_k : \nabla\underline{U}_k$, as has been noted previously.

The volume-averaged interfacial balance relations can be readily obtained from equations (3.6), (3.7), (3.9), and (3.10). They are:

- Mass Balance

$$\sum_{k,f}\Gamma_k = 0\ , \qquad (4.8)$$

since $\sum_{k,f}\dot{m}_k = 0$, and Γ_k is defined in equation (4.2).

- Linear Momentum Balance

$$v^{-1}\sum_{k,f}\int_{A_k}\left(\dot{m}_k\underline{U}_k + P_k\underline{n}_k - \underline{\underline{\tau}}_k\cdot\underline{n}_k\right)dA + v^{-1}\int_{A_k}\underline{F}_s\, dA = 0 \qquad (4.9)$$

- Total Energy Balance

$$v^{-1}\sum_{k,f}\int_{A_k}\left[\dot{m}_k E_k + \underline{J}_{qk}\cdot\underline{n}_k + \left(P_k\underline{U}_k - \underline{\underline{\tau}}_k\cdot\underline{U}_k\right)\cdot\underline{n}_k\right]dA$$

$$+ v^{-1}\int_{A_k}\underline{F}_s\cdot\underline{W}_s\, dA = 0\ . \qquad (4.10)$$

- Internal Energy Balance

$$v^{-1} \sum_{k,f} \int_{A_f} \left[\dot{m}_k u_k + \underline{J}_{qk} \cdot \underline{n}_k + \frac{1}{2} \dot{m}_k (\underline{U}_k - \underline{W}_s)^2 \right.$$

$$\left. + (P_k \underline{n}_k - \underline{\tau}_k \cdot \underline{n}_k) \cdot (\underline{U}_k - \underline{W}_s) \right] dA = 0 \qquad (4.11)$$

where $(\underline{U}_k - \underline{W}_s)^2 = (\underline{U}_k - \underline{W}_s) \cdot (\underline{U}_k - \underline{W}_s)$.

5. TIME AVERAGING OF LOCAL VOLUME-AVERAGED CONSERVATION EQUATIONS

5.1 Spatial and Time Decomposition of Dependent Variables

The local volume-averaged equations given in the preceding section are differential-integral equations. Before they can be used either for further analysis or for numerical computation, it is necessary (a) to express the volume averages of the product of the dependent variables in terms of the product of their volume averages and (b) to evaluate the interfacial transport integrals. For both cases, we need to relate the local values of the dependent variables to their corresponding averages.

To this end, the local dependent variable ρ_k is expressed as the sum of its local intrinsic volume average ${}^{i}\langle\rho_k\rangle$ and a spatial deviation $\tilde{\rho}_k$. Thus,

$$\rho_k = {}^{i}\langle\rho_k\rangle + \tilde{\rho}_k \,. \qquad (5.1.1)$$

Such a spatial decomposition scheme was first suggested by Gray [9]. Clearly, ${}^{i}\langle\tilde{\rho}_k\rangle = 0$. For dependent variables such as velocity $\underline{U}_k$ and internal energy u_k, we write

$$\underline{U}_k = {}^{i}\langle\underline{U}_k\rangle^{*} + \tilde{\underline{U}}_k \qquad (5.1.2)$$

$$u_k = {}^{i}\langle u_k\rangle^{*} + \tilde{u}_k \qquad (5.1.3)$$

where ${}^{i}\langle\underline{U}_k\rangle^{*}$ is the mass-weighted, volume-averaged velocity defined by equation (2.1.6a or b), and ${}^{i}\langle u_k\rangle^{*}$ is the mass-weighted, volume-averaged internal energy defined by equation (2.1.7).

While ${}^{i}\langle\tilde{\underline{U}}_k\rangle^{*} = 0$ and ${}^{i}\langle\tilde{u}_k\rangle^{*} = 0$, ${}^{i}\langle\tilde{\underline{U}}_k\rangle$ and ${}^{i}\langle\tilde{u}_k\rangle$ are not zero generally.

If ρ_k is uniform within v_k, then ${}^i\langle\cdot\rangle^* = {}^i\langle\cdot\rangle$.

For turbulent flows, we postulate that both the volume-averaged term and the spatial deviation term in equations (5.1.1), (5.1.2), and (5.1.3) would have a low-frequency component, to be denoted by the subscript LF, and a high-frequency component to be denoted by a prime. Thus, we write for ρ_k

$$ {}^i\langle\rho_k\rangle = {}^i\langle\rho_k\rangle_{LF} + {}^i\langle\rho_k\rangle' \ , \tag{5.1.4a} $$

and

$$ \tilde{\rho}_k = \tilde{\rho}_{kLF} + \tilde{\rho}_k' \ . \tag{5.1.4b} $$

In macroscopically steady flows, the low-frequency component is the temporal mean.

The time that characterizes the low-frequency component is of the order

$$ \tau_{LF} = L/(\Delta U)_c = \text{(characteristic dimension of the physical system)/(characteristic low-frequency speed variation)}. \tag{5.1.5a} $$

The characteristic time of the high-frequency component is of the order

$$ \tau_{HF} = \Lambda/(\text{rms } U') = \text{(integral length scale of high-frequency fluctuation)/(root mean square of the fluctuating velocity or turbulence intensity)} $$

$$ = 1/(\text{characteristic spectral frequency}). \tag{5.1.5b} $$

In performing time averaging, the duration T over which the averaging is made must satisfy the following inequalities:

$$ \boxed{\tau_{HF} \ll T \ll \tau_{LF}} \ . \tag{5.1.6} $$

If the temporal decomposition is applied to ρ_k from the outset, one writes

$$ \rho_k = \rho_{kLF} + \rho_k' \ , \tag{5.1.7} $$

which is the well-known Reynolds decomposition in turbulence analysis. For macroscopically steady flows, ρ_{kLF} becomes the temporal mean. When the time scale inequalities of equation (5.1.6) are satisfied, ρ_{kLF} and ρ_k' are

separable in the time or frequency domain. Substituting equations (5.1.4a,b) into (5.1.1) and comparing the result with equation (5.1.7) lead to

$$\rho_{kLF} = {}^{i}\langle\rho_k\rangle_{LF} + \tilde{\rho}_{kLF} \ , \tag{5.1.8a}$$

and

$$\rho_k' = {}^{i}\langle\rho_k\rangle' + \tilde{\rho}_k' \tag{5.1.8b}$$

in view of the fact that the low- and high-frequency entities are separable. Thus, we write

$$\rho_k = {}^{i}\langle\rho_k\rangle_{LF} + \tilde{\rho}_{kLF} + \rho_k' \ . \tag{5.1.9}$$

We shall soon demonstrate that ${}^{i}\langle\rho_k\rangle_{LF}$ closely approximates ${}^{i}\langle\rho_{kLF}\rangle$ when the high-frequency fluctuations of the volume v_k are negligible. We shall return to this point in Section 5.2. When ${}^{i}\langle\rho_k\rangle_{LF} = {}^{i}\langle\rho_{kLF}\rangle$, equation (5.1.8a) shows that

$${}^{i}\langle\tilde{\rho}_{kLF}\rangle = 0 \ . \tag{5.1.10}$$

The pressure P_k is to be decomposed in the same manner as ρ_k, i.e.,

$$P_k = {}^{i}\langle P_k\rangle_{LF} + \tilde{P}_{kLF} + P_k' \ . \tag{5.1.11}$$

For velocity $\underline{U}_k$ and internal energy u_k, we write

$$\underline{U}_k = {}^{i}\langle\underline{U}_k\rangle^{*}_{LF} + \tilde{\underline{U}}_{kLF} + \underline{U}_k' \tag{5.1.12}$$

and

$$u_k = {}^{i}\langle u_k\rangle^{*}_{LF} + \tilde{u}_{kLF} + u_k' \ . \tag{5.1.13}$$

The appropriate decomposition for the total energy E_k will be given later.

The local volume fraction α_k is a volume-averaged quantity. Hence, it should be decomposed simply as

$$\alpha_k = \alpha_{kLF} + \alpha_k' \ . \tag{5.1.14}$$

Intimately associated with equation (5.1.14) is the temporal decomposition

$$v_k = v_{kLF} + v_k' \ . \tag{5.1.15}$$

Obviously,

$$v_{kLF} = \alpha_{kLF}\, v, \quad \text{and} \quad v_k' = \alpha_k'\, v \ . \tag{5.1.16a,b}$$

Except for extreme conditions, high-frequency fluctuations in volume fraction seldom occur. Thus, α_k' can often be assumed negligible and will be ignored in the present analysis.

In multiphase flows, the fluid-fluid interface may not only translate, but also oscillate. Hence, in general, the interfacial area A_k, like α_k and v_k, can be decomposed as

$$A_k = A_{kLF} + A_k' \ . \tag{5.1.17}$$

However, as has been pointed out earlier, the present analysis is best suited for dispersed systems for which interfacial tension would normally play a prominent role and abrupt changes in surface curvature are not expected to occur. The familiar smooth and gentle shapes of oscillating bubbles and droplets are examples [24]. In such cases, A_k' can be ignored. The analysis of Ref. 12 also shows that if α_k' is deleted, A_k' should also be set to zero. Therefore, in this report, α_k', v_k', and A_k' are all neglected for consistent approximations.

The interfacial velocity $\underline{W}_s$ is not associated with ρ_k alone; it is decomposed as

$$\underline{W}_s = {}^{i}\langle \underline{W}_s \rangle_{LF} + \hat{\underline{W}}_s + \underline{W}_s' \ . \tag{5.1.18}$$

If the flow is such that the characteristic length of the dispersed phase d is large compared to the integral length scale of turbulence Λ, and if there is no vigorous interfacial mass transfer, then the high-frequency fluctuating component of the interface velocity $\underline{W}_s'$ would not be significant and should be deleted. On the other hand, if $d \ll \Lambda$, $\underline{W}_s'$ may not be ignored even though $A_k' =$

0. The case of turbulent flow of a suspension of small solid particles in a gas or liquid is an example. Hence, $\underline{W}'_s$ is retained in the analysis that follows.

5.2 Some Useful Observations

In deriving equation (5.1.10), we introduce the approximation that the low-frequency component of the intrinsic volume average of the density of phase k, ${}^{i}\langle\rho_k\rangle_{LF}$, is equal to the intrinsic volume average of the low-frequency component of the density, ${}^{i}\langle\rho_{kLF}\rangle$. We demonstrate in the following that the approximation is consistent with the stipulation that $v'_k \approx 0$, or more correctly, $|v'_k/v_{kLF}| \approx 0$. Denoting v'_k/v_{kLF} by ε, we can write

$${}^{i}\langle\rho_k\rangle_{LF} = \text{low-frequency component of}$$

$$\frac{1}{v_{kLF}}\left(1 - \varepsilon + \varepsilon^2 \cdots\right) \int_{v_{kLF}(1+\varepsilon)} \left(\rho_{kLF} + \rho'_k\right) dv$$

$$= \frac{1}{v_{kLF}} \int_{v_{kLF}} \rho_{kLF}\, dv \ , \tag{5.2.1}$$

and

$${}^{i}\langle\rho_{kLF}\rangle = \frac{1}{v_{kLF}}\left(1 - \varepsilon + \varepsilon^2 - \cdots\right) \int_{v_{kLF}(1+\varepsilon)} \rho_{kLF}\, dv$$

$$= \left(\frac{1}{v_{kLF}} \int_{v_{kLF}} \rho_{kLF}\, dv\right)\left[1 + O(\varepsilon)\right] . \tag{5.2.2}$$

Thus, ${}^{i}\langle\rho_k\rangle_{LF}$ and ${}^{i}\langle\rho_{kLF}\rangle$ differ by a small fraction of order ε.

Next, we proceed to examine ${}^{i}\langle\underline{U}_k\rangle^{*}_{LF}$ and ${}^{i}\langle\underline{U}_{kLF}\rangle^{*}$. It is straightforward to show that

$${}^{i}\langle\underline{U}_k\rangle^{*}_{LF} = \left(\int_{v_{kLF}} \rho_{kLF}\underline{U}_{kLF} dv\right) \Big/ \int_{v_{kLF}} \rho_{kLF} dv \ . \tag{5.2.3}$$

Now,

$$ {}^{i}\langle \underline{U}_{kLF} \rangle^{*} = \left(\int_{v_{kLF}(1+\varepsilon)} \rho_{kLF}(1 + \varepsilon_{\rho})\, \underline{U}_{kLF}\, dv \right) \Bigg/ \int_{v_{kLF}(1+\varepsilon)} \rho_{kLF}\,(1 + \varepsilon_{\rho})\, dv \quad , \qquad (5.2.4a) $$

where $\varepsilon_{\rho} = \rho_k'/\rho_{kLF}$ and $|\varepsilon_{\rho}| << 1$. Equation (5.2.4a) can be written as

$$ {}^{i}\langle \underline{U}_{kLF} \rangle^{*} = \frac{\int_{v_{kLF}} \rho_{kLF}\, \underline{U}_{kLF}\, dv}{\int_{v_{kLF}} \rho_{kLF}\, dv} \left[1 + 0_1(\varepsilon) + 0_2(\varepsilon_{\rho}) \right] . \qquad (5.2.4b) $$

Thus, ${}^{i}\langle \underline{U}_{k} \rangle^{*}_{LF}$ and ${}^{i}\langle \underline{U}_{kLF} \rangle^{*}$ may differ, at most, by a small fraction of order $|\varepsilon| + |\varepsilon_{\rho}|$. In the present analysis, the approximation ${}^{i}\langle \underline{U}_{k} \rangle^{*}_{LF} = {}^{i}\langle \underline{U}_{kLF} \rangle^{*}$ is adopted and hence,

$$ {}^{i}\langle \tilde{\underline{U}}_{kLF} \rangle^{*} = 0 . \qquad (5.2.5) $$

Likewise,

$$ {}^{i}\langle \tilde{u}_{kLF} \rangle^{*} = 0 . \qquad (5.2.6) $$

We also have occasion to use the following results.

Since

$$ \alpha_k \left\langle \nabla\, {}^{i}\langle \psi_k \rangle \right\rangle = \left\langle \nabla\, {}^{i}\langle \psi_k \rangle \right\rangle = \nabla\, \alpha_k\, {}^{i}\langle \psi_k \rangle + v^{-1} \int_{A_k} {}^{i}\langle \psi_k \rangle\, \underline{n}_k\, dA $$

$$ = \nabla\, \alpha_k\, {}^{i}\langle \psi_k \rangle - {}^{i}\langle \psi_k \rangle\, \nabla\, \alpha_k = \alpha_k\, \nabla\, {}^{i}\langle \psi_k \rangle , $$

it follows that

$$ {}^{i}\left\langle \nabla\, {}^{i}\langle \underline{\psi}_k \rangle \right\rangle = \nabla\, {}^{i}\langle \psi_k \rangle . \qquad (5.2.7a) $$

Likewise,

$$ {}^{i}\left\langle \nabla \cdot {}^{i}\langle \underline{\psi}_k \rangle \right\rangle = \nabla \cdot {}^{i}\langle \psi_k \rangle . \qquad (5.2.7b) $$

Also,

$$\left\langle \nabla\, {}^{i}\langle\psi_k\rangle^{*} \right\rangle = \nabla\, {}^{i}\langle\psi_k\rangle^{*} \quad \text{and} \quad \left\langle \nabla \cdot {}^{i}\langle\underline{\psi}_k\rangle^{*} \right\rangle = \nabla \cdot {}^{i}\langle\underline{\psi}_k\rangle^{*} \,. \tag{5.2.8a,b}$$

5.3 Volume Average of Products and Product of Volume Averages

From the definition of the mass-weighted, volume-averaged velocity, we have

$$\bullet \quad {}^{i}\langle\rho_k \underline{U}_k\rangle = {}^{i}\langle\rho_k\rangle\, {}^{i}\langle\underline{U}_k\rangle^{*} \,, \tag{5.3.1}$$

which is equation (2.1.6b). Using the decompositions $\rho_k = {}^{i}\langle\rho_k\rangle + \hat{\rho}_k$, and $\underline{U}_k = {}^{i}\langle\underline{U}_k\rangle^{*} + \tilde{\underline{U}}_k$, one readily obtains

$$ {}^{i}\langle\rho_k \underline{U}_k\rangle = {}^{i}\langle\rho_k\rangle\, {}^{i}\langle\underline{U}_k\rangle^{*} + {}^{i}\langle\rho_k\rangle\, {}^{i}\langle\tilde{\underline{U}}_k\rangle + {}^{i}\langle\tilde{\rho}_k \tilde{\underline{U}}_k\rangle \,. \tag{5.3.2}$$

Comparing equation (5.3.1) with equation (5.3.2) gives

$$\boxed{ {}^{i}\langle\rho_k\rangle\, {}^{i}\langle\tilde{\underline{U}}_k\rangle + {}^{i}\langle\hat{\rho}_k \tilde{\underline{U}}_k\rangle = 0 \,. } \tag{5.3.3}$$

It may be of interest to note that equation (5.3.3) can also be obtained by using the fact that ${}^{i}\langle\tilde{\underline{U}}_k\rangle^{*} = 0$ [†] .

$$\bullet \quad {}^{i}\langle\rho_k \underline{U}_k \underline{U}_k\rangle = {}^{i}\langle\rho_k\rangle {}^{i}\langle\underline{U}_k \underline{U}_k\rangle^{*} = {}^{i}\langle\rho_k\rangle \left({}^{i}\langle\underline{U}_k\rangle^{*}\, {}^{i}\langle\underline{U}_k\rangle^{*} + {}^{i}\langle\tilde{\underline{U}}_k\, \tilde{\underline{U}}_k\rangle^{*} \right) . \tag{5.3.4a}$$

Now,

$$ {}^{i}\langle\tilde{\underline{U}}_k \tilde{\underline{U}}_k\rangle^{*} = {}^{i}\langle\rho_k \tilde{\underline{U}}_k \tilde{\underline{U}}_k\rangle \Big/ {}^{i}\langle\rho_k\rangle = {}^{i}\langle\tilde{\underline{U}}_k \tilde{\underline{U}}_k\rangle + {}^{i}\langle\hat{\rho}_k \tilde{\underline{U}}_k \tilde{\underline{U}}_k\rangle \Big/ {}^{i}\langle\rho_k\rangle \,. \tag{5.3.5}$$

Thus, an alternative form of equation (5.3.4a) is

$$ {}^{i}\langle\rho_k \underline{U}_k \underline{U}_k\rangle = {}^{i}\langle\rho_k\rangle \left({}^{i}\langle\underline{U}_k\rangle^{*}\, {}^{i}\langle\underline{U}_k\rangle^{*} + {}^{i}\langle\tilde{\underline{U}}_k \tilde{\underline{U}}_k\rangle \right) + {}^{i}\langle\hat{\rho}_k \tilde{\underline{U}}_k \tilde{\underline{U}}_k\rangle \,. \tag{5.3.4b}$$

[†] ${}^{i}\langle\tilde{\underline{U}}_k\rangle^{*} = {}^{i}\langle\rho_k \tilde{\underline{U}}_k\rangle / {}^{i}\langle\rho_k\rangle = \left({}^{i}\langle\rho_k\rangle\, {}^{i}\langle\tilde{\underline{U}}_k\rangle + {}^{i}\langle\tilde{\rho}_k \tilde{\underline{U}}_k\rangle \right) \Big/ {}^{i}\langle\rho_k\rangle = 0.$

Since ${}^{i}\langle\rho_k\rangle$ is positive definite, the numerator of the third term must vanish. Hence, one obtains equation (5.3.3).

Corresponding relationships for internal energy are completely analogous:

$$^{i}\langle \rho_k u_k \rangle = {}^{i}\langle \rho_k \rangle \, {}^{i}\langle u_k \rangle^{*} , \tag{5.3.6}$$

$$^{i}\langle \rho_k \underline{U}_k u_k \rangle = {}^{i}\langle \rho_k \rangle {}^{i}\langle \underline{U}_k u_k \rangle^{*} = {}^{i}\langle \rho_k \rangle \left({}^{i}\langle \underline{U}_k \rangle^{*} \, {}^{i}\langle u_k \rangle^{*} + {}^{i}\langle \hat{\underline{U}}_k \hat{u}_k \rangle^{*} \right) \tag{5.3.7a}$$

$$= {}^{i}\langle \rho_k \rangle \left({}^{i}\langle \underline{U}_k \rangle^{*} \, {}^{i}\langle u_k \rangle^{*} + {}^{i}\langle \hat{\underline{U}}_k \hat{u}_k \rangle \right) + {}^{i}\langle \hat{\rho}_k \hat{\underline{U}}_k \hat{u}_k \rangle . \tag{5.3.7b}$$

Furthermore,

$$^{i}\langle \rho_k \rangle \, {}^{i}\langle \hat{u}_k \rangle + {}^{i}\langle \hat{\rho}_k \hat{u}_k \rangle = 0 \tag{5.3.8}$$

and

$$^{i}\langle \hat{\underline{U}}_k \hat{u}_k \rangle^{*} = {}^{i}\langle \hat{\underline{U}}_k \hat{u}_k \rangle + {}^{i}\langle \hat{\rho}_k \hat{\underline{U}}_k \hat{u}_k \rangle \Big/ {}^{i}\langle \rho_k \rangle . \tag{5.3.9}$$

For turbulent flows, the decomposition scheme of equation (5.1.9) for ρ_k and of equation (5.1.12) for $\underline{U}_k$ will be used. Accordingly,

$$^{i}\langle \rho_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} + {}^{i}\langle \rho_k' \rangle \tag{5.3.10}$$

and

$$^{i}\langle \underline{U}_k \rangle^{*} = {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{i}\langle \underline{U}_k' \rangle^{*} . \tag{5.3.11}$$

We reiterate that equation (5.3.10) follows from the relation $^{i}\langle \hat{\rho}_{kLF} \rangle = 0$ and that equation (5.3.11) follows from $^{i}\langle \hat{\underline{U}}_{kLF} \rangle^{*} = 0$. Both require that $v_k'/v_k \to 0$. The vanishing of $^{i}\langle \hat{\underline{U}}_{kLF} \rangle^{*}$ also requires that $\rho_k'/\rho_k \to 0$.

For convenience in writing, we shall hereafter write $\tilde{\rho}_k$ for $\hat{\rho}_{kLF}$, $\tilde{\underline{U}}_k$ for $\hat{\underline{U}}_{kLF}$, etc. This will not result in any confusion if it is understood that any quantity with a tilde is of low frequency.

Using the relations given by equations (5.3.10) and (5.3.11), equation (5.3.1) becomes

$$^{i}\langle \rho_k \underline{U}_k \rangle = \left({}^{i}\langle \rho_k \rangle_{LF} + {}^{i}\langle \rho_k' \rangle \right) {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}_k' \rangle^{*} . \tag{5.3.12}$$

We shall be interested in evaluating the time average of equation (5.3.12). Since $^{i}\langle \underline{U}_k \rangle^{*}_{LF}$ is, by definition, the low-frequency component of $^{i}\langle \underline{U}_k \rangle^{*}$,

$$\left\langle {}^{i}\langle \rho_k' \rangle \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right\rangle^{t} = 0.$$

Now,

$$ {}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}_k' \rangle^{*} = {}^{i}\langle \rho_k \underline{U}_k' \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k' \rangle + {}^{i}\langle \hat{\rho}_k \underline{U}_k' \rangle + {}^{i}\langle \rho_k' \underline{U}_k' \rangle \; . \qquad (5.3.13).$$

Therefore, its time average is

$$\left\langle {}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}_k' \rangle^{*} \right\rangle^{t} = {}^{ti}\langle \rho_k' \underline{U}_k' \rangle \; , \qquad (5.3.14)$$

which is a turbulent mass flux. Consequently, the time-average of ${}^{i}\langle \rho_k \underline{U}_k \rangle$ is

$$ {}^{ti}\langle \rho_k \underline{U}_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{ti}\langle \rho_k' \underline{U}_k' \rangle \; . \qquad (5.3.15)$$

A second expression for ${}^{i}\langle \rho_k \underline{U}_k \rangle$ can be obtained by multiplying equation (5.1.9) by equation (5.1.12), and then performing the indicated volume averaging. The result is

$$ {}^{i}\langle \rho_k \underline{U}_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{i}\langle \hat{\underline{U}}_k \rangle + {}^{i}\langle \underline{U}_k' \rangle \right) + {}^{i}\langle \hat{\rho}_k \hat{\underline{U}}_k \rangle + {}^{i}\langle \hat{\rho}_k \underline{U}_k' \rangle$$

$$ + {}^{i}\langle \rho_k' \rangle \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{i}\langle \rho_k' \hat{\underline{U}}_k \rangle + {}^{i}\langle \rho_k' \underline{U}_k' \rangle \; . \qquad (5.3.16)$$

Introducing equation (5.3.13) into equation (5.3.12) and comparing the result with equation (5.3.16) lead to

$$\boxed{ {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \hat{\underline{U}}_k \rangle + {}^{i}\langle \hat{\rho}_k \hat{\underline{U}}_k \rangle + {}^{i}\langle \rho_k' \hat{\underline{U}}_k \rangle = 0 \; . } \qquad (5.3.17a)$$

The first two terms of equation (5.3.17a) are of low frequency since v_k'/v_{kLF} is taken to be negligibly small, but the third term is of high frequency. Hence, they must independently vanish, i.e.,

$$ {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \hat{\underline{U}}_k \rangle + {}^{i}\langle \hat{\rho}_k \hat{\underline{U}}_k \rangle = 0 \; , \qquad (5.3.17b)$$

and

$$ {}^{i}\langle \rho'_k \underline{\hat{U}}_k \rangle = 0 \ . \qquad (5.3.17c) $$

The result that ρ'_k and $\underline{\hat{U}}_k$ are not spatially correlated is what one would intuitively expect. Equation (5.3.17b) may be compared with the corresponding result for the nonturbulent case, equation (5.3.3). It should be remembered, however, that in equations (5.3.17a,b,c) $\tilde{\rho}_k$ implies $\hat{\rho}_{kLF}$ and $\underline{\hat{U}}_k$ implies $\underline{\hat{U}}_{kLF}$.

Since $\underline{U}_k \underline{U}_k = {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \underline{\hat{U}}_k \underline{\hat{U}}_k + \underline{U}'_k \underline{U}'_k + 2 \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, \underline{\hat{U}}_k + 2 \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, \underline{U}'_k$

$+ 2 \, \underline{\hat{U}}_k \underline{U}'_k$, one readily obtains

$$ {}^{i}\langle \rho_k \underline{U}_k \underline{U}_k \rangle = {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}_k \underline{U}_k \rangle^{*} $$

$$ = {}^{i}\langle \rho_k \rangle \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{i}\langle \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle^{*} + {}^{i}\langle \underline{U}'_k \underline{U}'_k \rangle^{*} \right. $$

$$ \left. + 2 \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle \underline{U}'_k \rangle^{*} + 2 \, {}^{i}\langle \underline{\hat{U}}_k \underline{U}'_k \rangle^{*} \right) . \qquad (5.3.18) $$

It may be of interest to note that equation (5.3.18) can also be deduced by introducing the decomposition relations $\rho_k = {}^{i}\langle \rho_k \rangle_{LF} + \hat{\rho}_k + \rho'_k$ and $\underline{U}_k = {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \underline{\hat{U}}_k + \underline{U}'_k$, into the product $\rho_k \underline{U}_k \underline{U}_k$ from the very beginning, followed by taking the volume average and making use of equation (5.3.17a).

The time averages of the various terms in equation (5.3.18) are

$$ {}^{t}\left\langle {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right\rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \ , \qquad (5.3.19a) $$

$$ {}^{t}\left\langle {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle^{*} \right\rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle + {}^{i}\langle \hat{\rho}_k \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle $$

$$ = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle^{*}_{LF} \ , \qquad (5.3.19b) $$

where ${}^{i}\langle \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle^{*}_{LF}$ denotes the low-frequency component of ${}^{i}\langle \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle^{*}$.

$$ {}^{t}\left\langle {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}'_k \underline{U}'_k \rangle^{*} \right\rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}'_k \underline{U}'_k \rangle + {}^{ti}\langle \hat{\rho}_k \underline{U}'_k \underline{U}'_k \rangle + {}^{ti}\langle \rho'_k \underline{U}'_k \underline{U}'_k \rangle \ , \qquad (5.3.19c) $$

$$\left\langle {}^{t}\left({}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}_k' \rangle^{*} \right) \right\rangle = {}^{ti}\langle \rho_k' \underline{U}_k' \rangle \ , \tag{5.3.19d}$$

which is equation (5.3.14), and

$$\left\langle {}^{t}\left({}^{i}\langle \rho_k \rangle \; {}^{i}\langle \hat{\underline{U}}_k \underline{U}_k' \rangle^{*} \right) \right\rangle = {}^{ti}\langle \rho_k' \underline{U}_k' \hat{\underline{U}}_k \rangle \ . \tag{5.3.19e}$$

Accordingly,

$$\begin{aligned} \bullet \quad {}^{ti}\langle \rho_k \underline{U}_k \underline{U}_k \rangle &= {}^{i}\langle \rho_k \rangle_{LF} \left({}^{i}\langle \underline{U}_k \rangle_{LF}^{*} \; {}^{i}\langle \underline{U}_k \rangle_{LF}^{*} + {}^{i}\langle \hat{\underline{U}}_k \hat{\underline{U}}_k \rangle + {}^{ti}\langle \underline{U}_k' \underline{U}_k' \rangle \right) \\ &\quad + 2 \; {}^{ti}\langle \rho_k' \underline{U}_k' \rangle \; {}^{i}\langle \underline{U}_k \rangle_{LF}^{*} + {}^{i}\langle \hat{\rho}_k \hat{\underline{U}}_k \hat{\underline{U}}_k \rangle + {}^{ti}\langle \hat{\rho}_k \underline{U}_k' \underline{U}_k' \rangle \\ &\quad + 2 \; {}^{ti}\langle \rho_k' \underline{U}_k' \hat{\underline{U}}_k \rangle + {}^{ti}\langle \rho_k' \underline{U}_k' \underline{U}_k' \rangle \ . \end{aligned} \tag{5.3.20}$$

Equation (5.3.20) can also be deduced by forming the product $\rho_k \underline{U}_k \underline{U}_k$, followed by performing the intrinsic volume averaging, making use of the relation given by equation (5.3.17a), and then time averaging.

Analogous expressions for internal energy are

$$\boxed{ {}^{i}\langle \rho_k \rangle_{LF} \; {}^{i}\langle \hat{u}_k \rangle + {}^{i}\langle \hat{\rho}_k \hat{u}_k \rangle + {}^{i}\langle \rho_k' \hat{u}_k \rangle = 0 \ , } \tag{5.3.21a}$$

$$\text{with } {}^{i}\langle \rho_k \rangle_{LF} \; {}^{i}\langle \hat{u}_k \rangle + {}^{i}\langle \hat{\rho}_k \hat{u}_k \rangle = 0, \text{ and } {}^{i}\langle \rho_k' \hat{u}_k \rangle = 0 \ . \tag{5.3.21b,c}$$

$$\begin{aligned} \bullet \quad {}^{i}\langle \rho_k u_k \rangle &= {}^{i}\langle \rho_k \rangle \; {}^{i}\langle u_k \rangle^{*} \\ &= \left({}^{i}\langle \rho_k \rangle_{LF} + {}^{i}\langle \rho_k' \rangle \right) {}^{i}\langle u_k \rangle_{LF}^{*} + {}^{i}\langle \rho_k \rangle \; {}^{i}\langle u_k' \rangle^{*} \end{aligned} \tag{5.3.22}$$

$$\bullet \quad {}^{ti}\langle \rho_k u_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \; {}^{i}\langle u_k \rangle_{LF}^{*} + {}^{ti}\langle \rho_k' u_k' \rangle \tag{5.3.23}$$

$$\bullet \quad {}^{i}\langle \rho_k \underline{U}_k u_k \rangle = {}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}_k u_k \rangle^{*}$$

$$= {}^{i}\langle\rho_k\rangle \left({}^{i}\langle \underline{U}_k\rangle^{*}_{LF}\, {}^{i}\langle u_k\rangle^{*}_{LF} + {}^{i}\langle \tilde{\underline{U}}_k \tilde{u}_k\rangle^{*} + {}^{i}\langle \underline{U}'_k u'_k\rangle^{*} + {}^{i}\langle \underline{U}_k\rangle^{*}_{LF}\, {}^{i}\langle u'_k\rangle^{*} \right.$$

$$\left. + {}^{i}\langle \underline{U}'_k\rangle^{*}\, {}^{i}\langle u_k\rangle^{*}_{LF} + {}^{i}\langle \tilde{\underline{U}}_k u'_k\rangle^{*} + {}^{i}\langle \underline{U}'_k \tilde{u}_k\rangle^{*} \right) \qquad (5.3.24)$$

$$\bullet \quad {}^{ti}\langle \rho_k \underline{U}_k u_k\rangle = {}^{i}\langle\rho_k\rangle_{LF} \left({}^{i}\langle \underline{U}_k\rangle^{*}_{LF}\, {}^{i}\langle u_k\rangle^{*}_{LF} + {}^{i}\langle \tilde{\underline{U}}_k \tilde{u}_k\rangle + {}^{ti}\langle \underline{U}'_k u'_k\rangle \right)$$

$$+ {}^{ti}\langle \rho'_k \underline{U}'_k\rangle {}^{i}\langle u_k\rangle^{*}_{LF} + {}^{ti}\langle \rho'_k u'_k\rangle {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} + {}^{i}\langle \tilde{\rho}_k \tilde{\underline{U}}_k \tilde{u}_k\rangle$$

$$+ {}^{ti}\langle \tilde{\rho}_k \underline{U}'_k u'_k\rangle + {}^{ti}\langle \tilde{\underline{U}}_k \rho'_k u'_k\rangle + {}^{ti}\langle \tilde{u}_k \rho'_k \underline{U}'_k\rangle + {}^{ti}\langle \rho'_k \underline{U}'_k u'_k\rangle \,. \qquad (5.3.25)$$

For the total energy, we may also write

$$E_k = {}^{i}\langle E_k\rangle^{*}_{LF} + \tilde{E}_k + E'_k \,. \qquad (5.3.26)$$

However, the defining relations for ${}^{i}\langle E_k\rangle^{*}_{LF}$, $\tilde{E}_k$, and E'_k require careful consideration. Specifically, ${}^{i}\langle E_k\rangle^{*}_{LF}$ is of low frequency and of length scale ℓ, $\tilde{E}_k$ is also of low frequency and must satisfy the condition ${}^{i}\langle \tilde{E}_k\rangle^{*}_{LF} = 0$, and finally the time-average of E'_k should vanish. Since $E_k = u_k + \frac{1}{2}\underline{U}_k \cdot \underline{U}_k$, and by using the already defined relations $u_k = {}^{i}\langle u_k\rangle^{*}_{LF} + \tilde{u}_k + u'_k$ and $\underline{U}_k = {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} + \tilde{\underline{U}}_k + \underline{U}'_k$, we obtain

$$E_k = {}^{i}\langle u_k\rangle^{*}_{LF} + \tilde{u}_k + u'_k + \frac{1}{2}\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \cdot {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} + \frac{1}{2}\tilde{\underline{U}}_k \cdot \tilde{\underline{U}}_k + \frac{1}{2}\underline{U}'_k \cdot \underline{U}'_k$$

$$+ {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \cdot \tilde{\underline{U}}_k + {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \cdot \underline{U}'_k + \tilde{\underline{U}}_k \cdot \underline{U}'_k \,. \qquad (5.3.27)$$

We define

$${}^{i}\langle E_k\rangle^{*}_{LF} = {}^{i}\langle u_k\rangle^{*}_{LF} + \frac{1}{2}\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \cdot {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} + \frac{1}{2}\, {}^{i}\langle \tilde{\underline{U}}_k \cdot \tilde{\underline{U}}_k\rangle^{*} + \frac{1}{2}\, {}^{ti}\langle \underline{U}'_k \cdot \underline{U}'_k\rangle \,, \qquad (5.3.28)$$

$$\tilde{E}_k = \tilde{u}_k + {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \cdot \tilde{\underline{U}}_k + \frac{1}{2}\left(\tilde{\underline{U}}_k \cdot \tilde{\underline{U}}_k - {}^{i}\langle \tilde{\underline{U}}_k \cdot \tilde{\underline{U}}_k\rangle^{*} \right) , \qquad (5.3.29)$$

$$E'_k = u'_k + {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \cdot \underline{U}'_k + \tilde{\underline{U}}_k \cdot \underline{U}'_k + \frac{1}{2}\left(\underline{U}'_k \cdot \underline{U}'_k - {}^{ti}\langle \underline{U}'_k \cdot \underline{U}'_k\rangle \right) . \qquad (5.3.30)$$

It can be readily verified that equation (5.3.26) with the defining relations for ${}^{i}\langle E_k \rangle_{LF}^{*}$, $\grave{E}_k$, and E_k', given respectively by equations (5.3.28), (5.3.29), and (5.3.30), is identical to equation (5.3.27). Furthermore, the constraints just cited for ${}^{i}\langle E_k \rangle_{LF}^{*}$ and $\grave{E}_k$ are also satisfied. If we assume that ${}^{t}\langle \underline{U}_k' \cdot \underline{U}_k' \rangle \simeq {}^{ti}\langle \underline{U}_k' \cdot \underline{U}_k' \rangle$, then ${}^{t}\langle E_k' \rangle = 0$. For the equations which follow, this approximation is used. Accordingly,

$$\boxed{{}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \grave{E}_k \rangle + {}^{i}\langle \grave{\rho}_k \grave{E}_k \rangle + {}^{i}\langle \rho_k' \grave{E}_k \rangle = 0} \tag{5.3.31}$$

and ${}^{i}\langle \rho_k E_k \rangle$, ${}^{ti}\langle \rho_k E_k \rangle$, ${}^{i}\langle \rho_k \underline{U}_k E_k \rangle$, and ${}^{ti}\langle \rho_k \underline{U}_k E_k \rangle$ can be written down from equations (5.3.22), (5.3.23), (5.3.24), and (5.3.25) by simply replacing u_k by E_k with ${}^{i}\langle E_k \rangle_{LF}^{*}$ defined by equation (5.3.28), $\grave{E}_k$ by equation (5.3.29), and E_k' by equation (5.3.30). Finally, we note that the decomposition of E_k given in Ref. 12 was incorrect.

It is seen from the foregoing that our objective of expressing the volume average of products in terms of products of volume averages is only partially achieved. For the nonturbulent case, it is necessary to assess terms like ${}^{i}\langle \grave{\underline{U}}_k \grave{\underline{U}}_k \rangle^{*}$ in the momentum equation, ${}^{i}\langle \grave{\underline{U}}_k \grave{E}_k \rangle^{*}$ in the total energy equation, and ${}^{i}\langle \grave{\underline{U}}_k \grave{u}_k \rangle^{*}$ in the internal energy equation. A number of additional terms appear for the turbulent case. The assessment or modeling of these terms constitutes, in part, the closure problem.

Simplifications

Except in extraordinary circumstances, $\grave{\rho}_k / {}^{i}\langle \rho_k \rangle_{LF} \ll 1$, and, furthermore, $\grave{\rho}_k$ changes sign within an averaging volume. Hence, one may usually assume

$${}^{i}\langle \grave{\rho}_k \grave{\underline{U}}_k \grave{\underline{U}}_k \rangle \ll {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \grave{\underline{U}}_k \grave{\underline{U}}_k \rangle \, , \tag{5.3.32}$$

$${}^{ti}\langle \grave{\rho}_k \underline{U}_k' \underline{U}_k' \rangle \ll {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}_k' \underline{U}_k' \rangle \, , \tag{5.3.33}$$

$${}^{i}\langle \grave{\rho}_k \grave{\underline{U}}_k \grave{u}_k \rangle \ll {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \grave{\underline{U}}_k \grave{u}_k \rangle \, , \tag{5.3.34}$$

$${}^{ti}\langle \grave{\rho}_k \underline{U}_k' u_k' \rangle \ll {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}_k' u_k' \rangle \, , \tag{5.3.35}$$

and similar inequalities for total energy.

Furthermore, if ${}^{ti}\langle \rho_k' \underline{U}_k' \rangle \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle_{LF}^{*} \ll 1$, which is often the case, then equations (5.3.15), (5.3.20), (5.3.23), and (5.3.25) can be simplified to

$$\bullet \quad {}^{ti}\langle \rho_k \underline{U}_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, , \tag{5.3.36}$$

$$\bullet \quad {}^{ti}\langle \rho_k \underline{U}_k U_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle U_k \rangle^{*}_{LF} + {}^{i}\langle \tilde{\underline{U}}_k \tilde{U}_k \rangle + {}^{ti}\langle \underline{U}'_k U'_k \rangle \right), \tag{5.3.37}$$

$$\bullet \quad {}^{ti}\langle \rho_k u_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF} \, , \tag{5.3.38}$$

$$\bullet \quad {}^{ti}\langle \rho_k \underline{U}_k u_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF} + {}^{i}\langle \tilde{\underline{U}}_k \tilde{u}_k \rangle + {}^{ti}\langle \underline{U}'_k u'_k \rangle \right) . \tag{5.3.39}$$

Likewise,

$$\bullet \quad {}^{ti}\langle \rho_k E_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF} \, , \tag{5.3.40}$$

$$\bullet \quad {}^{ti}\langle \rho_k \underline{U}_k E_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF} + {}^{i}\langle \tilde{\underline{U}}_k \tilde{E}_k \rangle + {}^{ti}\langle \underline{U}'_k E'_k \rangle \right) . \tag{5.3.41}$$

Under the stronger restriction of constant ρ_k, we have $\tilde{\rho}_k = 0$, $\rho'_k = 0$, and ${}^{i}\langle \rho_k \rangle_{LF} = \rho_k$. Also, ${}^{i}\langle \underline{U}_k \rangle^{*}_{LF} = {}^{i}\langle \underline{U}_k \rangle_{LF}$, ${}^{i}\langle u_k \rangle^{*}_{LF} = {}^{i}\langle u_k \rangle_{LF}$, and ${}^{i}\langle E_k \rangle^{*}_{LF} = {}^{i}\langle E_k \rangle_{LF}$.

Finally, we note that from equation (2.2.5), we can write

$$\frac{\partial \alpha_k}{\partial t} = v^{-1} \int_{A_k} \left({}^{i}\langle \underline{W}_s \rangle_{LF} + \tilde{\underline{W}}_s \right) \cdot \underline{n}_k \, dA$$

$$+ v^{-1} \int_{A_k} \underline{W}'_s \cdot \underline{n}_k \, dA \, . \tag{5.3.42}$$

Since α'_k is assumed to be negligible, the second area integral must vanish, i.e.,

$$\int_{A_k} \underline{W}'_s \cdot \underline{n}_k \, dA = 0 \, . \tag{5.3.43}$$

5.4 Time- and Volume-Averaged Mass Conservation Equation and Interfacial Mass Balance Equation

The local volume-averaged mass conservation equation is given by equation (4.1). By introducing the decomposition scheme for ρ_k defined by equation (5.1.9), $\underline{U}_k$ by equation (5.1.12) and $\underline{W}_s$ by equation (5.1.18), followed by time-averaging, one obtains for negligible α_k' and A_k',

$$\left\langle \alpha_k \, {}^{i}\langle \rho_k \rangle \right\rangle^{t} = \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, , \tag{5.4.1}$$

$$\left\langle \alpha_k \, {}^{i}\langle \rho_k \underline{U}_k \rangle \right\rangle^{t} = \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \alpha_k \, {}^{ti}\langle \rho_k' \underline{U}_k' \rangle \, , \tag{5.4.2}$$

which is taken directly from equation (5.3.15). For convenience, the turbulent mass flux ${}^{ti}\langle \rho_k' \underline{U}_k' \rangle$ will be denoted by $\underline{\psi}_{mk}$, i.e.,

$$\underline{\psi}_{mk} = {}^{ti}\langle \rho_k' \underline{U}_k' \rangle \, . \tag{5.4.3}$$

The time-averaged interfacial mass generation rate of phase k within v is

$${}^{t}\langle \Gamma_k \rangle = - v^{-1} \int_{A_k} {}^{t}\langle \rho_k (\underline{U}_k - \underline{W}_s) \rangle \cdot \underline{n}_k \, dA \, ,$$

$$= {}^{i}\langle \rho_k \rangle_{LF} \left(\frac{\partial \alpha_k}{\partial t} + {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \nabla \alpha_k \right) + (MTI)_k \tag{5.4.4}$$

where $(MTI)_k$ stands for the interfacial mass transfer integral defined by

$$(MTI)_k = - v^{-1} \, {}^{i}\langle \rho_k \rangle_{LF} \int_{A_k} \hat{\underline{U}}_k \cdot \underline{n}_k \, dA$$

$$- v^{-1} \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} - {}^{i}\langle \underline{W}_s \rangle_{LF} \right) \cdot \int_{A_k} \hat{\rho}_k \, \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \hat{\rho}_k \left(\hat{\underline{U}}_k - \hat{\underline{W}}_s \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} {}^{t}\langle \rho_k' (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA \, . \qquad (5.4.5a)$$

In deriving equations (5.4.4) and (5.4.5a), use has been made of equations (2.2.4), (5.3.42), and (5.3.43), as well as the fact that volume-averaged entities such as ${}^{i}\langle \rho_k \rangle_{LF}$ and ${}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$ can be taken outside of the area integrals. When $\hat{\rho}_k / {}^{i}\langle \rho_k \rangle_{LF} << 1$, which is usually the case, the two terms involving $\hat{\rho}_k$ in equation (5.4.5a) may be deleted. Accordingly,

$$(MTI)_k \hat{=} - v^{-1} \, {}^{i}\langle \rho_k \rangle_{LF} \int_{A_k} \hat{\underline{U}}_k \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} {}^{t}\langle \rho_k' (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA \, . \qquad (5.4.5b)$$

Performing the time averaging of equation (4.1), followed by introducing the results given in equations (5.4.1), (5.4.2), (5.4.3), and (5.4.4), leads to the desired time- and volume-averaged mass conservation equation:

$$\frac{\partial}{\partial t} \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} + \nabla \cdot \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \underline{\Psi}_{mk} = {}^{t}\langle \Gamma_k \rangle \, . \qquad (5.4.6a)$$

An equivalent form of equation (5.4.6a) is

$$\alpha_k \left(\frac{\partial}{\partial t} \, {}^{i}\langle \rho_k \rangle_{LF} + \nabla \cdot {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) + \nabla \cdot \alpha_k \underline{\Psi}_{mk} = (MTI)_k \, . \qquad (5.4.6b)$$

When ρ_k = constant, simplification is seen in that $\underline{\Psi}_{mk} = 0$, and $(MTI)_k$ becomes

$$(MTI)_k^{\circ} = - v^{-1} \rho_k \int_{A_k} \hat{\underline{U}}_k \cdot \underline{n}_k \, dA \, . \qquad (5.4.7)$$

Accordingly, the time- and volume-averaged mass conservation equation for constant ρ_k is

$$\alpha_k \nabla \cdot {}^{i}\langle \underline{U}_k \rangle_{LF} = - v^{-1} \int_{A_k} \hat{\underline{U}}_k \cdot \underline{n}_k \, dA \, . \qquad (5.4.8)$$

It is straightforward to demonstrate that equation (5.4.8) can also be deduced by performing the time- and volume-averaging of $\nabla \cdot \underline{U}_k = 0$.

The local volume-averaged mass balance equation for the interface is given by equation (4.8). Performing the time-averaging gives

$$\sum_{k,f} {}^{t}\langle \Gamma_k \rangle = 0 \tag{5.4.9a}$$

or

$${}^{t}\langle \Gamma_k \rangle + {}^{t}\langle \Gamma_f \rangle = 0 \ , \tag{5.4.9b}$$

where ${}^{t}\langle \Gamma_f \rangle$ is given by equation (5.4.4) with subscript k replaced by f.

5.5 Time- and Volume-Averaged Linear Momentum Conservation Equation and Interfacial Momentum Balance Equation

The local volume-averaged linear momentum conservation equation for constant field force is given by equation (4.4). The time-average of the several terms in the equation have already been given; others can be obtained from a procedure similar to that used in the previous section. To conserve space, we present only the results of those terms that require further discussion.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{i}\langle \rho_k \underline{U}_k \underline{U}_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + 2 \, \alpha_k \, \underline{\Psi}_{mk} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$$

$$- \alpha_k \left({}^{ti}\langle \underline{\underline{\tau}}^{T}_{k} \rangle + {}^{i}\langle \underline{\underline{\tilde{\tau}}}^{D}_{k} \rangle + {}^{ti}\langle \underline{\underline{\tilde{\tau}}}^{T}_{k} \rangle \right) , \tag{5.5.1}$$

in which $\underline{\Psi}_{mk}$ is given by equation (5.4.3). The last three terms are, respectively,

(a) the time- and volume-averaged Reynolds stress tensor ${}^{ti}\langle \underline{\underline{\tau}}^{T}_{k} \rangle$ defined by

$${}^{ti}\langle \underline{\underline{\tau}}^{T}_{k} \rangle = - {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}'_k \underline{U}'_k \rangle - {}^{ti}\langle \tilde{\rho}_k \underline{U}'_k \underline{U}'_k \rangle - {}^{ti}\langle \rho'_k \underline{U}'_k \underline{U}'_k \rangle \tag{5.5.2a}$$

$$= - \left\langle {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}'_k \underline{U}'_k \rangle^{*} \right\rangle , \tag{5.5.2b}$$

(b) the volume-averaged dispersive stress tensor ${}^{i}\langle \underline{\underline{\tilde{\tau}}}^{D}_{k} \rangle$ defined by

$${}^{i}\langle \underline{\underline{\tilde{\tau}}}^{D}_{k} \rangle = - {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \tilde{\underline{U}}_k \tilde{\underline{U}}_k \rangle - {}^{i}\langle \tilde{\rho}_k \tilde{\underline{U}}_k \tilde{\underline{U}}_k \rangle \tag{5.5.3a}$$

$$= - {}^{i}\langle \rho_k \rangle_{LF} \; {}^{i}\langle \tilde{\underline{U}}_k \tilde{\underline{U}}_k \rangle^{*}_{LF} \; , \tag{5.5.3b}$$

and

(c) the time- and volume-averaged turbulent, dispersive stress tensor ${}^{ti}\langle \tilde{\underline{\underline{\tau}}}^{T}_{k} \rangle$ defined by

$${}^{ti}\langle \tilde{\underline{\underline{\tau}}}^{T}_{k} \rangle = - 2 \; {}^{ti}\langle \rho'_k \; \underline{U}'_k \tilde{\underline{U}}_k \rangle \tag{5.5.4a}$$

$$= - 2 \left\langle {}^{t}\langle {}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}'_k \tilde{\underline{U}}_k \rangle^{*} \right\rangle . \tag{5.5.4b}$$

Under the simplifying conditions listed near the end of Section 5.3, $\underline{\psi}_{mk} = 0$, and the three stress tensors defined in equations (5.5.2), (5.5.3), and (5.5.4) become, respectively,

(a') $${}^{ti}\langle \underline{\underline{\tau}}^{T}_{k} \rangle \tilde{=} - {}^{i}\langle \rho_k \rangle_{LF} \; {}^{ti}\langle \underline{U}'_k \underline{U}'_k \rangle \; , \tag{5.5.2c}$$

(b') $${}^{ti}\langle \tilde{\underline{\underline{\tau}}}^{D}_{k} \rangle \tilde{=} - {}^{i}\langle \rho_k \rangle_{LF} \; {}^{i}\langle \tilde{\underline{U}}_k \tilde{\underline{U}}_k \rangle \; , \tag{5.5.3c}$$

(c') $${}^{ti}\langle \tilde{\underline{\underline{\tau}}}^{T}_{k} \rangle \tilde{=} 0 \; . \tag{5.5.4c}$$

- ${}^{t}\left\langle \alpha_k \; {}^{i}\langle \underline{\underline{\tau}}_k \rangle \right\rangle$ where $\underline{\underline{\tau}}_k$ is the viscous stress tensor.

For Newtonian fluids,

$$\underline{\underline{\tau}}_k = \left(\lambda_k - \frac{2}{3} \mu_k \right) (\nabla \cdot \underline{U}_k) \; \underline{\underline{I}} + \mu_k \left[\nabla \underline{U}_k + (\nabla \underline{U}_k)_c \right] \tag{5.5.5}$$

where λ_k is the bulk viscosity, $\nabla \underline{U}_k$ is a dyad and the subscript c denotes conjugate. Using the decomposition $\underline{U}_k = {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \tilde{\underline{U}}_k + \underline{U}'_k$, equation (5.5.5) becomes

$$\underline{\underline{\tau}}_k = \left(\lambda_k - \frac{2}{3} \mu_k \right) \left(\nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) \underline{\underline{I}} + \mu_k \left[\nabla \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \left(\nabla \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right)_c \right]$$

$$+ \left(\lambda_k - \frac{2}{3} \mu_k \right) (\nabla \cdot \tilde{\underline{U}}_k) \; \underline{\underline{I}} + \mu_k \left[\nabla \tilde{\underline{U}}_k + (\nabla \tilde{\underline{U}}_k)_c \right]$$

$$+ \left(\lambda_k - \frac{2}{3}\mu_k\right)(\nabla \cdot \underline{U}_k')\,\underline{\underline{I}} + \mu_k\left[\nabla \underline{U}_k' + (\nabla \underline{U}_k')_c\right] . \qquad (5.5.6)$$

Since

$$\left\langle \nabla \cdot {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \right\rangle = \nabla \cdot \alpha_k\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} + v^{-1}\int_{A_k} {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \cdot \underline{n}_k\, dA = \alpha_k\, \nabla \cdot {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} ,$$

$$\left\langle \nabla\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \right\rangle = \alpha_k\, \nabla\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} .$$

Also,

$$\left\langle (\nabla\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF})_c \right\rangle = \left\langle \nabla\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} \right\rangle_c = \alpha_k\, (\nabla\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF})_c ,$$

$$\langle \nabla \cdot \hat{\underline{U}}_k\rangle = \nabla \cdot \alpha_k\, {}^{i}\langle \hat{\underline{U}}_k\rangle + v^{-1}\int_{A_k} \hat{\underline{U}}_k \cdot \underline{n}_k\, dA ,$$

$$\langle \nabla\, \hat{\underline{U}}_k\rangle = \nabla\, \alpha_k\, {}^{i}\langle \hat{\underline{U}}_k\rangle + v^{-1}\int_{A_k} \hat{\underline{U}}_k\, \underline{n}_k\, dA ,$$

etc. Thus, we have for uniform λ_k and μ_k,

$$\langle \underline{\underline{\tau}}_k\rangle = \alpha_k\, {}^{i}\langle \underline{\underline{\tau}}_k\rangle$$

$$= \alpha_k \left\{ \left(\lambda_k - \frac{2}{3}\mu_k\right)\left(\nabla \cdot {}^{i}\langle \underline{U}_k\rangle^{*}_{LF}\right)\underline{\underline{I}} + \mu_k\left[\nabla\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF} + \left(\nabla\, {}^{i}\langle \underline{U}_k\rangle^{*}_{LF}\right)_c\right]\right\}$$

$$+ \left(\lambda_k - \frac{2}{3}\mu_k\right)\left(\nabla \cdot \alpha_k\, {}^{i}\langle \hat{\underline{U}}_k\rangle + v^{-1}\int_{A_k} \hat{\underline{U}}_k \cdot \underline{n}_k\, dA\right)\underline{\underline{I}}$$

$$+ \mu_k\left[\nabla\, \alpha_k\, {}^{i}\langle \hat{\underline{U}}_k\rangle + \left(\nabla\, \alpha_k\, {}^{i}\langle \hat{\underline{U}}_k\rangle\right)_c + v^{-1}\int_{A_k} \hat{\underline{U}}_k\, \underline{n}_k\, dA + v^{-1}\int_{A_k} (\underline{U}_k\, \underline{n}_k)_c\, dA\right]$$

$$+ \text{terms which vanish upon time-averaging.}$$

It may be noted that $\langle (\nabla\, \hat{\underline{U}}_k)_c\rangle = \langle \nabla\, \hat{\underline{U}}_k\rangle_c$, i.e., the volume average of the conjugate is the same as the conjugate of the volume average.

Accordingly, the desired result is

$$
\begin{aligned}
&{}^{t}\left\langle \alpha_k \, {}^{i}\langle \underline{\underline{\tau}}_k \rangle \right\rangle \\
&= \alpha_k \left\{ \left(\lambda_k - \frac{2}{3}\mu_k \right) \left(\nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) \underline{\underline{I}} + \mu_k \left[\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \left(\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right)_c \right] \right\} \\
&\quad + \left(\lambda_k - \frac{2}{3}\mu_k \right) \left(\nabla \cdot \alpha_k \, {}^{i}\langle \tilde{\underline{U}}_k \rangle + v^{-1} \int_{A_k} \tilde{\underline{U}}_k \cdot \underline{n}_k \, dA \right) \underline{\underline{I}} \\
&\quad + \mu_k \left[\nabla \, \alpha_k \, {}^{i}\langle \tilde{\underline{U}}_k \rangle + \left(\nabla \, \alpha_k \, {}^{i}\langle \tilde{\underline{U}}_k \rangle \right)_c + v^{-1} \int_{A_k} \tilde{\underline{U}}_k \, \underline{n}_k \, dA \right. \\
&\quad \left. + v^{-1} \int_{A_k} \left(\tilde{\underline{U}}_k \, \underline{n}_k \right)_c \, dA \right] \qquad (5.5.7a)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_k \left\{ \left(\lambda_k - \frac{2}{3}\mu_k \right) \left(\nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) \underline{\underline{I}} + \mu_k \left[\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \left(\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right)_c \right] \right. \\
&\quad \left. + \left(\lambda_k - \frac{2}{3}\mu_k \right) {}^{i}\langle \nabla \cdot \tilde{\underline{U}}_k \rangle \underline{\underline{I}} + \mu_k \left({}^{i}\langle \nabla \, \tilde{\underline{U}}_k \rangle + {}^{i}\langle \nabla \, \tilde{\underline{U}}_k \rangle_c \right) \right\} . \qquad (5.5.7b)
\end{aligned}
$$

When ρ_k = constant, ${}^{i}\langle \underline{U}_k \rangle^{*}_{LF} = {}^{i}\langle \underline{U}_k \rangle_{LF}$, ${}^{i}\langle \tilde{\underline{U}}_k \rangle = 0$, and furthermore, $\alpha_k \nabla \cdot {}^{i}\langle \underline{U}_k \rangle_{LF} + v^{-1} \int_{A_k} \tilde{\underline{U}}_k \cdot \underline{n}_k \, dA = 0$ according to equation (5.4.8). Hence, equation (5.5.7a,b) reduces for constant ρ_k to

$$
\begin{aligned}
{}^{t}\left\langle \alpha_k \, {}^{i}\langle \underline{\underline{\tau}}_k \rangle \right\rangle &= \alpha_k \mu_k \left[\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \left(\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right)_c \right] \\
&\quad + \mu_k v^{-1} \int_{A_k} \left[\underline{U}_k \, \underline{n}_k + \left(\underline{U}_k \, \underline{n}_k \right)_c \right] dA . \qquad (5.5.8)
\end{aligned}
$$

For convenience in subsequent discussion, it is desirable to express $\underline{\underline{\tau}}_k$ in the form

$$
\underline{\underline{\tau}}_k = {}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \tilde{\underline{\underline{\tau}}}_k + \underline{\underline{\tau}}'_k . \qquad (5.5.9a)
$$

Since ${}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} = {}^{t}\left\langle {}^{i}\langle \underline{\underline{\tau}}_k \rangle \right\rangle$, it follows from equation (5.5.7b) that

$$
{}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} = \left(\lambda_k - \frac{2}{3}\mu_k \right) \left(\nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) \underline{\underline{I}} + \mu_k \left[\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \left(\nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right)_c \right]
$$

$$\left(\kappa - \frac{2}{3}\mu_k\right)\,{}^{i}\langle\nabla\cdot\underline{\tilde{U}}_k\rangle\,\underline{\underline{I}} + \mu_k\left({}^{i}\langle\nabla\,\underline{\tilde{U}}_k\rangle + {}^{i}\langle\nabla\,\underline{\tilde{U}}_k\rangle_c\right). \quad (5.5.9b)$$

Substituting equation (5.5.9b) into equation (5.5.9a), followed by comparing the result with equation (5.5.6), leads to

$$\underline{\underline{\tilde{\tau}}}_k = \left(\lambda_k - \frac{2}{3}\mu_k\right)\left(\nabla\cdot\underline{\tilde{U}}_k - {}^{i}\langle\nabla\cdot\underline{\tilde{U}}_k\rangle\right)\underline{\underline{I}}$$

$$+ \mu_k\left[\nabla\,\underline{\tilde{U}}_k - {}^{i}\langle\nabla\,\underline{\tilde{U}}_k\rangle + \left(\nabla\,\underline{\tilde{U}}_k\right)_c - {}^{i}\langle\nabla\,\underline{\tilde{U}}_k\rangle_c\right], \quad (5.5.9c)$$

and

$$\underline{\underline{\tau}}'_k = \left(\lambda_k - \frac{2}{3}\mu_k\right)\left(\nabla\cdot\underline{U}'_k\right)\underline{\underline{I}} + \mu_k\left[\nabla\,\underline{U}'_k + \left(\nabla\,\underline{U}'_k\right)_c\right], \quad (5.5.9d)$$

which is not to be confused with the Reynolds stresses. Clearly, ${}^{t}\langle\underline{\underline{\tau}}'_k\rangle = 0$. Also, ${}^{i}\langle\underline{\underline{\tilde{\tau}}}_k\rangle = 0$, as it must. It is pertinent to note that expressions for ${}^{i}\langle\underline{\underline{\tau}}_k\rangle_{LF}$, $\underline{\underline{\tilde{\tau}}}_k$, and $\underline{\underline{\tau}}'_k$ are incorrectly given in Ref. 12.

In highly turbulent flows, the viscous stresses are usually insignificant relative to the turbulent stresses and so they can be ignored in the momentum equation.

$$\bullet \quad v^{-1}\left\langle{}^{t}\int_{A_k}\left(-P_k\,\underline{\underline{I}} + \underline{\underline{\tau}}_k\right)\cdot\underline{n}_k\,dA\right\rangle$$

$$= {}^{i}\langle p_k\rangle_{LF}\,\nabla\alpha_k - {}^{i}\langle\underline{\underline{\tau}}_k\rangle_{LF}\cdot\nabla\alpha_k + (PTI)_k - (VSTI)_k, \quad (5.5.10)$$

where ${}^{i}\langle\underline{\underline{\tau}}_k\rangle_{LF}$ is given by equation (5.5.9b), $(PTI)_k$ stands for the interfacial pressure transfer integral defined by

$$(PTI)_k = -v^{-1}\int_{A_k}\tilde{P}_k\underline{n}_k\,dA, \quad (5.5.11)$$

and (VSTI) stands for the interfacial viscous stress transfer integral defined by

$$(VSTI)_k = -v^{-1}\int_{A_k}\underline{\underline{\tilde{\tau}}}_k\cdot\underline{n}_k\,dA, \quad (5.5.12)$$

in which $\tilde{\underline{\underline{\tau}}}_k$ is given by equation (5.5.9c). While equation (5.5.10) is rigorously valid for a dispersed system for which the length scale inequality, equation (2.1.4), is satisfied, the integral

$$\int_{A_k} \left(- P_k \underline{\underline{I}} + \underline{\underline{\tau}}_k\right) \cdot \underline{n}_k \, dA$$

may be prudently evaluated from the Lagrangian equation of motion for the dispersed phase. More research is needed to assess the merit of this approach.

$$- v^{-1} \left\langle \int_{A_k} \rho_k \underline{U}_k \left(\underline{U}_k - \underline{W}_s\right) \cdot \underline{n}_k \, dA \right\rangle = {}^t\langle \Gamma_k \rangle \, {}^i\langle \underline{U}_k \rangle^*_{LF} + \left(MMTI\right)_k , \quad (5.5.13)$$

where ${}^t\langle \Gamma_k \rangle$ is given by equation (5.4.4) and $(MMTI)_k$ stands for the interfacial momentum transfer integral defined by

$$\left(MMTI\right)_k = - v^{-1} \int_{A_k} \left[\left({}^i\langle \rho_k \rangle_{LF} + \tilde{\rho}_k\right) \tilde{\underline{U}}_k + {}^t\langle \rho_k' \underline{U}_k' \rangle\right]$$

$$\left({}^i\langle \underline{U}_k \rangle^*_{LF} + \tilde{\underline{U}}_k - {}^i\langle \underline{W}_s \rangle_{LF} - \tilde{\underline{W}}_s\right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \left({}^i\langle \rho_k \rangle_{LF} + \tilde{\rho}_k\right) {}^t\langle \underline{U}_k' \left(\underline{U}_k' - \underline{W}_s'\right) \rangle \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \tilde{\underline{U}}_k \, {}^t\langle \rho_k' \left(\underline{U}_k' - \underline{W}_s'\right) \rangle \cdot \underline{n}_k \, dA . \quad (5.5.14a)$$

It is seen from equations (5.5.13) and (5.5.14a) that the time- and volume-averaged interfacial momentum transfer consists of two parts: (1) transfer associated with interfacial mass generation ${}^t\langle \Gamma_k \rangle \, {}^i\langle \underline{U}_k \rangle^*_{LF}$, and (2) extraneous transfer due to spatial deviation of density and fluid velocity at the interface, as well as that due to time correlations of fluctuating velocities and density.

As has been previously indicated, under usual circumstances $\tilde{\rho}_k / {}^i\langle \rho_k \rangle_{LF} \ll 1$. Hence, a valid approximation for $(MMTI)_k$ is

$$(MMTI)_k \cong - v^{-1} \int_{A_k} \left({}^{i}\langle \rho_k \rangle_{LF} \, \tilde{\underline{U}}_k + {}^{t}\langle \rho_k' \underline{U}_k' \rangle \right) \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \tilde{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \tilde{\underline{W}}_s \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \, {}^{i}\langle \rho_k \rangle_{LF} \int_{A_k} {}^{t}\langle \underline{U}_k' \left(\underline{U}_k' - \underline{W}_s' \right) \rangle \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \tilde{\underline{U}}_k \, {}^{t}\langle \rho_k' \left(\underline{U}_k' - \underline{W}_s' \right) \rangle \cdot \underline{n}_k \, dA \, . \qquad (5.5.14b)$$

A further approximation can be made if the correlation of density and velocity fluctuations is insignificant. It is

$$(MMTI)_k \cong -v^{-1} \, {}^{i}\langle \rho_k \rangle_{LF} \int_{A_k} \left[\tilde{\underline{U}}_k \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \tilde{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \tilde{\underline{W}}_s \right) \right.$$

$$\left. + {}^{t}\langle \underline{U}_k' \left(\underline{U}_k' - \underline{W}_s' \right) \rangle \right] \cdot \underline{n}_k \, dA \, . \qquad (5.5.14c)$$

When ρ_k = constant, $(MMTI)_k$ becomes

$$(MMTI)_k^{\circ} = -v^{-1} \, \rho_k \int_{A_k} \left[\tilde{\underline{U}}_k \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \tilde{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \tilde{\underline{W}}_s \right) + {}^{t}\langle \underline{U}_k' \left(\underline{U}_k' - \underline{W}_s' \right) \rangle \right] \cdot \underline{n}_k dA \qquad (5.5.15)$$

The time- and volume-averaged interfacial momentum source per unit volume, ${}^{t}\langle \underline{M}_k \rangle$, resulting from pressure and viscous stresses at the interface as well as that associated with interfacial mass generation, is

$${}^{t}\langle \underline{M}_k \rangle = {}^{i}\langle P_k \rangle_{LF} \nabla \alpha_k - {}^{i}\langle \underline{\underline{\Gamma}}_k \rangle_{LF} \cdot \nabla \alpha_k + {}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$$

$$+ (PTI)_k - (VSTI)_k + (MMTI)_k \, . \qquad (5.5.16)$$

It is recognized that equation (5.5.16) is the equivalent of

$${}^{t}\langle \underline{M}_k \rangle = - v^{-1} \left\langle \int_{A_k} \left(\dot{m}_k \underline{U}_k + P_k \underline{n}_k - \underline{\underline{\Gamma}}_k \cdot \underline{n}_k \right) \right. \qquad (5.5.17)$$

where the three terms of the integrand are precisely those of the first integral in equation (4.9).

Using the foregoing results, we obtain the time- and volume-averaged momentum conservation equation:

$$\frac{\partial}{\partial t}\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF} + \nabla\cdot\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}$$

$$+\frac{\partial}{\partial t}\alpha_k\,\underline{\psi}_{mk} + 2\,\nabla\cdot\alpha_k\,\underline{\psi}_{mk}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}$$

$$= -\nabla\alpha_k\,{}^{i}\langle P_k\rangle_{LF} + \nabla\cdot\alpha_k\,{}^{i}\langle\underline{\underline{\tau}}_k\rangle_{LF} + \nabla\cdot\alpha_k\left({}^{ti}\langle\underline{\underline{\tau}}_k^{T}\rangle + {}^{i}\langle\underline{\underline{\tau}}_k^{D}\rangle + {}^{ti}\langle\underline{\underline{\hat{\tau}}}_k^{T}\rangle\right)$$

$$+\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,\underline{f} + {}^{t}\langle\underline{M}_k\rangle\,, \tag{5.5.18a}$$

in which ${}^{i}\langle\underline{\underline{\tau}}_k\rangle_{LF}$ is defined by equation (5.5.9b) and ${}^{t}\langle\underline{M}_k\rangle$ is given by equation (5.5.16). The r.h.s. of equation (5.5.18a) can be recast into the following form:

$$\text{r.h.s.} = -\alpha_k\nabla\,{}^{i}\langle P_k\rangle_{LF} + \alpha_k\nabla\cdot{}^{i}\langle\underline{\underline{\tau}}_k\rangle_{LF} + \nabla\cdot\alpha_k\left({}^{ti}\langle\underline{\underline{\tau}}_k^{T}\rangle + {}^{i}\langle\underline{\underline{\tau}}_k^{D}\rangle + {}^{ti}\langle\underline{\underline{\hat{\tau}}}_k^{T}\rangle\right)$$

$$+\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,\underline{f} + {}^{t}\langle\underline{\underline{\tau}}_k\rangle\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF} + (PTI)_k - (VSTI)_k + (MMTI)_k\,. \tag{5.5.19}$$

Multiplying equation (5.4.6a) by ${}^{i}\langle\underline{U}_k\rangle^{*}_{LF}$ and introducing the result into equation (5.5.18a), followed by combining terms, one obtains an alternative form of the time- and volume-averaged momentum conservation equation:

$$\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\left(\frac{\partial\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}}{\partial t} + {}^{i}\langle\underline{U}_k\rangle^{*}_{LF}\cdot\nabla\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}\right)$$

$$+\frac{\partial}{\partial t}\alpha_k\,\underline{\psi}_{mk} + \nabla\cdot\alpha_k\,\underline{\psi}_{mk}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF} + \alpha_k\,\underline{\psi}_{mk}\cdot\nabla\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}$$

$$= -\alpha_k\nabla\,{}^{i}\langle P_k\rangle_{LF} + \alpha_k\nabla\cdot{}^{i}\langle\underline{\underline{\tau}}_k\rangle_{LF} + \nabla\cdot\alpha_k\left({}^{ti}\langle\underline{\underline{\tau}}_k^{T}\rangle + {}^{i}\langle\underline{\underline{\tau}}_k^{D}\rangle + {}^{ti}\langle\underline{\underline{\hat{\tau}}}_k^{T}\rangle\right)$$

$$+\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,\underline{f} + (PTI)_k - (VSTI)_k + (MMTI)_k\,. \tag{5.5.18b}$$

If the turbulent mass flux $\underline{\psi}_{mk}\left(= {}^{ti}\langle\rho_k\underline{U}_k'\rangle\right)$ is negligible, which is usually the case, then ${}^{ti}\langle\underline{\underline{\hat{\tau}}}_k^{T}\rangle$ is also negligible, and equation (5.5.18a) simplifies to

$$\frac{\partial}{\partial t}\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF} + \nabla\cdot\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}$$

$$= -\alpha_k\nabla\,{}^{i}\langle P_k\rangle_{LF} + \alpha_k\nabla\cdot{}^{i}\langle\underline{\underline{I}}_k\rangle_{LF} + \nabla\cdot\alpha_k\left({}^{ti}\langle\underline{\underline{I}}^{T}_k\rangle + {}^{i}\langle\underline{\underline{I}}^{D}_k\rangle\right)$$

$$+ \alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,\underline{f} + {}^{t}\langle\Gamma_k\rangle\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF} + (PTI)_k - (VSTI)_k + (MMTI)_k \ . \qquad (5.5.20a)$$

Equivalently,

$$\alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\left(\frac{\partial\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}}{\partial t} + {}^{i}\langle\underline{U}_k\rangle^{*}_{LF}\cdot\nabla\,{}^{i}\langle\underline{U}_k\rangle^{*}_{LF}\right)$$

$$= -\alpha_k\nabla\,{}^{i}\langle P_k\rangle_{LF} + \alpha_k\nabla\cdot{}^{i}\langle\underline{\underline{I}}_k\rangle_{LF} + \nabla\cdot\alpha_k\left({}^{ti}\langle\underline{\underline{I}}^{T}_k\rangle + {}^{i}\langle\underline{\underline{I}}^{D}_k\rangle\right)$$

$$+ \alpha_k\,{}^{i}\langle\rho_k\rangle_{LF}\,\underline{f} + (PTI)_k - (VSTI)_k + (MMTI)_k \ . \qquad (5.5.20b)$$

In highly turbulent flows, ${}^{i}\langle\underline{\underline{I}}_k\rangle_{LF} \ll {}^{ti}\langle\underline{\underline{I}}^{T}_k\rangle$, and hence both $\alpha_k\nabla\cdot{}^{i}\langle\underline{\underline{I}}_k\rangle_{LF}$ and $(VSTI)_k$ in equations (5.5.20a,b) can be dropped.

For a single-phase system, $\alpha_k = 1$ and all interfacial integrals vanish. Furthermore, the inequality condition, equation (2.1.4), should be dropped and one can choose the averaging volume v as small as one pleases. Consequently, ${}^{i}\langle\rho_k\rangle \to \rho_k$, ${}^{i}\langle\underline{U}_k\rangle^{*}_{LF} \to \underline{U}_k$, ${}^{i}\langle\underline{\underline{I}}_k\rangle_{LF} \to \underline{\underline{I}}_k$ and all spatial deviations vanish. Also, ${}^{t}\langle\Gamma_k\rangle$, $(PTI)_k$, $(VSTI)_k$, and $(MMTI)_k$ do not exist. We thus recover the familiar form of the linear momentum conservation equation for <u>single phase</u> turbulent flow from either equation (5.5.20a) or (5.5.20b).

The time- and volume-averaged momentum equation for the special case of constant ρ_k can be readily obtained. It is of the same form as equation (5.5.20b) with ${}^{i}\langle\rho_k\rangle_{LF}$ replaced by ρ_k, ${}^{i}\langle\underline{U}_k\rangle^{*}_{LF}$ replaced by ${}^{i}\langle\underline{U}_k\rangle_{LF}$ and $(MMTI)_k$ replaced by $(MMTI)^{\circ}_k$, which is given by equation (5.5.15).

The local volume-averaged linear momentum balance equation for the interface between fluid phases k and f is given by equation (4.9). Upon time averaging, it becomes

$$\sum_{k,f}{}^{t}\langle\underline{M}_k\rangle = v^{-1}\int_{A_k}{}^{t}\langle\underline{F}_s\rangle\,dA = v^{-1}\int_{A_k}\left(\nabla_s\sigma_{LF} - 2H_{kLF}\sigma_{LF}\underline{n}_k\right)dA \ , \qquad (5.5.21a)$$

for which the relation given by equation (5.5.17) has been used. The last integral on the r.h.s. of equation (5.5.21a) can be written as

$$v^{-1} \int_{A_f} \left(\nabla_s \sigma_{LF} - 2 H_{fLF} \sigma_{LF} \underline{n}_f \right) dA ,$$

since $H_{kLF} = - H_{fLF}$, $\underline{n}_k = - \underline{n}_f$, and $A_k = A_f$. An equivalent form of equation (5.5.21a) is

$$\sum_{k,f} \left[{}^{i}\langle P_k \rangle_{LF} \nabla \alpha_k - {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} \cdot \nabla \alpha_k + {}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right.$$

$$\left. + (PTI)_k - (VSTI)_k + (MMTI)_k \right] = v^{-1} \int_{A_k} {}^{t}\langle \underline{F}_s \rangle \, dA. \qquad (5.5.21b)$$

That is, the group of terms in the square bracket of equation (5.5.21b), which appears in the time-volume averaged equation for phase k, and the corresponding group of terms in the time-volume averaged equation for phase f must add up to a value equal to $v^{-1} \int_{A_k} {}^{t}\langle \underline{F}_s \rangle \, dA$. The evaluation or assessment of these interfacial integrals constitutes another part of the closure problem.

We consider a gas-liquid system at rest with uniform σ. Equation (5.5.21b) simplifies to

$$\sum_{k,f} \left[{}^{i}\langle P_k \rangle \nabla \alpha_k + (PTI)_k \right] = - v^{-1} \int_{A_k} 2 H_k \sigma \underline{n}_k \, dA . \qquad (5.5.22)$$

The l.h.s. of equation (5.5.22) can be replaced by

$$- \sum_{k,f} \left(v^{-1} \int_{A_k} P_k \underline{n}_k \, dA \right) ,$$

since $P_k = {}^{i}\langle P_k \rangle + \hat{P}_k$ and $\nabla \alpha_k = - v^{-1} \int_{A_k} \underline{n}_k \, dA$. In view of the fact that $\underline{n}_f = - \underline{n}_k$, and $A_f = A_k$, equation (5.5.22) can be written as

$$\int_{A_k} \left(P_k - P_f - 2 H_k \sigma \right) \underline{n}_k \, dA = 0 , \qquad (5.5.23a)$$

which is valid for any A_k. Accordingly,

$$P_k - P_f = 2 H_k \sigma . \qquad (5.5.23b)$$

If phase k denotes gas bubbles in a liquid or liquid droplets in a gas, H_k is positive and equals $\frac{1}{R}$, R being the local mean radius of curvature of the interface. Hence, the pressure difference across the interface is

$$P_k - P_f = \frac{2\sigma}{R} , \tag{5.5.24}$$

which is well known. Needless to say, P_k and P_f refer to the static pressure in the bulk fluid as the interface is approached.

5.6 Time- and Volume-Averaged Total Energy Conservation Equation and Interfacial Total Energy Balance Equation

The local volume-averaged total energy conservation equation is given by equation (4.5). Subsequent time-averaging requires the use of the following results:

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{i}\langle \rho_k E_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF} + \alpha_k \, \Phi_{Ek} , \tag{5.6.1}$$

where ${}^{i}\langle E_k \rangle^{*}_{LF}$ is given by equation (5.3.28) and Φ_{Ek} is a scalar total energy function defined by

$$\Phi_{Ek} = {}^{ti}\langle \rho_k' E_k' \rangle \tag{5.6.2}$$

with E_k' given by equation (5.3.30).

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{i}\langle \rho_k \underline{U}_k E_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF} + \alpha_k \, \underline{\psi}_{mk} \, {}^{i}\langle E_k \rangle^{*}_{LF}$$

$$+ \alpha_k \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \Phi_{Ek} + \alpha_k \left({}^{ti}\langle \underline{E}_k^{T} \rangle + {}^{i}\langle \underline{\hat{E}}_k^{D} \rangle + {}^{ti}\langle \underline{\hat{E}}_k^{T} \rangle \right) , \tag{5.6.3}$$

in which $\underline{\psi}_{mk}$ is given by equation (5.4.3). The last three terms are, respectively,

(a) the time- and volume-averaged turbulent total energy flux ${}^{ti}\langle \underline{E}_k^{T} \rangle$ defined by

$${}^{ti}\langle \underline{E}_k^{T} \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}_k' E_k' \rangle + {}^{ti}\langle \hat{\rho}_k \underline{U}_k' E_k' \rangle + {}^{ti}\langle \rho_k' \underline{U}_k' E_k' \rangle \tag{5.6.4a}$$

$$= {}^{t}\left\langle {}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}'_k E'_k{}^{*} \rangle \right\rangle . \qquad (5.6.4b)$$

Referring to equation (5.3.30), we may write

$$E'_k = u'_k + K'_k , \qquad (5.6.5)$$

where K'_k denotes the high-frequency fluctuating component of the kinetic energy, defined by

$$K'_k = {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \underline{U}'_k + \tilde{\underline{U}}_k \cdot \underline{U}'_k + \frac{1}{2}\left(\underline{U}'_k \cdot \underline{U}'_k - {}^{ti}\langle \underline{U}'_k \cdot \underline{U}'_k \rangle \right) . \qquad (5.6.6)$$

Accordingly,

$${}^{ti}\langle \underline{E}^{T}_k \rangle = {}^{ti}\langle \underline{u}^{T}_k \rangle + {}^{ti}\langle \underline{K}^{T}_k \rangle \qquad (5.6.7)$$

in which the time- and volume-averaged turbulent internal energy flux ${}^{ti}\langle \underline{u}^{T}_k \rangle$ is defined by

$${}^{ti}\langle \underline{u}^{T}_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \; {}^{ti}\langle \underline{U}'_k u'_k \rangle + {}^{ti}\langle \hat{\rho}_k \underline{U}'_k u'_k \rangle + {}^{ti}\langle \rho'_k \underline{U}'_k u'_k \rangle \qquad (5.6.8a)$$

$$= {}^{t}\left\langle {}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}'_k u'_k \rangle^{*} \right\rangle , \qquad (5.6.8b)$$

and the time- and volume-averaged turbulent kinetic energy flux ${}^{ti}\langle \underline{K}^{T}_k \rangle$ is defined by

$${}^{ti}\langle \underline{K}^{T}_k \rangle = {}^{i}\langle \rho_k \rangle_{LF} \; {}^{ti}\langle \underline{U}'_k \, K'_k \rangle + {}^{ti}\langle \hat{\rho}_k \, \underline{U}'_k \, K'_k \rangle$$

$$+ {}^{ti}\langle \rho'_k \, \underline{U}'_k \, K'_k \rangle \qquad (5.6.9a)$$

$$= {}^{t}\left\langle {}^{i}\langle \rho_k \rangle \; {}^{i}\langle \underline{U}'_k \, K'_k \rangle^{*} \right\rangle . \qquad (5.6.9b)$$

The turbulent internal energy flux ${}^{ti}\langle \underline{u}^{T}_k \rangle$ can be conveniently expressed in terms of eddy diffusivity for internal energy transfer D^{T}_{uk} according to

$$ {}^{ti}\langle \underline{u}_k^T \rangle = - {}^{i}\langle \rho_k \rangle_{LF} \, D_{uk}^T \, \nabla \, {}^{i}\langle u_k \rangle_{LF}^{*} \, . \tag{5.6.10a} $$

Also, one may define a volume-averaged specific heat at constant volume c_{vk} such that

$$ \nabla \, {}^{i}\langle u_k \rangle_{LF}^{*} = c_{vk} \, \nabla \, {}^{i}\langle T_k \rangle_{LF} \, . \tag{5.6.11} $$

It follows, then, that

$$ {}^{ti}\langle \underline{u}_k^T \rangle = - \kappa_k^T \, \nabla \, {}^{i}\langle T_\kappa \, {}_{LF} \, , \tag{5.6.10b} $$

where the turbulent conductivity κ_k^T is related to D_{uk}^T according to

$$ \kappa_k^T = {}^{i}\langle \rho_k \rangle_{LF} \, c_{vk} \, D_{uk}^T \, . \tag{5.6.12} $$

(b) the volume-averaged dispersive total energy flux ${}^{i}\langle \underline{\hat{E}}_k^D \rangle$ defined by

$$ {}^{i}\langle \underline{\hat{E}}_k^D \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{\hat{U}}_k \hat{E}_k \rangle + {}^{i}\langle \hat{\rho}_k \underline{\hat{U}}_k \hat{E}_k \rangle \tag{5.6.13a} $$

$$ = {}^{i}\langle \underline{\hat{u}}_k^D \rangle + {}^{i}\langle \underline{\hat{K}}_k^D \rangle \, , \tag{5.6.13b} $$

where ${}^{i}\langle \underline{\hat{u}}_k^D \rangle$ is the volume-averaged dispersive internal energy flux given by

$$ {}^{i}\langle \underline{\hat{u}}_k^D \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{\hat{U}}_k \hat{u}_k \rangle + {}^{i}\langle \hat{\rho}_k \underline{\hat{U}}_k \hat{u}_k \rangle \, , \tag{5.6.14a} $$

and ${}^{i}\langle \underline{\hat{K}}_k^D \rangle$ is the volume-averaged dispersive kinetic energy flux given by

$$ {}^{i}\langle \underline{\hat{K}}_k^D \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{\hat{U}}_k \hat{K}_k \rangle + {}^{i}\langle \hat{\rho}_k \underline{\hat{U}}_k \hat{K}_k \rangle \, , \tag{5.6.15a} $$

in which $\hat{K}_k = {}^{i}\langle \underline{U}_k \rangle_{LF}^{*} \cdot \underline{\hat{U}}_k + \frac{1}{2} \left(\underline{\hat{U}}_k \cdot \underline{\hat{U}}_k - {}^{i}\langle \underline{\hat{U}}_k \cdot \underline{\hat{U}}_k \rangle^{*} \right)$, (5.6.16)

as is obvious from equation (5.3.29).

(c) the time- and volume-averaged turbulent, dispersive total energy flux ${}^{ti}\langle \underline{\hat{E}}_k^T \rangle$ defined by

$$ {}^{ti}\langle \underline{\hat{E}}_k^T \rangle = {}^{ti}\langle \rho_k' \underline{U}_k' \hat{E}_k + \rho_k' \underline{\hat{U}}_k E_k' \rangle \tag{5.6.17a} $$

$$ = {}^{ti}\langle \underline{\hat{u}}_k^T \rangle + {}^{ti}\langle \underline{\hat{K}}_k^T \rangle \tag{5.6.17b} $$

where ${}^{ti}\langle \underline{\hat{U}}_k^T \rangle$ is the time- and volume-averaged turbulent, dispersive internal energy flux given by

$$ {}^{ti}\langle \underline{\hat{u}}_k^T \rangle = {}^{ti}\langle \rho_k' \underline{U}_k' \hat{u}_k + \rho_k' \underline{\hat{U}}_k u_k' \rangle \tag{5.6.18a} $$

and ${}^{ti}\langle \underline{\hat{K}}_k^T \rangle$ is the time- and volume-averaged turbulent, dispersive kinetic energy flux defined by

$$ {}^{ti}\langle \underline{\hat{K}}_k^T \rangle = {}^{ti}\langle \rho_k' \underline{U}_k' \hat{K}_k + \rho_k' \underline{\hat{U}}_k K_k' \rangle \, . \tag{5.6.19a} $$

Under the usual circumstance, $\hat{\rho}_k / {}^{i}\langle \rho_k \rangle_{LF} \ll 1$, the triple correlations involving ρ_k' may be neglected, the spatial deviations $\hat{u}_k$ and $\hat{K}_k$ have little correlation with $\rho_k' \underline{U}_k'$, and $\underline{\hat{U}}_k$ has little correlation with $\rho_k' u_k'$ and $\rho_k' K_k'$. Accordingly, equations (5.6.8a), (5.6.14a), and (5.6.18a) can be approximated by

(a') $$ {}^{ti}\langle \underline{u}_k^T \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}_k' u_k' \rangle \, , \tag{5.6.8c} $$

(b') $$ {}^{i}\langle \underline{u}_k^D \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{\hat{U}}_k \hat{u}_k \rangle \, , \tag{5.6.14b} $$

(c') $$ {}^{ti}\langle \underline{\hat{u}}_k^T \rangle = 0 \, , \tag{5.6.18b} $$

and equations (5.6.9a), (5.6.15a), and (5.6.19a) reduce to

(a") $$ {}^{ti}\langle \underline{K}_k^T \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}_k' K_k' \rangle \, , \tag{5.6.9c} $$

$$(b'') \quad {}^{i}\langle \underline{\hat{R}}^{D} \rangle = {}^{i}\langle \rho_k \rangle_{LF} \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot {}^{i}\langle \underline{\hat{U}}_k \underline{\hat{U}}_k \rangle \; , \tag{5.6.15b}$$

$$(c'') \quad {}^{ti}\langle \underline{\hat{R}}^{T}_{k} \rangle = 0 \; . \tag{5.6.19b}$$

$$\bullet \quad {}^{t}\left\langle \nabla \cdot \alpha_k \; {}^{i}\langle P_k \underline{U}_k \rangle \right\rangle = \nabla \cdot \alpha_k \; {}^{i}\langle P_k \rangle_{LF} \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \; \underline{\Psi}_{Pk} \tag{5.6.20}$$

where $\underline{\Psi}_{Pk}$ is a vector pressure work function defined by

$$\underline{\Psi}_{Pk} = {}^{i}\langle P_k \rangle_{LF} \; {}^{i}\langle \underline{\hat{U}}_k \rangle + {}^{i}\langle \hat{P}_k \underline{\hat{U}}_k \rangle + {}^{ti}\langle P'_k \underline{U}'_k \rangle \; . \tag{5.6.21}$$

$$\bullet \quad {}^{t}\left\langle \nabla \cdot \alpha_k \; {}^{i}\langle \underline{\underline{\tau}}_k \cdot \underline{U}_k \rangle \right\rangle = \nabla \cdot \alpha_k \; {}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \; \underline{\Psi}_{\tau k} \tag{5.6.22}$$

where $\underline{\Psi}_{\tau k}$ is a vector viscous stress work function defined by

$$\underline{\Psi}_{\tau k} = {}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{\hat{U}}_k \rangle + {}^{i}\langle \hat{\underline{\underline{\tau}}}_k \cdot \underline{\hat{U}}_k \rangle + {}^{ti}\langle \underline{\underline{\tau}}'_k \cdot \underline{U}'_k \rangle \tag{5.6.23}$$

where $\hat{\underline{\underline{\tau}}}_k$ is defined by equation (5.5.9c) and $\underline{\underline{\tau}}'_k$ is defined by equation (5.5.9d).

$$\bullet \quad {}^{t}\left\langle \alpha_k \; {}^{i}\langle J_{Ek} \rangle \right\rangle = \alpha_k \; {}^{i}\langle J_{Ek} \rangle_{LF} \tag{5.6.24}$$

$$\bullet \quad {}^{t}\left\langle \nabla \cdot \alpha_k \; {}^{i}\langle \underline{J}_{qk} \rangle \right\rangle = \nabla \cdot \alpha_k \; {}^{i}\langle \underline{J}_{qk} \rangle_{LF} \tag{5.6.25}$$

$$\bullet \quad {}^{t}\left\langle \alpha_k \; {}^{i}\langle \rho_k \underline{U}_k \rangle \right\rangle \cdot \underline{f} = \alpha_k \; {}^{i}\langle \rho_k \rangle_{LF} \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \underline{f} + \alpha_k \; \underline{\Psi}_{mk} \cdot \underline{f} \tag{5.6.26}$$

$$\bullet \quad {}^{t}\langle \dot{Q}_k \rangle = - \; v^{-1} \left\langle \int_{A_k} \underline{J}_{qk} \cdot \underline{n}_k \; dA \right\rangle$$

$$= {}^{i}\langle \underline{J}_{qk} \rangle_{LF} \cdot \nabla \alpha_k + (HTI)_k \tag{5.6.27}$$

where $(HTI)_k$ stands for the heat transfer integral defined by

$$\left(\mathrm{HTI}\right)_k = - v^{-1} \int_{A_k} \tilde{\underline{J}}_{qk} \cdot \underline{n}_k \, dA \, . \tag{5.6.28}$$

It may be noted that ${}^{i}\langle \underline{J}_{qk} \rangle_{LF} \cdot \nabla \alpha_k$ in equation (5.6.27) may be combined with $- \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF}$ of equation (5.6.25) to give $- \alpha_k \nabla \cdot {}^{i}\langle \underline{J}_{qk} \rangle_{LF}$.

$$\bullet \quad v^{-1} \left\langle \int_A \left(- P_k \underline{U}_k + \underline{\underline{\tau}}_k \cdot \underline{U}_k \right) \cdot \underline{n}_k \, dA \right\rangle^{t}$$

$$= {}^{i}\langle P_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \nabla \alpha_k - \left({}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) \cdot \nabla \alpha_k$$

$$+ \left(\mathrm{PWI}\right)_k - \left(\mathrm{VWI}\right)_k \, , \tag{5.6.29}$$

in which the interfacial pressure work integral is

$$\left(\mathrm{PWI}\right)_k = - v^{-1} \, {}^{i}\langle P_k \rangle_{LF} \int_{A_k} \tilde{\underline{U}}_k \cdot \underline{n}_k \, dA$$

$$- v^{-1} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \int_{A_k} \tilde{P}_k \, \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \tilde{P}_k \tilde{\underline{U}}_k + {}^{t}\langle P'_k \underline{U}'_k \rangle \cdot \underline{n}_k \, dA \tag{5.6.30}$$

and the interfacial viscous stress work integral is

$$\left(\mathrm{VWI}\right)_k = - v^{-1} \int_{A_k} \left({}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot \tilde{\underline{U}}_k \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \left(\tilde{\underline{\underline{\tau}}}_k \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \left(\tilde{\underline{\underline{\tau}}}_k \cdot \tilde{\underline{U}}_k + {}^{t}\langle \underline{\underline{\tau}}'_k \cdot \underline{U}'_k \rangle \right) \cdot \underline{n}_k \, dA \, . \tag{5.6.31}$$

Finally,

$$- v^{-1} \left\langle \int_{A_k} \rho_k E_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \right\rangle = {}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle E_k \rangle^{*}_{LF} + (TETI)_k \qquad (5.6.32)$$

where ${}^{i}\langle E_k \rangle^{*}_{LF}$ is defined in equation (5.3.28) and the interfacial total energy transfer integral is

$$(TETI)_k = - v^{-1} \int_{A_k} \left[\left({}^{i}\langle \rho_k \rangle_{LF} + \hat{\rho}_k \right) \hat{E}_k + {}^{t}\langle \rho_k' E_k' \rangle \right]$$

$$\left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \hat{\underline{W}}_s \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \left({}^{i}\langle \rho_k \rangle_{LF} + \hat{\rho}_k \right) {}^{t}\langle E_k' (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \hat{E}_k \, {}^{t}\langle \rho_k' (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA \, . \qquad (5.6.33a)$$

The derivation of equation (5.6.32) parallels that of equation (5.3.13), although they differ in details. The time- and volume-averaged interfacial total energy transfer is seen to consist of (a) a transfer that is directly related to interfacial mass generation ${}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle E_k \rangle^{*}_{LF}$ and (b) extraneous transfer to account for various spatial deviations and time correlations of density, energy, and velocity fluctuations. When $\hat{\rho}_k / {}^{i}\langle \rho_k \rangle_{LF} \ll 1$, as is usually the case, equation (5.6.33a) reduces to

$$(TETI)_k \cong - v^{-1} \int_{A_k} \left({}^{i}\langle \rho_k \rangle_{LF} \hat{E}_k + {}^{t}\langle \rho_k' E_k' \rangle \right) \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \hat{\underline{W}}_s \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \, {}^{i}\langle \rho_k \rangle_{LF} \int_{A_k} {}^{t}\langle E_k' \, (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \hat{E}_k \, {}^{t}\langle \rho_k' \, (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA \, . \qquad (5.6.33b)$$

If, furthermore, all time correlations with ρ_k' are small, then

$$(TETI)_k \cong - v^{-1} \, {}^{i}\langle \rho_k \rangle_{LF} \int_{A_k} \left[\hat{E}_k \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \hat{\underline{W}}_s \right) \right.$$

$$+ {}^{t}\langle E_k'\left(\underline{U}_k' - \underline{W}_s'\right)\rangle\Big] \cdot \underline{n}_k \, dA \, . \qquad (5.6.33c)$$

An examination of the foregoing results shows that the time- and volume-averaged total energy source ${}^{t}\langle \xi_k \rangle$ consists of interfacial heat transfer ${}^{t}\langle Q_k \rangle$, work done by pressure and viscous forces at the interface and transport associated with interfacial mass generation $\dot{m}_k$. Thus,

$$ {}^{t}\langle \xi_k \rangle = {}^{t}\langle Q_k \rangle + {}^{i}\langle P_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \nabla \alpha_k - \left({}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right) \cdot \nabla \alpha_k $$

$$ + {}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle E_k \rangle^{*}_{LF} + \left(PWI\right)_k - \left(VWI\right)_k + \left(TETI\right)_k \, . \qquad (5.6.34)$$

It is recognized that

$$ {}^{t}\langle \xi_k \rangle = - V^{-1} \int_{A_k} {}^{t}\langle \dot{m}_k E_k + \underline{J}_{qk} \cdot \underline{n}_k + \left(P_k \underline{U}_k - \underline{\underline{I}}_k \cdot \underline{U}_k \right) \cdot \underline{n}_k \rangle \, dA \qquad (5.6.35)$$

wherein all terms of the integrand are precisely those in the first integral of equation (4.10). Hence

$$ \sum_{k,f} {}^{t}\langle \xi_k \rangle = V^{-1} \int_{A_k} {}^{t}\langle \underline{F}_s \cdot \underline{W}_s \rangle \, dA \, . \qquad (5.6.36)$$

Using the foregoing results, we obtain the desired time- and volume-averaged total energy equation.

$$ \frac{\partial}{\partial t} \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF} $$

$$ + \frac{\partial}{\partial t} \alpha_k \, \Phi_{Ek} + \nabla \cdot \alpha_k \, \Phi_{Ek} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \, \underline{\Psi}_{mk} \, {}^{i}\langle E_k \rangle^{*}_{LF} $$

$$ = - \nabla \cdot \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} - \nabla \cdot \alpha_k \, \underline{\Psi}_{pk} + \nabla \cdot \alpha_k \, {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} $$

$$ + \nabla \cdot \alpha_k \, \underline{\Psi}_{Tk} - \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{\underline{T}}_k \rangle + {}^{i}\langle \underline{\underline{D}}_k \rangle + {}^{ti}\langle \underline{\underline{T}}_k \rangle \right) $$

$$ - \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF} + \alpha_k \, {}^{i}\langle I_{Ek} \rangle_{LF} + \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \underline{f} $$

$$+ \alpha_k \underline{\psi}_{mk} \cdot \underline{f} + {}^{t}\langle \xi_k \rangle \cdot \qquad (5.6.37a)$$

An alternative form of equation (5.6.37a) can be obtaind by multiplying equation (5.4.6a) by ${}^{i}\langle E_k \rangle^{*}_{LF}$ and introducing the result into equation (5.6.37a), followed by combining terms. The result is

$$\alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \left(\frac{\partial \, {}^{i}\langle E_k \rangle^{*}_{LF}}{\partial t} + {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \nabla \, {}^{i}\langle E_k \rangle^{*}_{LF} \right)$$

$$+ \frac{\partial}{\partial t} \alpha_k \Phi_{Ek} + \nabla \cdot \alpha_k \Phi_{Ek} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \alpha_k \underline{\psi}_{mk} \cdot \nabla \, {}^{i}\langle E_k \rangle^{*}_{LF}$$

$$= - \alpha_k \nabla \cdot {}^{i}\langle P_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} - \nabla \cdot \alpha_k \underline{\psi}_{Pk} + \alpha_k \nabla \cdot {}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$$

$$+ \nabla \cdot \alpha_k \underline{\psi}_{\tau k} - \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{E}^{T}_{k} \rangle + {}^{i}\langle \underline{\tilde{E}}^{D}_{k} \rangle + {}^{ti}\langle \underline{\tilde{E}}^{T}_{k} \rangle \right)$$

$$- \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF} + \alpha_k \, {}^{i}\langle J_{Ek} \rangle_{LF} + \alpha_k \, {}^{i}\langle \rho_k \rangle^{*}_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \underline{f} + \alpha_k \underline{\psi}_{mk} \cdot \underline{f}$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + (PWI)_k - (VWI)_k + (TETI)_k \, . \qquad (5.6.37b)$$

The turbulent mass flux $\underline{\psi}_{mk} = {}^{ti}\langle \rho_k' \, \underline{U}_k' \rangle$, the scalar total energy function $\Phi_{Ek} \left[= {}^{ti}\langle \rho_k' \, E_k' \rangle = {}^{ti}\langle \rho_k' u_k' \rangle + {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \underline{\tilde{U}}_k \cdot \underline{\psi}_{mk} \right]$, the vector pressure work function $\underline{\psi}_{Pk}$, and the vector viscous stress work function $\underline{\psi}_{\tau k}$ are usually small and can be ignored. Furthermore, the stated approximations following equation (5.6.19a) often hold. Under these conditions, equations (5.6.37a) and (5.6.37b) simplify, respectively, to

$$\frac{\partial}{\partial t} \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle E_k \rangle^{*}_{LF}$$

$$= - \nabla \cdot \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \, {}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$$

$$- \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{E}^{T}_{k} \rangle + {}^{i}\langle \underline{\tilde{E}}^{D}_{k} \rangle \, {}^{ti}\langle \underline{\tilde{E}}^{T}_{k} \rangle \right)$$

$$- \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF} + \alpha_k \, {}^{i}\langle J_{Ek} \rangle_{LF} + \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \underline{f} + {}^{t}\langle \xi_k \rangle, \qquad (5.6.38a)$$

and

$$\alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \left(\frac{\partial \, {}^{i}\langle E_k \rangle^{*}_{LF}}{\partial t} + {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \nabla \, {}^{i}\langle E_k \rangle^{*}_{LF} \right)$$

$$= - \alpha_k \nabla \cdot {}^{i}\langle P_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \alpha_k \nabla \cdot {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$$

$$- \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{E}^{T}_k \rangle + {}^{i}\langle \underline{\hat{E}}^{D}_k \rangle + {}^{ti}\langle \underline{\hat{E}}^{T}_k \rangle \right)$$

$$- \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle + \alpha_k \, {}^{i}\langle J_{Ek} \rangle_{LF} + \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \underline{f}$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + (PWI)_k - (VWI)_k + (TETI)_k \, . \qquad (5.6.38b)$$

In these equations, the following approximations should be used:

$${}^{ti}\langle \underline{E}^{T}_k \rangle \simeq {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}'_k u'_k \rangle + {}^{i}\langle \rho_k \rangle_{LF} \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k \right) \cdot {}^{ti}\langle \underline{U}'_k U'_k \rangle \, , \qquad (5.6.38c)$$

$${}^{i}\langle \underline{\hat{E}}^{D}_k \rangle \simeq {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \hat{\underline{U}}_k \hat{u}_k \rangle + {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot {}^{i}\langle \hat{\underline{U}}_k \hat{\underline{U}}_k \rangle \, , \qquad (5.6.38d)$$

$${}^{ti}\langle \underline{\hat{E}}^{T}_k \rangle \simeq 0 \, , \qquad (5.6.38e)$$

and $(TETI)_k$ is given by equation (5.6.33c).

The time- and volume-averaged interfacial balance equation for total energy is given by equation (5.6.36).

5.7 Time- and Volume-Averaged Internal Energy Conservation Equation and Interfacial Internal Energy Balance Equation

The local volume-averaged internal energy conservation equation is given by equation (4.7). Subsequent time averaging can be made by following the procedure used in Sections 5.4, 5.5, and 5.6, and the following results are readily obtained. However, the presence of (1) $\langle P_k \nabla \cdot \underline{U}_k \rangle$ and (2) $\langle \phi_k \rangle = \langle \underline{\underline{I}}_k : \nabla \underline{U}_k \rangle$ in equation (4.7) renders the analysis incomplete. This will be made clear later.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{i}\langle \rho_k u_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF} + \alpha_k \, \phi_{uk} \qquad (5.7.1)$$

where ϕ_{uk} is a scalar internal energy function defined by

$$\Phi_{uk} = {}^{ti}\langle \rho_k' u_k' \rangle \ . \tag{5.7.2}$$

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{i}\langle \rho_k \underline{U}_k u_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF} + \alpha_k \, \underline{\psi}_{mk} \, {}^{i}\langle u_k \rangle^{*}_{LF}$$

$$+ \alpha_k \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, \Phi_{uk} + \alpha_k \left({}^{ti}\langle \underline{u}_k^{T} \rangle + {}^{i}\langle \underline{\tilde{u}}_k^{D} \rangle + {}^{ti}\langle \underline{\tilde{u}}_k^{T} \rangle \right) \tag{5.7.3}$$

where $\underline{\psi}_{mk}$ is defined by equation (5.4.3). The last three terms are, respectively,

(a) the time- and volume-averaged turbulent internal energy flux ${}^{ti}\langle \underline{u}_k^{T} \rangle$ defined by

$${}^{ti}\langle \underline{u}_k^{T} \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}_k' u_k' \rangle + {}^{ti}\langle \tilde{\rho}_k \underline{U}_k' u_k' \rangle + {}^{ti}\langle \rho_k' \underline{U}_k' u_k' \rangle \tag{5.7.4a}$$

$$= \left\langle {}^{t}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}_k' u_k' \rangle^{*} \right\rangle , \tag{5.7.4b}$$

which have been given previously as equations (5.6.8a) and (5.6.8b);

(b) the volume-averaged dispersive internal energy flux ${}^{i}\langle \underline{\tilde{u}}_k^{D} \rangle$ defined by

$${}^{i}\langle \underline{\tilde{u}}_k^{D} \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{\tilde{U}}_k \tilde{u}_k \rangle + {}^{i}\langle \tilde{\rho}_k \underline{\tilde{U}}_k \tilde{u}_k \rangle \ , \tag{5.7.5a}$$

which is equation (5.6.14a); and

(c) the time- and volume-averaged turbulent, dispersive internal energy flux ${}^{ti}\langle \underline{\tilde{u}}_k^{T} \rangle$ defined by

$${}^{ti}\langle \underline{\tilde{u}}_k^{T} \rangle = {}^{ti}\langle \rho_k' \underline{U}_k' \tilde{u}_k + \rho_k' u_k' \underline{\tilde{U}}_k \rangle \ , \tag{5.7.6a}$$

which is equation (5.6.18a).

Again, under the usual circumstances, $\tilde{\rho}_k / {}^{i}\langle \rho_k \rangle_{LF} \ll 1$, the triple correlations involving ρ_k' may be neglected, the spatial deviation $\tilde{u}_k$ has little correlation with $\rho_k' \underline{U}_k'$, and $\underline{\tilde{U}}_k$ has little correlation with $\rho_k' u_k'$. Accordingly, equations (5.7.4a), (5.7.5a), and (5.7.6a) reduce, respectively, to

$$(a') \quad {}^{ti}\langle \underline{u}_k^T \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{ti}\langle \underline{U}_k' \underline{u}_k' \rangle \, , \tag{5.7.4c}$$

$$(b') \quad {}^{i}\langle \underline{\tilde{u}}_k^D \rangle = {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \tilde{\underline{U}}_k \tilde{\underline{u}}_k \rangle \, , \tag{5.7.5b}$$

$$(c') \quad {}^{ti}\langle \underline{\tilde{u}}_k^T \rangle \simeq 0 \, . \tag{5.7.6b}$$

These approximate relations have also been given previously. In addition, $\underline{\psi}_{mk}$ and Φ_{uk} in equation (5.7.3) can ordinarily be ignored.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{i}\langle P_k \nabla \cdot \underline{U}_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle_{LF}^{*} + \alpha_k \, \Phi_{pk} - (PWI)_k^{(u)} \tag{5.7.7}^{\dagger}$$

where Φ_{pk} is a scalar pressure work function defined by

$$\Phi_{pk} = {}^{i}\langle \tilde{P}_k \nabla \cdot \tilde{\underline{U}}_k \rangle + {}^{ti}\langle P_k' \nabla \cdot \underline{U}_k' \rangle \tag{5.7.8}$$

and $(PWI)_k^{(u)}$, the interfacial reversible pressure work integral associated with internal energy, is defined by

$$(PWI)_k^{(u)} = - v^{-1} \, {}^{i}\langle P_k \rangle_{LF} \int_{A_k} \tilde{\underline{U}}_k \cdot \underline{n}_k \, dA \, , \tag{5.7.9}$$

which is the first term on the r.h.s. of equation (5.6.30). The appearance of Φ_{pk} (which is ${}^{i}\langle \tilde{P}_k \nabla \cdot \tilde{\underline{U}}_k \rangle$ plus the time average of ${}^{i}\langle P_k' \nabla \cdot \underline{U}_k' \rangle$) in equation (5.7.8) suggests that $(PWI)_k^{(u)}$ does not fully account for the reversible work done on the fluid phase k at all interfaces within the averaging volume v_k.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{i}\langle \phi_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} : \nabla \, {}^{i}\langle \underline{U}_k \rangle_{LF}^{*} + \alpha_k \, \Phi_{\tau k} + (VDI)_k \tag{5.7.10}^{\dagger\dagger}$$

where ${}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF}$ is defined by equation (5.5.9b) and $\Phi_{\tau k}$ is a scalar viscous dissipation function defined by

$$\Phi_{\tau k} = {}^{i}\langle \tilde{\underline{\underline{\tau}}}_k : \nabla \tilde{\underline{U}}_k \rangle + {}^{ti}\langle \underline{\underline{\tau}}_k' : \nabla \underline{U}_k' \rangle \, , \tag{5.7.11}$$

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†A derivation of equation (5.7.7) is presented in Appendix A.

††Appendix B presents some details of the derivation of equation (5.7.10).

in which $\hat{\underline{\tau}}_k$ and $\underline{\tau}'_k$ are given by equations (5.5 9c) and (5.5.9d), respectively. The interfacial viscous dissipation integral $(VDI)_k$ is defined by

$$(VDI)_k = v^{-1}\, {}^{i}\langle \underline{\tau}_k \rangle_{LF} : \int_{A_k} \hat{\underline{U}}_k \underline{n}_k \, dA \quad . \tag{5.7.12}$$

For the reason analogous to that for $(PWI)_k^{(u)}$, the integral $(VDI)_k$ defined in equation (5.7.12) does not fully account for the irreversible conversion to thermal energy of the mechanical work done by the viscous stresses at the interfaces. These difficulties prevent us from formulating an expression for the interfacial internal energy source that is in complete agreement with all the terms in the square bracket of equation (4.11). Research is in progress to resolve this difficulty.

Time-averaging of $\nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle$ formally leads to $\nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF}$. A useful constitutive relation between ${}^{i}\langle \underline{J}_{qk} \rangle_{LF}$ and ${}^{i}\langle \underline{u}_k \rangle^{*}_{LF}$ for an isotropic medium is

$${}^{i}\langle \underline{J}_{qk} \rangle_{LF} = - \left(k_k / c_{vk} \right) \nabla \, {}^{i}\langle u_k \rangle^{*}_{LF} \quad , \tag{5.7.13}$$

where k_k is the volume-averaged thermal conductivity of the fluid. The interfacial heat transfer rate per unit volume of the mixture $\hat{Q}_k$ and particularly its time-average ${}^{t}\langle \hat{Q}_k \rangle$ can best be correlated by experimental data. However, the relation given by equation (5.6.27) may be useful. It is

$${}^{t}\langle \hat{Q}_k \rangle = {}^{i}\langle \underline{J}_{qk} \rangle_{LF} \cdot \nabla \alpha_k + (HTI)_k \quad , \tag{5.7.14}$$

where $(HTI)_k$ stands for the heat transfer integral defined by

$$(HTI)_k = - v^{-1} \int_{A_k} \tilde{\underline{J}}_{qk} \cdot \underline{n}_k \, dA \quad . \tag{5.7.15}$$

Finally, the time-average of the last term in equation (4.7) is

$$v^{-1} \left\langle \int_{A_k} \rho_k u_k (\underline{U}_k - \underline{W}_s) \cdot \underline{n}_k \, dA \right\rangle = {}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle u_k \rangle^{*}_{LF} + (IETI)_k \tag{5.7.16}$$

where $(IETI)_k$ stands for the interfacial internal energy transfer integral defined by

$$(\mathrm{IETI})_k = - v^{-1} \int_{A_k} \left[\left({}^{i}\langle \varphi_k \rangle_{LF} + \hat{\rho}_k \right) \hat{u}_k + {}^{t}\langle \rho_k' u_k' \rangle \right]$$

$$\left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \hat{\underline{W}}_s \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \left({}^{i}\langle \varphi_k \rangle_{LF} + \hat{\rho}_k \right) {}^{t}\langle u_k' (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \hat{u}_k \, {}^{t}\langle \rho_k' (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA \, . \qquad (5.7.17a)$$

The time- and volume-averaged interfacial internal energy transfer is seen to consist of two parts: (1) a transfer associated with interfacial mass generation ${}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle u_k \rangle^{*}_{LF}$ and (2) extraneous transfer to account for the spatial deviation of density, velocity, and internal energy at the interface as well as that due to the time correlations of their fluctuations.

When $\hat{\rho}_k / {}^{i}\langle \varphi_k \rangle_{LF} \ll 1$, which is usually the case, equation (5.7.17a) reduces to

$$(\mathrm{IETI})_k \cong - v^{-1} \int_{A_k} \left({}^{i}\langle \varphi_k \rangle_{LF} \hat{u}_k + {}^{t}\langle \rho_k' u_k' \rangle \right) \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \hat{\underline{W}}_s \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \, {}^{i}\langle \varphi_k \rangle \int_{A_k} {}^{t}\langle u_k' \, (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \hat{u}_k \, {}^{t}\langle \rho_k' \, (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA \, , \qquad (5.7.17b)$$

which can be further simplified as follows if all time correlations with fluctuating density are insignificant:

$$(\mathrm{IETI})_k \cong - v^{-1} \, {}^{i}\langle \varphi_k \rangle_{LF} \int_{A_k} \left[\hat{u}_k \left({}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k - {}^{i}\langle \underline{W}_s \rangle_{LF} - \hat{\underline{W}}_s \right) \right.$$

$$\left. + {}^{t}\langle u_k' (\underline{U}_k' - \underline{W}_s') \rangle \right] \cdot \underline{n}_k \, dA \, . \qquad (5.7.17c)$$

An examination of the foregoing results shows that the time- and volume-averaged interfacial internal energy source ${}^{t}\langle Q_k \rangle$ consists of interfacial pressure work $(\mathrm{PWI})^{(u)}_k$, viscous dissipation $(\mathrm{VDI})_k$, heat transfer ${}^{t}\langle \dot{Q}_k \rangle$,

transfer associated with mass generation ${}^{t}\langle\Gamma_k\rangle\, {}^{i}\langle u_k\rangle^{*}_{LF}$, and extraneous transfer $(IETI)_k$. Thus,

$$ {}^{t}\langle \mathcal{Q}_k \rangle = {}^{t}\langle \dot{Q}_k \rangle + {}^{t}\langle \Gamma_k \rangle\, {}^{i}\langle u_k \rangle^{*}_{LF} + (PWI)^{(u)}_k + (VDI)_k + (IETI)_k \quad . \tag{5.7.18} $$

As pointed out earlier, $(PWI)^{(u)}_k$ as given by equation (5.7.9) does not account completely for the reversible work done at the interface, nor does $(VDI)_k$ as given by equation (5.7.12) account completely for the viscous dissipation. Consequently, ${}^{t}\langle \mathcal{Q}_k \rangle$ differs from

$$ - v^{-1} \int_{A_k} {}^{t}\left\langle \dot{m}_k u_k + \underline{J}_{qk} \cdot \underline{n}_k + \frac{1}{2} \dot{m}_k (\underline{U}_k - \underline{W}_s)^2 + \left(P_k \underline{n}_k - \underline{\tau}_k \cdot \underline{n}_k \right) \cdot \left(\underline{U}_k - \underline{W}_s \right) \right\rangle dA \ , $$

which can be inferred from equation (4.11). However, it is recognized that

$$ - v^{-1} \int_{A_k} {}^{t}\langle \dot{m}_k \, u_k \rangle \, dA = {}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle u_k \rangle^{*}_{LF} + (IETI)_k \tag{5.7.19} $$

and

$$ - v^{-1} \int_{A_k} {}^{t}\langle \underline{J}_{qk} \rangle \cdot \underline{n}_k \, dA = {}^{t}\langle \dot{Q}_k \rangle \ . \tag{5.7.20} $$

By using the preceding results, the time- and volume-averaged internal energy conservation equation can be obtained. The result is

$$ \frac{\partial}{\partial t} \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF} $$

$$ + \frac{\partial}{\partial t} \alpha_k \, \Phi_{uk} + \nabla \cdot \alpha_k \, \Phi_{uk} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \, \underline{\Psi}_{mk} \, {}^{i}\langle u_k \rangle^{*}_{LF} $$

$$ = - \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} - \alpha_k \, \Phi_{Pk} $$

$$ + \alpha_k \, {}^{i}\langle \underline{\tau}_k \rangle_{LF} : \nabla \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \alpha_k \, \Phi_{\tau k} $$

$$ - \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{u}^{T}_k \rangle + {}^{i}\langle \hat{\underline{u}}^{D}_k \rangle + {}^{ti}\langle \hat{\underline{u}}^{T}_k \rangle \right) $$

$$- \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF} + \alpha_k \, {}^{i}\langle \underline{J}_{Ek} \rangle_{LF} + {}^{t}\langle \mathcal{D}_k \rangle \, . \qquad (5.7.21a)$$

The third and fourth terms on the r.h.s., taken together, denote the time- and volume-averaged viscous dissipation per unit volume of phase k. They are written in the indicated form to bring out their similarity to the reversible pressure work terms.

An alternative form of equation (5.7.21a) can be obtained by using equation (5.4.6a) to eliminate ${}^{t}\langle \Gamma_k \rangle$ in the expression for ${}^{t}\langle \mathcal{D}_k \rangle$. The result is

$$\alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \left(\frac{\partial \, {}^{i}\langle u_k \rangle^{*}_{LF}}{\partial t} + {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \nabla \, {}^{i}\langle u_k \rangle^{*}_{LF} \right)$$

$$+ \frac{\partial}{\partial t} \alpha_k \, \Phi_{uk} + \nabla \cdot \alpha_k \, \Phi_{uk} \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \alpha_k \, \underline{\Psi}_{mk} \cdot \nabla \, {}^{i}\langle u_k \rangle^{*}_{LF}$$

$$= - \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} - \alpha_k \, \Phi_{Pk} + \alpha_k \, {}^{i}\langle \underline{\underline{\tau}}_k \rangle_{LF} : \nabla \, {}^{i}\langle \underline{U}_k \rangle_{LF} + \alpha_k \, \Phi_{\tau k}$$

$$- \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{u}^{T}_k \rangle + {}^{i}\langle \hat{\underline{u}}^{D}_k \rangle + {}^{ti}\langle \hat{\underline{u}}^{T}_k \rangle \right) - \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF} + \alpha_k \, {}^{i}\langle J_{Ek} \rangle_{LF}$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + \left(PWI \right)^{(u)}_k + \left(VDI \right)_k + \left(IETI \right)_k \, . \qquad (5.7.21b)$$

The turbulent mass flux $\underline{\Psi}_{mk}$ and the scalar internal energy function Φ_{uk} $\left(= {}^{ti}\langle \rho'_k u'_k \rangle \right)$ are often small and can be ignored. So is the viscous dissipation. Under these conditions, equations (5.7.21a) and (5.7.21b) become, respectively,

$$\frac{\partial}{\partial t} \alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF} + \nabla \cdot \alpha_k \, {}^{i}\langle \rho_k \rangle \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, {}^{i}\langle u_k \rangle^{*}_{LF}$$

$$= - \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} - \alpha_k \, \Phi_{Pk} - \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{u}^{T}_k \rangle + {}^{i}\langle \hat{\underline{u}}^{D}_k \rangle \right)$$

$$- \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF} + \alpha_k \, {}^{i}\langle J_{Ek} \rangle_{LF} + {}^{t}\langle \dot{Q}_k \rangle + {}^{t}\langle \Gamma_k \rangle \, {}^{i}\langle u_k \rangle^{*}_{LF}$$

$$+ \left(PWI \right)^{(u)}_k + \left(IETI \right)_k \, , \qquad (5.7.22a)$$

and

$$\alpha_k \, {}^{i}\langle \rho_k \rangle_{LF} \left(\frac{\partial \, {}^{i}\langle u_k \rangle^{*}_{LF}}{\partial t} + {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \nabla \, {}^{i}\langle u_k \rangle^{*}_{LF} \right)$$

$$= - \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle'_{LF} - \alpha_k \, \Phi_{Pk} - \nabla \cdot \alpha_k \left({}^{ti}\langle \underline{u}_k^{T} \rangle + {}^{i}\langle \underline{\tilde{u}}_k^{D} \rangle \right)$$

$$- \nabla \cdot \alpha_k \, {}^{i}\langle \underline{J}_{qk} \rangle_{LF} + \alpha_k \, {}^{i}\langle J_{Ek} \rangle_{LF} + {}^{t}\langle \dot{Q}_k \rangle + \left(PWI \right)_k^{(u)} + \left(IETI \right)_k . \quad (5.7.22b)$$

In equations (5.7.22a,b), ${}^{ti}\langle \underline{u}_k^{T} \rangle$, ${}^{i}\langle \underline{\tilde{u}}_k^{D} \rangle$, and $\left(IETI \right)_k$ are given by equations (5.7.4c), (5.7.5b), and (5.7.17c).

The time- and volume-averaged interfacial balance equation for internal energy can be obtained by performing the time averaging of equation (4.11). However, additional work is necessary to resolve the difference between the time- and volume-averaged interfacial internal energy source ${}^{t}\langle \quad _k \rangle$ as defined in equation (5.7.18) and that based on equation (3.10), i.e.,

$$- v^{-1} \sum_{k,f} \int_{A_k} \Big\langle \dot{m}_k \, u_k + \underline{J}_{qk} \cdot \underline{n}_k + \frac{1}{2} \dot{m}_k \left(\underline{U}_k - \underline{W}_s \right) \cdot \left(\underline{U}_k - \underline{W}_s \right)$$

$$+ \left(P_k \, \underline{n}_k - \underline{\tau}_k \cdot \underline{n}_k \right) \cdot \left(\underline{U}_k - \underline{W}_s \right) \Big\rangle \, dA = 0 . \quad (5.7.23)$$

6. SIMPLIFICATIONS AND COMPARISONS WITH OTHER "ACCEPTED" TWO-PHASE FLOW GOVERNING EQUATIONS

A number of two-phase flow equations are available in the literature. It is instructive to compare them with the set of equations given in Chapter 5 of this report. Ishii's monograph [1] considers time averaging only; hence, the results cannot be compared directly. The local volume fraction α_k of phase k that appears in the present set of equations is generally not the same as the local time fraction in Ishii's equations. Nevertheless, the corresponding conservation equations and interfacial balance relations bear a striking resemblance.

6.1 Comparison with Ishii's Two-Fluid Equations

Since Ishii's analysis was based on time averaging only, comparison with the present results may be made by ignoring all contributions due to spatial deviations $\tilde{\psi}_k$. However, the averaging volume would remain finite, containing dispersed interfaces and satisfying length scale restrictions. We are mindful that the arbitrary deletion of $\tilde{\psi}_k$ cannot be justified in general.

For convenience, in this chapter we denote ${}^{i}\langle \psi_k \rangle^{*}_{LF}$ or ${}^{i}\langle \psi_k \rangle_{LF}$ by $\bar{\psi}_k$, implying low frequency or a temporal mean of the intrinsic volume average of

ψ_k with or without weighting by mass. Furthermore, for the purpose of comparison, we assume that all correlations involving ρ_k' are negligible. With the foregoing simplifications, the time- and volume-averaged conservation equations for mass and momentum, and their interfacial balance relations, reduce to the following:

- Mass Conservation Equation. From equation (5.4.6a), one obtains

$$\frac{\partial}{\partial t} \alpha_k \bar{\rho}_k + \nabla \cdot \alpha_k \bar{\rho}_k \bar{\underline{U}}_k = {}^t\langle \Gamma_k \rangle^s \ , \tag{6.1.1}$$

since $\underline{\psi}_{mk} = 0$. In addition, since $(MTI)_k = 0$, the interfacial mass generation rate becomes

$${}^t\langle \Gamma_k \rangle^s = \bar{\rho}_k \left(\frac{\partial \alpha_k}{\partial t} + \bar{\underline{U}}_k \cdot \nabla \alpha_k \right) \tag{6.1.2}$$

where the superscript s denotes simplification. For a two-fluid system, k = 1 or 2, Ishii gave the following equation for void propagation (Eq. VII 3.15 in Ref. 1):

$$\Gamma_k^I = \frac{\bar{\bar{\rho}}_1 \bar{\bar{\rho}}_2}{\sum_{k=1}^{2} \alpha_k^I \bar{\bar{\rho}}_k} \left(\frac{\partial \alpha_k^I}{\partial t} + \underline{C}_k \cdot \nabla \alpha_k^I \right) , \tag{6.1.3}$$

in which superscript (I) refers to Ishii and superscript (=) denotes Ishii's phase average. The local time fraction, α_k^I, relates the phase average $\bar{\bar{\rho}}_k$ to the Eulerian time average $\bar{\rho}_k$ according to

$$\alpha_k^I \bar{\bar{\rho}}_k = \bar{\rho}_k \ . \tag{6.1.4}$$

In equation (6.1.3), $\underline{C}_k$ is Ishii's kinematic wave velocity and Γ_k^I is the rate of production of phase k at the interface, for which Ishii wrote (Eq. V 2.1 in Ref. 1)

$$\Gamma_k^I = -\frac{1}{\Delta t} \sum_j \frac{1}{W_{sn}} \rho_k (\underline{U}_k - \underline{W}_s) \cdot \underline{n}_k \ , \tag{6.1.5}$$

where Δt is a **fixed** time interval for averaging, W_{sn} is the magnitude of the normal component of the interfacial velocity, and index j refers to the number

of times the interface passes through a fixed observation point during Δt. Ishii's time-averaged mass conservation equation (Eq. IX 1.1 in Ref. 1) is

$$\frac{\partial}{\partial t} \alpha_k^I \bar{\rho}_k + \nabla \cdot \alpha_k^I \bar{\rho}_k \bar{\underline{U}}_k^I = \Gamma_k^I , \tag{6.1.6}$$

where $\bar{\underline{U}}_k^I$ is the mass-weighted mean velocity. Although the similarities between equations (6.1.1) and (6.1.6) and those between equations (6.1.2) and (6.1.3) are apparent, precise comparison is difficult owing to differences in the definitions used for the various terms involved.

The time- and volume-averaged interfacial mass balance equation can be written down from equation (5.4.9a). It is

$$\sum_{k,f} {}^t\langle \Gamma_k \rangle^s = 0 . \tag{6.1.7}$$

In conjunction with equation (6.1.6), Ishii gave

$$\sum_{k=1}^{2} \Gamma_k^I = 0 . \tag{6.1.8}$$

We note that our $\sum_{k,f}$ is equivalent to Ishii's $\sum_{k=1}^{2}$. Both equations (6.1.7) and (6.1.8) express the conservation of mass at the interface.

• Linear Momentum Equation. Under the stated simplifying assumptions, equation (5.5.18a) becomes, for $\underline{f} = \underline{g}$,

$$\frac{\partial}{\partial t} \alpha_k \bar{\rho}_k \bar{\underline{U}}_k + \nabla \cdot \alpha_k \bar{\rho}_k \bar{\underline{U}}_k \bar{\underline{U}}_k$$

$$= - \nabla \alpha_k \bar{P}_k + \nabla \cdot \alpha_k \left(\bar{\underline{\underline{\tau}}}_k + \bar{\underline{\underline{\tau}}}_k^T \right) + \alpha_k \bar{\rho}_k \underline{g} + {}^t\langle \underline{M}_k \rangle^s , \tag{6.1.9}$$

in which ${}^t\langle \underline{M}_k \rangle^s$ denotes the simplified time- and volume-averaged interfacial momentum source. It can be written down from equation (5.5.16). Thus,

$${}^t\langle \underline{M}_k \rangle^s = {}^t\langle \Gamma_k \rangle^s \bar{\underline{U}}_k + \bar{P}_k \nabla \alpha_k - \bar{\underline{\underline{\tau}}}_k \cdot \nabla \alpha_k + \left(\text{MMTi} \right)_k^s , \tag{6.1.10}$$

where ${}^t\langle \Gamma_k \rangle^s$ is defined by equation (6.1.2) and

$$(\text{MMTI})_k^s = - v^{-1} \bar{\rho}_k \int_{A_k} {}^t\langle \underline{U}_k'(\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_k \, dA \, . \tag{6.1.11}$$

It is seen that $(\text{MMTI})_k^s$ represents a "modified" Reynolds stress due to interfacial turbulence.

The time-averaged linear momentum equation given by Ishii (Eq. IX 1.14 in Ref. 1) is

$$\frac{\partial}{\partial t} \alpha_k^I \bar{\bar{\rho}}_k \bar{\underline{U}}_k^I + \nabla \cdot \alpha_k^I \bar{\bar{\rho}}_k \bar{\underline{U}}_k^I \bar{\underline{U}}_k^I$$

$$= - \nabla \alpha_k^I \bar{\bar{P}}_k + \nabla \cdot \alpha_k^I \left(\bar{\bar{\tau}}_k + \bar{\bar{\tau}}_k^T \right) + \alpha_k^I \bar{\bar{\rho}}_k \underline{g} + \underline{M}_k^I \, , \tag{6.1.12}$$

in which $\bar{\bar{\tau}}_k$ and $\bar{\bar{\tau}}_k^T$ are the viscous and turbulent stress tensors and $\underline{M}_k^I$ is the interfacial momentum source given by

$$\underline{M}_k^I = \Gamma_k^I \bar{\underline{U}}_{ki}^I + \bar{\bar{P}}_{ki} \nabla \alpha_k^I - \bar{\bar{\tau}}_i \cdot \nabla \alpha_k^I + \underline{M}_k^d \, , \tag{6.1.13}$$

where subscript i refers to interface and $\underline{M}_k^d$ is the total drag force. In equation (6.1.13), the last term was added in accordance with Refs. 25 and 26. $\underline{M}_k^d$ consists of a tangential component giving rise to the skin friction drag and a normal component which gives rise to the form drag.

Despite the difference between our α_k and Ishii's α_k^I, it is instructive to compare the simplified momentum equation (6.1.9) with Ishii's result, equation (6.1.12). If we assume that ${}^t\langle \Gamma_k \rangle^s = \Gamma_k^I$, $\bar{\rho}_k = \bar{\bar{\rho}}_k$, $\bar{P}_k = \bar{\bar{P}}_k$, etc., and $\alpha_k = \alpha_k^I$, then equation (6.1.9) becomes formally identical to equation (6.1.12), provided that

$${}^t\langle \underline{M}_k \rangle^s = \underline{M}_k^I \, , \tag{6.1.14a}$$

or, equivalently,

$$\underline{M}_k^d + (\bar{P}_{ki} - \bar{P}_k) \nabla \alpha_k + {}^t\langle \Gamma_k \rangle^s (\bar{\underline{U}}_{ki} - \bar{\underline{U}}_k)$$

$$- (\bar{\bar{\tau}}_i - \bar{\bar{\tau}}_k) \cdot \nabla \alpha_k = (\text{MMTI})_k^s \, . \tag{6.1.14b}$$

• Interfacial Momentum Balance Equation. Equation (5.5.21a), when simplified for conditions consistent with those used in deriving equation (6.1.9), becomes

$$\sum_{k,f} {}^{t}\langle \underline{M}_k \rangle^{s} = v^{-1} \int_{A_k} \left(\nabla_s \bar{\sigma} - 2\bar{\sigma}\, \bar{H}_k\, \underline{n}_k \right) dA \cong \frac{A_k}{v} \nabla_s \bar{\sigma} + 2\,\bar{\sigma}\, \bar{H}_k\, \nabla \bar{\alpha}_k \,, \qquad (6.1.15a)$$

where ∇_s is the interfacial surface gradient operator, $\bar{\sigma}$ is the temporal mean interfacial tension, $\bar{H}_k$ is the temporal mean of the average principal curvature of the interface, and subscript k can be either 1 or 2, referring to fluids on either side of the interface. The first term on the right side accounts for the variation of surface tension along the interface, which is probably small in dispersed systems owing to the random nature of its distribution over the particles. The second term accounts for the mean curvature effect. When both contributions are small, equation (6.1.15a) becomes

$$\sum_{k,f} {}^{t}\langle \underline{M}_k \rangle^{s} = 0 \,. \qquad (6.1.15b)$$

The interfacial transfer equation associated with equation (6.1.12) has been given by Ishii (Eqs. IX 1.12, IX 1.13, and VIII 2.7 in Ref. 1). It is

$$\sum_{k=1}^{2} \underline{M}_k^I = \sum_j \frac{1}{L_j} 2\left(H_{21} - \bar{\bar{H}}_{21}\right) \bar{\bar{\sigma}}\, \underline{n}_1 + 2\bar{\bar{\sigma}} \bar{\bar{H}}_{21} \nabla \alpha_2^I + \text{force due to } \nabla_i \bar{\bar{\sigma}} \,, \qquad (6.1.16a)$$

where L_j^{-1} denotes the area concentration per unit volume. The first term on the right side of equation (6.1.16a) accounts for the effect of the change in mean curvature, which is not included in equation (6.1.15a). When the terms on the right side of equation (6.1.16a), taken collectively, are small, one can write

$$\sum_{k=1}^{2} \underline{M}_k^I = 0 \,. \qquad (6.1.16b)$$

When the assumptions used in establishing equations (6.1.14a) and (6.1.16a) are valid, equations (6.1.15b) and (6.1.16b) become identical.

In two recent publications, Ishii and Mishima [25] and Ishii and Kocamustafaogullari [26] gave the following "simplified" form of the time-averaged momentum equation (Eq. 2 in Refs. 25 and 26):

$$\frac{\partial}{\partial t} \alpha_k^I \bar{\bar{\rho}}_k \bar{\underline{U}}_k + \nabla \cdot \alpha_k^I \bar{\bar{\rho}}_k \bar{\underline{U}}_k \bar{\underline{U}}_k$$

$$= - \alpha_k^I \nabla \bar{\bar{P}}_k + \nabla \cdot \alpha_k^I \left(\bar{\bar{\tau}}_k + \bar{\bar{\tau}}_k^T \right)$$

$$+ \alpha_k^I \bar{\bar{\rho}}_k \underline{g} + \Gamma_k^I \underline{\bar{U}}_{ki} - \underline{\tau}_i \cdot \nabla \alpha_k^I + \underline{M}_{ik} , \qquad (6.1.17)$$

in which $\underline{\tau}_i$ is the interfacial shear stress and $\underline{M}_{ik}$ is the "generalized" interfacial drag. By comparing equation (6.1.12) with equation (6.1.17), one sees that $\underline{M}_k^d + (\bar{\bar{P}}_{ki} - \bar{\bar{P}}_k) \nabla \alpha_k^I = \underline{M}_{ik}$. The authors of Refs. 25 and 26 suggested that for the dispersed phase,

$\underline{M}_{ik}$ = sum of standard drag force, virtual mass force, and Basset force, all computed on the basis of a unit volume. (6.1.18)*

Equation (6.1.18) is not a derived result. While it is physically meaningful, there is no assurance that it is complete, nor there is agreement among researchers how the three forces should be mathematically represented.

The interfacial momentum balance equation associated with equation (6.1.17) as given in Ref. 26 is

$$\sum_{k=1}^{2} \underline{M}_{ik} = 0 . \qquad (6.1.19)$$

Again, if we assume that ${}^t\langle \Gamma_k \rangle^s = \Gamma_k^I$, $\bar{\rho}_k = \bar{\bar{\rho}}_k$, $\alpha_k = \alpha_k^I$, etc., we can readily demonstrate that equations (6.1.9) and (6.1.17) become formally identical if

$$- {}^t\langle \Gamma_k \rangle^s (\underline{\bar{U}}_{ki} - \underline{\bar{U}}_k) + (\underline{\tau}_i - \bar{\underline{\tau}}_k) \cdot \nabla \alpha_k + (\mathrm{MMTI})_k^s = \underline{M}_{ik} , \qquad (6.1.20)$$

which is equivalent to equation (6.1.14b) since $\underline{M}_{ik} = \underline{M}_k^d + (\bar{P}_{ki} - \bar{P}_k) \nabla \alpha_k$.

6.2 Comparison with Energy Conservation Equation Used in TRAC Computer Code

The field equations describing the two-phase, two-fluid flow used in the Transient Reactor Analysis Code (TRAC) were based on the mixture mass equation, vapor mass equation, vapor equation of motion, liquid equation of motion, mixture energy equation, and vapor energy equation [27]. Since the energy equation is written in terms of internal energy, it is selected for comparison.

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*The "generalized" drag force was represented by $\underline{M}_{id}$ in equation (6) of Ref. 25 and in equation (9) of Ref. 26 instead of $\underline{M}_{ik}$. Presumably the subscript d refers to the dispersed phase.

When the simplifications introduced in the beginning of this chapter are used, equation (5.7.21a) reduces to

$$\frac{\partial}{\partial t}\alpha_k \bar{\rho}_k \bar{u}_k + \nabla \cdot \alpha_k \bar{\rho}_k \underline{\bar{U}}_k \bar{u}_k$$

$$= -\alpha_k \bar{P}_k \nabla \cdot \underline{\bar{U}}_k - \nabla \cdot \alpha_k \left(\underline{\bar{J}}_{qk} + \underline{\bar{u}}_k^T \right) + \alpha_k \bar{J}_{Ek}$$

$$+ {}^t\langle \dot{Q}_k \rangle^s + {}^t\langle \Gamma_k \rangle^s \bar{u}_k + \left(\mathrm{IETI} \right)_k^s , \tag{6.2.1}$$

in which contributions due to reversible mechanical work associated with fluctuating pressure and velocity and those due to irreversible viscous dissipation are neglected. The time- and volume-averaged turbulent internal energy flux $\underline{\bar{u}}_k^T$ is given by

$$\underline{\bar{u}}_k^T = \bar{\rho}_k \, {}^{ti}\langle \underline{U}_k' u_k' \rangle \tag{6.2.2}$$

according to equation (5.7.4c), and the time-averaged interfacial heat transfer rate per unit volume may be expressed as

$${}^t\langle \dot{Q}_k \rangle^s = -\frac{A_k}{V} \frac{1}{A_k} \int_{A_k} {}^t\langle \underline{J}_{qk} \rangle \cdot \underline{n}_k \, dA$$

$$= \frac{A_k}{V} q_{ki} , \tag{6.2.3}$$

where the scalar q_{ki} is the area-averaged temporal mean interfacial heat flux **entering** phase k. By deleting the term involving $\hat{u}_k$ in equation (5.7.17c), we obtain

$$\left(\mathrm{IETI} \right)_k^s = -V^{-1} \bar{\rho}_k \int_{A_k} {}^t\langle u_k' (\underline{U}_k' - \underline{W}_s') \rangle \cdot \underline{n}_k \, dA , \tag{6.2.4}$$

which denotes the internal energy transfer due to interfacial turbulence.

Using the expression for ${}^t\langle \Gamma_k \rangle^s$ given in equation (6.1.2), one finds that

$$^{t}\langle\Gamma_k\rangle^{s}\ \bar{u}_k = {}^{t}\langle\Gamma_k\rangle^{s}\left(\bar{h}_k - \frac{\bar{P}_k}{\bar{\rho}_k}\right)$$

$$= {}^{t}\langle\Gamma_k\rangle^{s}\ \bar{h}_k - \bar{P}_k\left(\frac{\partial\ \alpha_k}{\partial t} + \underline{\bar{U}}_k \cdot \nabla\ \alpha_k\right). \qquad (6.2.5)$$

Substituting equation (6.2.5) into equation (6.2.1), followed by combining and rearranging terms, yields the internal energy equation:

$$\frac{\partial}{\partial t}\ \alpha_k\ \bar{\rho}_k\ \bar{u}_k + \nabla \cdot \alpha_k\ \bar{\rho}_k\ \underline{\bar{U}}_k\ \bar{u}_k$$

$$= -\ \bar{P}_k\ \frac{\partial\ \alpha_k}{\partial t} - \bar{P}_k\ \nabla \cdot \alpha_k\ \underline{\bar{U}}_k - \nabla \cdot \alpha_k\left(\underline{\bar{J}}_{qk} + \underline{\bar{u}}^{f}_{k}\right)$$

$$+\ {}^{t}\langle\dot{Q}_k\rangle^{s} + {}^{t}\langle\Gamma_k\rangle^{s}\ \bar{h}_k + \alpha_k\ \bar{J}_{Ek} + \left(\mathrm{IETI}\right)^{s}_{k}\ . \qquad (6.2.6)$$

The molecular and turbulent conduction fluxes can be expressed in terms of molecular and eddy diffusivities for internal energy transfer, D_{uk} and D^{T}_{uk}. Thus,

$$-\ \nabla \cdot \alpha_k\left(\underline{\bar{J}}_{qk} + \underline{\bar{u}}^{T}_{k}\right) = \nabla \cdot \alpha_k\ \bar{\rho}_k\left(D_{uk} + D^{T}_{uk}\right)\nabla\ \bar{u}_k\ . \qquad (6.2.7)$$

The vapor internal energy equation used in the TRAC code (equation (65) in Ref. 27), written in the present notation, is (with subscript g replaced by k)

$$\frac{\partial}{\partial t}\ \alpha_k\ \bar{\rho}_k\ \bar{u}_k + \nabla \cdot \alpha_k\ \bar{\rho}_k\ \underline{\bar{U}}_k\ \bar{u}_k$$

$$= -\ \bar{P}_k\ \frac{\partial\ \alpha_k}{\partial t} - \bar{P}_k\ \nabla \cdot \alpha_k\ \underline{\bar{U}}_k + {}^{t}\langle\dot{Q}_k\rangle^{s} + {}^{t}\langle\Gamma_k\rangle^{s}\ \bar{h}_{ki} + \dot{Q}_{wk}\ , \qquad (6.2.8)$$

where $\dot{Q}_{wk}$ denotes the wall heat transfer rate per unit volume. Equation (6.2.6) becomes identical to equation (6.2.8) when the following assumptions are made:

a. Molecular and turbulent conduction are negligible,

b. $\alpha_k \bar{J}_{Ek} = \dot{Q}_{wk}$, and

c. ${}^t\langle T_k \rangle^s \left(\bar{h}_{ki} - \bar{h}_k\right) - \left(\text{IETI}\right)_k^s = 0$.

In boiling water reactor applications, the first condition is probably quite reasonable. The second is merely a statement that the wall heat transfer (such as that from fuel rods) is treated as a distributed heat source. The third states that in the presence of interfacial mass generation, the transport of enthalpy from the interface to the bulk of phase k is via interfacial turbulence. The validity of this conclusion requires confirmation.

We have demonstrated in this section that when the contributions due to spatial deviations of the dependent variables are dropped, and when several additional simplifications are introduced, the set of rigorously derived conservation equations presented in Chapter 5 reduces to various forms that compare reasonably well with those given in Refs. 1, 25, 26, and 27. Perhaps the agreement found by this comparison represents a less important finding then the differences that are revealed.

7. DISCUSSION AND CONCLUDING REMARKS

This report presents a set of rigorously derived time- and volume-averaged conservation equations for dispersed multiphase flow. The starting point of the analysis is the well-established phasic conservation equations of mass, momentum, and energy, and their interfacial balance relations. The local volume averaging is performed first, followed by time averaging. In this way, the identity of the dynamic phases is preserved. The result is a set of differential equations involving area integrals. An examination of these equations reveals immediately that they are **incomplete** in that (1) the need of expressing the volume average of products in terms of product of volume averages is only partially met and (2) constitutive relations for the diffusive, dispersive, turbulent, and interfacial transport also need to be developed. Collectively, this constitutes the closure problem, which is not unlike that in the analysis of turbulent flow, but with additional complications. In the absence of turbulence, a closure scheme for the determination of the spatial deviation of the dependent variable for systems involving only diffusion and first-order chemical reaction was given recently by Crapiste, Rotstein, and Whitaker [28]. A rigorous approach to treat the general closure problem, including convective transport and turbulence, will no doubt remain a challenge for some time to come.

If the flow and thermodynamic conditions are such that the spatial deviations of the dependent variables, denoted by ($\tilde{}$), are small and can be deleted, and if, in addition, all time correlations involving ρ_k' and P_k' are negligible, then the resulting set of simplified equations reduces to a form

closely resembling Ishii's time-averaged equations, although differences remain. These differences are not unexpected since Ishii's local time fraction is not identical to the local volume fraction. The internal energy equation used in the TRAC code [27] has also been demonstrated to be in reasonable agreement with the internal energy equation obtained in the present study following simplifications suitable for applications to nuclear reactor systems.

At present, the evaluation of the interfacial transfer integrals in the time- and volume-averaged conservation equations is not generally known. One of the fundamental problems in the analysis of multiphase flow is the lack of knowledge of mass, momentum, and energy transfer at the interface. The quantification of transport properties such as eddy and dispersive diffusivities of mass, heat, and momentum transfer is also urgently needed. It is hoped that the present study may provide some guidance for the development of the constitutive equations as well as correlations for interfacial transport.

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APPENDIX A

DERIVATION OF $\left\langle \alpha_k \, {}^{i}\langle P_k \nabla \cdot \underline{U}_k \rangle \right\rangle^{t}$

Beginning with the decomposition $P_k = {}^{i}\langle P_k \rangle_{LF} + \hat{P}_k + P_k'$ and $\underline{U}_k = {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{\underline{U}}_k + \underline{U}_k'$, we obtain

$$
\begin{aligned}
P_k \nabla \cdot \underline{U}_k = {}& {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{i}\langle P_k \rangle_{LF} \nabla \cdot \hat{\underline{U}}_k + {}^{i}\langle P_k \rangle_{LF} \nabla \cdot \underline{U}_k' \\
& + \hat{P}_k \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \hat{P}_k \nabla \cdot \hat{\underline{U}}_k + \hat{P}_k \nabla \cdot \underline{U}_k' \\
& + P_k' \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + P_k' \nabla \cdot \hat{\underline{U}}_k + P_k' \nabla \cdot \underline{U}_k' \, . \qquad \text{(A.1)}
\end{aligned}
$$

Since we are interested in evaluating the time average of ${}^{i}\langle P_k \nabla \cdot \underline{U}_k \rangle$, the following terms in equation (A.1) require no consideration since they vanish upon time-averaging: ${}^{i}\langle P_k \rangle_{LF} \nabla \cdot \underline{U}_k'$, $\hat{P}_k \nabla \cdot \underline{U}_k'$, $P_k' \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$ and $P_k' \nabla \cdot \hat{\underline{U}}_k$. Now

$$
\begin{aligned}
\alpha_k \, {}^{i}\left\langle {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right\rangle &= {}^{i}\langle P_k \rangle_{LF} \left\langle \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right\rangle = {}^{i}\langle P_k \rangle_{LF} \nabla \cdot \alpha_k \, {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \\
&\quad + {}^{i}\langle P_k \rangle_{LF} \, v^{-1} \int_{A_k} {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \cdot \underline{n}_k \, dA \\
&= \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \, . \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_k \, {}^{i}\left\langle {}^{i}\langle P_k \rangle_{LF} \nabla \cdot \tilde{\underline{U}}_k \right\rangle &= {}^{i}\langle P_k \rangle_{LF} \langle \nabla \cdot \hat{\underline{U}}_k \rangle \\
&= {}^{i}\langle P_k \rangle_{LF} \nabla \cdot \alpha_k \, {}^{i}\langle \hat{\underline{U}}_k \rangle \\
&\quad + v^{-1} \, {}^{i}\langle P_k \rangle_{LF} \int_{A_k} \hat{\underline{U}}_k \cdot \underline{n}_k \, dA \, . \qquad \text{(A.3)}
\end{aligned}
$$

$$
\alpha_k \, {}^{i}\left\langle \hat{P}_k \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} \right\rangle = 0 \qquad \text{(A.4)}
$$

Unfortunately, $\alpha_k \, {}^{i}\langle \hat{P}_k \nabla \cdot \tilde{\underline{U}}_k \rangle$ and $\alpha_k \, {}^{ti}\langle P'_k \nabla \cdot \underline{U}'_k \rangle$ cannot be reduced further.

Strictly speaking, ${}^{i}\langle \tilde{\underline{U}}_k \rangle$ vanishes only if ρ_k is uniform in the averaging volume. Nevertheless, it is usually small compared to ${}^{i}\langle \underline{U}_k \rangle^{*}_{LF}$ under the ordinary circumstance. Thus, it appears reasonable to ignore the first term on the r.h.s. of equation (A.3). Consequently, we write

$$ {}^{t}\left\langle \alpha_k \, {}^{i}\langle P_k \nabla \cdot \underline{U}_k \rangle \right\rangle = \alpha_k \, {}^{i}\langle P_k \rangle_{LF} \nabla \cdot {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \alpha_k \, \Phi_{Pk} - (PWI)^{(u)}_k \qquad (A.5) $$

where the scalar pressure work function Φ_{Pk} is defined by

$$ \Phi_{Pk} = {}^{i}\langle \hat{P}_k \nabla \cdot \tilde{\underline{U}}_k \rangle + {}^{ti}\langle P'_k \nabla \cdot \underline{U}'_k \rangle \qquad (A.6) $$

and the interfacial pressure work integral $(PWI)^{(u)}_k$ is

$$ (PWI)^{(u)}_k = - v^{-1} \, {}^{i}\langle P_k \rangle_{LF} \int_{A_k} \tilde{\underline{U}}_k \cdot \underline{n}_k \, dA \, . \qquad (A.7) $$

One may stipulate that a portion of Φ_{Pk} also contributes to the interfacial work.

APPENDIX B

DERIVATION OF $\overset{t}{\left\langle \alpha_k \; {}^{i}\langle \phi_k \rangle \right\rangle}$

We recall that $\phi_k = \underline{\underline{I}}_k : \nabla \underline{U}_k$. Using the decomposition

$$\underline{\underline{I}}_k = {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} + \tilde{\underline{\underline{I}}}_k = \underline{\underline{I}}'_k \tag{B.1}$$

where ${}^{i}\langle \underline{\underline{I}}_k \rangle_{LF}$ is defined by equation (5.5.9b), $\tilde{\underline{\underline{I}}}_k$ by equation (5.5.9c), and $\underline{\underline{I}}'_k$ by equation (5.5.9d), and carrying out the indicated scalar multiplication of the tensor $\underline{\underline{I}}_k$ and the dyad $\nabla \underline{U}_k$, one obtains

$$\begin{aligned} \underline{\underline{I}}_k : \nabla \underline{U}_k = {} & {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} : \nabla \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} : \nabla \, \tilde{\underline{U}}_k + {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} : \nabla \, \underline{U}'_k \\ & + \tilde{\underline{\underline{I}}}_k : \nabla \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \tilde{\underline{\underline{I}}}_k : \nabla \, \tilde{\underline{U}}_k + \tilde{\underline{\underline{I}}}_k : \nabla \, \underline{U}'_k \\ & + \underline{\underline{I}}'_k : \nabla \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \underline{\underline{I}}'_k : \nabla \, \tilde{\underline{U}}_k + \underline{\underline{I}}'_k : \nabla \, \underline{U}'_k \; . \end{aligned} \tag{B.2}$$

Following a procedure completely analogous to that detailed in Appendix A, we obtain

$$\overset{t}{\left\langle \alpha_k \; {}^{i}\langle \phi_k \rangle \right\rangle} = \alpha_k \; {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} : \nabla \; {}^{i}\langle \underline{U}_k \rangle^{*}_{LF} + \alpha_k \, \phi_{\tau k} + (VDI)_k \tag{B.3}$$

where

$$\phi_{\tau k} = {}^{i}\langle \tilde{\underline{\underline{I}}}_k : \nabla \, \tilde{\underline{U}}_k \rangle + {}^{ti}\langle \underline{\underline{I}}'_k : \nabla \, \underline{U}'_k \rangle \tag{B.4}$$

and

$$(VDI)_k = v^{-1} \; {}^{i}\langle \underline{\underline{I}}_k \rangle_{LF} : \int_{A_k} \tilde{\underline{U}}_k \, \underline{n}_k \, dA \; . \tag{B.5}$$

Note that the integrand $\tilde{\underline{U}}_k \, \underline{n}_k$ is a dyad.

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Conservation equations of mass, momentum, and energy for multiphase flow, formulated on the basis of local volume averaging followed by time-averaging for turbulent flows, are presented. They are differential equations of transport with area integrals associated with interfacial transport. Because the local averaging theorems used in the analysis are subject to certain length scale restrictions, the resulting equations are best suited for dispersed systems.

The local instantaneous variable is decomposed as a linear combination of its local intrinsic volume average and a spatial deviation. Use of the mass-weighted, volume-averaged velocity and internal energy simplified certain relationships between the volume average of products and the product of volume averages. Recognition of the fact that the spatial deviation component takes on positive and negative values within the averaging volume makes further simplifications feasible. Inasmuch as information is always lost as a result of averaging, be it volume-averaging or time-averaging or both, the lost information must somehow be replaced before the equations can be solved. This is commonly done by the development of appropriate constitutive relations, which, however, is not treated in this report.

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