NUREG/CR-3989
ANL-84-66

NUREG/CR-3989
ANL-84-66

TIME- AND VOLUME-AVERAGED CONSERVATION EQUATIONS FOR MULTIPHASE FLOW

PART ONE: SYSTEM WITHOUT INTERNAL SOLID STRUCTURES

by

W. T. Sha, B. T. Chao,
and S. L. Soo

ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS
Operated by THE UNIVERSITY OF CHICAGO

Prepared for the Office of Nuclear Regulatory Research
U. S. NUCLEAR REGULATORY COMMISSION
under interagency Agreement DOE 40-550-75

NUREG/CR-3989
ANL-84-66

Distribution
Category: R7

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

TIME- AND VOLUME-AVERAGED CONSERVATION EQUATIONS FOR MULTIPHASE FLOW

PART ONE: SYSTEM WITHOUT INTERNAL SOLID STRUCTURES

by

W. T. Sha, B. T. Chao,*
and S. L. Soo*

Components Technology Division

December 1984

Prepared for

Office of Nuclear Regulatory Research
U. S. Nuclear Regulatory Commission
Washington, D.C. 20555
under Interagency Agreement DOE 40-550-75

NRC FIN No. A2045

*Consultants: Department of Mechanical Engineering,
University of Illinois, Urbana

TIME- AND VOLUME-AVERAGED CONSERVATION EQUATIONS FOR MULTIPHASE FLOW

PART ONE: SYSTEM WITHOUT INTERNAL SOLID STRUCTURES

by

W. T. Sha, B. T. Chao, and S. L. Soo

ABSTRACT

Local volume averaging of phasic conservation equations of mass, momentum, and energy for a multiphase system yields equations in terms of local volume averaged products of density, velocity, energy, stresses, etc., together with interface transfer integrals. These averaging relations are subject to the following length scale restrictions:

$$d \ll \ell \ll L,$$

where d is a characteristic length of the pores or dispersed phases, ℓ is a characteristic length of the averaging volume, and L is a characteristic length of the physical system.

Solutions of local-volume-averaged conservation equations call for expressing these local volume-averaged products in terms of products of averages. In nonturbulent flows, this can be achieved by expressing the "point" variable as the sum of its intrinsic volume average and a spatial deviation. In turbulent flows, such a determination can be made via time averaging over a duration T such that

$$\tau_{HF} \ll T \ll \tau_{LF},$$

where τ_{HF} is a characteristic time of high-frequency fluctuation and τ_{LF} is a characteristic time of low-frequency fluctuation. In this case, the "point" variable is decomposed into a low-frequency component of the intrinsic volume average and the associated spatial deviation and a high-frequency component. This procedure reduces the volume-averaged products to products of averages plus terms representing eddy and dispersive diffusivities of mass, Reynolds and dispersive stresses, eddy and dispersive conductivities of heat, etc. These terms arise from both local spatial deviations and high-frequency fluctuating components. Time averaging after local volume averaging preserves the identity of dynamic phases. The resulting conservation equations are in the form of differential-integral equations of transport with probability integrals depending on phase configurations, interfacial velocities, and other factors. When the flow conditions are such that the local averaging volume can be made sufficiently small that the effect of spatial deviations can be ignored, and in addition, the contributions due to high-frequency fluctuations in local volume fraction and fluid density are also negligible, then the proposed set of rigorously derived conservation equations reduces closely to various forms that are currently "accepted" for thermal hydraulic analysis of nuclear reactors as well as systems involving two-phase flows in general.

In Part One of this study, a rigorous derivation of a set of conservation equations of mass, momentum, and energy for a multiphase system without internal solid structures via time-volume averaging is presented. Similar derivation is presented in Part Two of this study (a separate publication) with consideration of the presence of stationary internal solid structures, for which the concepts of volume porosity, directional surface porosities, distributed resistance and distributed heat source or sink are introduced. The concept of directional surface porosities is new and it greatly facilitates modeling anisotropic flow and temperature field for such systems. Specific attention is given to the numerical computation of flow and temperature fields.

FIN No. | Title

A2045 | 3-D Time-Dependent Code Development

CONTENTS

$\underline{q}$ Heat flux vector

r_c Characteristic length of local averaging volume v

r_o Characteristic length of physical system

$\underline{M}_k$ Interfacial momentum source per unit volume, ${}^t\langle \underline{M}_k \rangle$ is defined by Eq. 6.5.22

$\underline{n}$ Unit outward normal vector of interface as illustrated in Fig. 2

p Static pressure

q Interfacial heat transfer rate per unit volume

t Time

T Temperature; also averaging time interval

u Internal energy per unit mass

$3i\langle \underline{u}_k^T \rangle$ is the volume-averaged turbulent internal energy flux of phase k, defined by Eq. 6.9.4

$3i\langle \underline{\tilde{u}}_k \rangle$ is the volume-averaged dispersive internal energy flux of phase k, defined by Eq. 6.9.7

$3i\langle \underline{\tilde{u}}_k^T \rangle$ is the volume-averaged turbulent, dispersive internal energy flux of phase k, defined by Eq. 6.9.8

$\underline{v}$ Velocity

V Local averaging volume; also volume in general

$\underline{w}$ Interface velocity

z Cartesian coordinates; z is also elevation

ε Local volume fraction

ε_a Surface porosity

ε_v Volume porosity

Γ_k Interfacial mass source per unit volume due to phase change, ${}^t\langle \Gamma_k \rangle$ is defined by Eq. 6.3.10

$\mathcal{E}_k$ Interfacial total energy source per unit volume, ${}^t\langle \mathcal{E}_k \rangle$ is defined by Eq. 6.7.32

$\mathcal{H}_k$ Interfacial enthalpy source per unit volume, ${}^t\langle \mathcal{H}_k \rangle$ is defined by Eq. 6.11.16

FIGURES

NOMENCLATURE

NOMENCLATURE

A — Area: A_e is the free flow area of enveloping surface of local averaging volume v; A_k is the total interfacial area associated with phase k inside v

c_v — Specific heat at constant volume

d — Characteristic length of a dispersed phase

D — Diffusivity

- D^T_{mk} is the eddy diffusivity for mass transfer of phase k, defined by Eq. 6.3.3
- $\breve{D}^T_{mk}$ is the dispersive diffusivity for mass transfer of phase k, defined by Eq. 6.3.4
- D_{uk} is the molecular thermal diffusivity of phase k, defined by Eq. 6.9.20
- D^T_{uk} is the turbulent diffusivity for internal energy transfer of phase k, defined by Eq. 6.9.5a

E — Total energy per unit mass, $= u + (1/2)\,\underline{U}_k \cdot \underline{U}_k$

- $3i_{\langle \underline{E}^T_k \rangle}$ is the volume-averaged turbulent total energy flux vector of phase k, defined by Eq. 6.7.5
- $3i_{\langle \breve{\underline{E}}_k \rangle}$ is the volume-averaged dispersive total energy flux vector of phase k, defined by Eq. 6.7.6
- $3i_{\langle \breve{\underline{E}}^T_k \rangle}$ is the volume-averaged turbulent, dispersive total energy flux vector of phase k, defined by Eq. 6.7.7

$\underline{f}$ — Field force per unit mass

g — Gravitational acceleration

h — Enthalpy per unit mass

- $3i_{\langle \underline{h}^T_k \rangle}$ is the volume-averaged turbulent enthalpy flux vector of phase k, defined by Eq. 6.11.4
- $3i_{\langle \breve{\underline{h}}_k \rangle}$ is the volume-averaged dispersive enthalpy flux vector of phase k, defined by Eq. 6.11.5
- $3i_{\langle \breve{\underline{h}}^T_k \rangle}$ is the volume-averaged turbulent dispersive enthalpy flux vector of phase k, defined by Eq. 6.11.6

H_{kf} — Mean curvature of interface between phases k and f

$\underline{\underline{I}}$ — Unitary tensor

J_E — Internal heat source per unit volume

$\underline{J}_q$

ℓ

L

$\underline{M}$

n

P

$\dot{Q}$

t

T

u

$\underline{U}$

v

$\underline{W}$

$x, y,$

α

γ_A

γ_v

Γ

ε

$\mathcal{H}$

Transport properties associated with high-frequency fluctuation

Refers to the case of constant density

The following superscripts appear in Section 8 only:

Same as $(\ )_{LF}$

Mass weighted mean defined in Ref. 13

Refers to Ishii

Further simplification of $(\ )^{zd}$

Refers to zero spatial deviation

Subscripts

Capillarity; also characteristic quantity

Phase f

High frequency

Interface

Phase k

Interface of fluid phases k and f; $A_k = A_{kf} = A_{fk} = A_f$

Low frequency

Mass; also mixture

Components in the x-, y-, and z-direction

Symbols

Vector

Tensor, second order

Area average, local

$^{2m}\langle\ \rangle$ denotes average over free flow area for the fluid mixture

$^{2i}\langle\ \rangle$ denotes intrinsic average over free flow area for a phase

Volume average, local

$^{3m}\langle\ \rangle$ denotes volume average over fluid mixture

$^{3i}\langle\ \rangle$ denotes intrinsic volume average of a phase

Time average

$\mathcal{Q}$	Interfacial internal energy source per unit volume $^{t}\mathcal{Q}_{k}>$ is defined by Eq. 6.9.18
κ	Thermal conductivity
λ	Bulk viscosity
μ	Dynamic viscosity
ρ	Density
σ	Surface tension
τ	Characteristic time
$\underline{\underline{\tau}}$	Viscous stress
	$^{3i}<\underline{\underline{\tau}}_{k}^{T}>$ is the volume-averaged Reynolds stress of phase k, defined by Eq. 6.5.3
	$^{3i}<\underline{\underline{\tilde{\tau}}}_{k}>$ is the volume-averaged dispersive stress of phase k, defined by Eq. 6.5.4
	$^{3i}<\underline{\underline{\tilde{\tau}}}_{k}^{T}>$ is the volume-averaged turbulent, dispersive stress of phase k, defined by Eq. 6.5.5
ϕ	Dissipation function
Φ_{Ek}	Scalar total energy function defined by Eq. 6.7.3
Φ_{hk}	Scalar enthalpy function defined by Eq. 6.11.2
Φ_{Pk}	Scalar pressure work function defined by Eq. 6.9.10
Φ_{uk}	Scalar internal energy function defined by Eq. 6.9.2
$\Phi_{\tau k}$	Scalar viscous dissipation function defined by Eq. 6.9.13
ψ	Intensive property
$\underline{\Psi}_{mk}$	Vector mass flux function defined by Eq. 6.3.5
$\underline{\Psi}_{Pk}$	Vector pressure work function defined by Eq. 6.7.9
$\underline{\Psi}_{\tau k}$	Vector viscous stress work function defined by Eq. 6.7.11

Superscripts

$(\tilde{\ })$	Local spatial deviation
$(')$	High-frequency fluctuation

$( )^{T}$	
$^{o}( )$	
$(\bar{\ })$	
$(\bar{\bar{\ }})$	
$( )^{I}$	
$( )^{s}$	
$( )^{zd}$	

c	
f	
HF	
i	
k	
kf	
LF	
m	
x,y,z	
$(\underline{\ })$	
$(\underline{\ })$	
$^{2}< >$	
$^{3}< >$	
$^{t}< >$	

Operators

∇	Gradient
$\nabla \cdot$	Divergence
$\nabla,$	Dyad
$(\nabla,)_c$	Conjugate of dyad
∇_{kf}	Surface gradient along interface between phases k and f
$\frac{d}{dt_k}$	Substantive time derivative, $= \frac{\partial}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla$

Acronyms

(EPYTI)	Interfacial enthalpy transfer integral, defined by Eq. 6.11.15
(HTI)	Interfacial heat transfer integral, defined by Eq. 6.7.20
(IETI)	Interfacial internal energy transfer integral, defined by Eq. 6.9.17
(MTI)	Interfacial mass transfer integral, defined by Eq. 6.3.11
(MMTI)	Interfacial momentum transfer integral, defined by Eq. 6.5.21
(PTI)	Interfacial pressure transfer integral, defined by Eq. 6.5.15
(PWI)	Interfacial pressure work integral, defined by Eq. 6.7.25 $(PWI)^{(h)}$, defined by Eq. 6.11.12, is associated with enthalpy production $(PWI)^{(u)}$, defined by Eq. 6.9.11, is associated with internal energy production
(TETI)	Interfacial total energy transfer integral, defined by Eq. 6.7.31
(VDI)	Interfacial viscous dissipation integral, defined by Eq. 6.9.14
(VSTI)	Interfacial viscous stress transfer integral, defined by Eq. 6.5.16
(VWI)	Interfacial viscous stress work integral, defined by Eq. 6.7.26

All equations referred to in the above list are written for phase k. For phase f, it is necessary only to change subscript k to f for the entries that appear in the equations.

TIME- AND VOLUME-AVERAGED CONSERVATION EQUATIONS FOR MULTIPHASE FLOW

PART ONE: SYSTEM WITHOUT INTERNAL SOLID STRUCTURES

by

W. T. Sha, B. T. Chao, and S. L. Soo

EXECUTIVE SUMMARY

Multiphase flows consist of interacting phases that are dispersed randomly in space and in time. An additional complication arises from the fact that the flow region of interest often contains irregularly shaped structures. While, in principle, the intraphase conservation equations for mass, momentum, and energy, and their initial and boundary conditions can be written, the cost of detailed fluid flow and heat transfer analysis with explicit treatment of these internal structures often is prohibitive, if not impossible. In most engineering applications, all that is required is to capture the essential features of the system and to express the flow and temperature field in terms of local, global quantities while sacrificing some of the details. The present study is an attempt to achieve this goal by applying time averaging after local volume averaging.

Local volume averaging of conservation equations of mass, momentum, and energy for a multiphase system yields equations in terms of local-volume-averaged products of density, velocity, energy, stresses, and field forces, together with interface transfer integrals. These averaging relations are subject to the following length scale restrictions:

$$d \ll \ell \ll L,$$

where d is a characteristic length of the pores or dispersed phases, ℓ is a characteristic length of the averaging volume, and L is a characteristic length of the physical system.

Solutions of local-volume-averaged conservation equations call for expressing these local volume-averaged products in terms of products of averages. In nonturbulent flows, this can be achieved by expressing the "point" variable as the sum of its intrinsic volume average and a spatial deviation. In turbulent flows, the same can be achieved via subsequent time averaging over a duration T such that

$$\tau_{HF} \ll T \ll \tau_{LF},$$

where τ_{HF} is a characteristic time of high-frequency fluctuation, and τ_{LF} is a characteristic time of low-frequency fluctuation. In this case, an instantaneous "point" variable ψ_k of phase k is decomposed into a low-frequency component ψ_{kLF} and a high-frequency component ψ_k', similar to Reynolds analysis of turbulent flow. The low-frequency component consists of the sum of the local intrinsic volume average $\langle\psi_k\rangle_{LF}$ and its local spatial deviation $\tilde{\psi}_{kLF}$. Time averaging then reduces the volume averaged products to products of

averages plus terms representing eddy and dispersive diffusivities of mass, Reynolds and dispersive stresses, and eddy and dispersive conductivities of heat, etc. These terms arise from both high-frequency fluctuations and local spatial deviations. This procedure of time averaging after local volume averaging leads to a set of differential-integral equations of conservation for multiphase flow. If the flow conditions are such that the local averaging volume can be made sufficiently small that $\tilde{\psi}_{kLF}$ can be ignored, and in addition, the contributions due to high-frequency fluctuations of local volume fraction and fluid density are also negligible, then the rigorously derived set of equations presented in this report reduces, as approximations, to the various forms that are currently "accepted" for thermal hydraulic analysis of nuclear reactors and systems involving two-phase flow in general.

The sequence of performing local volume averaging and time averaging cannot be chosen arbitrarily. Local volume averaging should first be performed in order to preserve the different dynamic phases such as bubbles or drops of different sizes. Time averaging of the phasic conservation equations from the very beginning will remove the distinction of dynamic phases, unless suitable conditional averaging is used. Time averaging leads to fraction of residence time of a phase rather than volume fraction of a phase.

Part One of this study presents the detailed derivation of a set of conservation equations of mass, momentum, and energy for a multiphase system without internal solid structures via time and volume averaging. Similar derivation will be presented in Part Two (a separate publication), which treats systems having stationary internal solid structures, for which use is made of the concepts of volume and directional surface porosities, distributed resistance, and distributed heat source or sink. The concept of directional surface porosities, which is new, greatly facilitates modeling anisotropic flow and temperature field in such systems. Specific attention is given to the numerical computation of flow and temperature fields.

1. INTRODUCTION

Multiphase flows consist of interacting phases that are dispersed randomly in space and in time. Additional complications arise from the fact that the flow system of interest often contains irregularly shaped structures. While, in principle, the intraphase conservation equations for mass, momentum, and energy, and their associated initial and boundary conditions can be written, the problem is far too complicated to permit detailed solutions. In fact, they are seldom needed in engineering applications. A more realistic approach is to express the essential dynamics and thermodynamics of such a system in terms of local, global quantities. This may be achieved by applying some sort of averaging process, such as time averaging, space averaging, statistical averaging, etc. The present work begins with local volume averaging to be followed by time averaging. In this way, the identity of dynamic phases is preserved.

In an earlier report[1], the local volume-averaged transport equations for multiphase flow in regions containing stationary, distributed solid structures are derived. Further time averaging of these equations is presented in Ref. 2. A significant step in the development of these averaged equations is the introduction of the concept of volume porosity and directional surface porosity* associated with immersed stationary solid structures. This concept

*Referred to as directional surface permeability in Refs. 1 and 2.

greatly facilitates the treatment of flow and temperature fields in anisotropic media, and significant savings in computational effort are realized in many cases. However, recently we noted that certain assumptions introduced in Ref. 2 regarding the decomposition of the point values of the dependent variables, such as density, velocity, total energy, internal energy, etc., could be improved. Furthermore, the approximations introduced in evaluating the time averages of the interfacial integrals are not entirely consistent with the inclusion of the high-frequency fluctuating component of the local volume fraction in the analysis. To present a consistent set of time-volume-averaged equations, we begin by considering in Part One of this report a system that is without stationary internal structures. The governing conservation equations for multiphase flow derived here via time-volume averaging are subject only to the length scale restrictions inherent in the local volume averaging theorems used[3], and the time scale restrictions prescribed in Ref. 2.

In Part Two of this report, which will be issued as a separate document, the conservation equations of mass, momentum, and energy for multiphase systems with stationary internal solid structures via time-volume averaging will be presented. Advantage is taken of the use of volume porosity, directional surface porosity, distributed resistance, and distributed heat source or sink. The concept of directional surface porosity is new and has significant practical utility in modeling flow and temperature fields in anisotropic media.

This report--Parts One and Two--supercedes all our previous work reported in Refs. 1 and 2.

2. SIGNIFICANCE OF PHASE CONFIGURATIONS IN MULTIPHASE FLOW

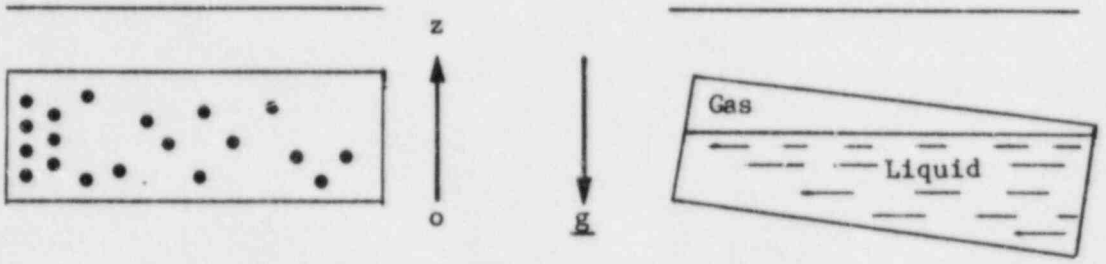
The configuration of phases plays a major role in determining the dynamics of multiphase flows and the concomitant heat and mass transport processes when they occur. This is illustrated in Fig. 1 for the two extreme cases of the highly dispersed flow and the ideally stratified flow which, by definition, has a plane interface. The figure is, to a large extent, self-explanatory. It may be noted that the mixture velocity U_m is based on the barycentric frame of reference. Only simple arithmetic is required to demonstrate that if the Bernoulli relationship for the ideal mixture in highly dispersed flow is written as Eq. 2.1, then that for the individual phase must be given by Eq. 2.2. For the ideally stratified flow, the Bernoulli relationship for the individual phase is given by Eq. 2.3. It follows then, that for the mixture is given by Eq. 2.4. The corresponding Bernoulli equations for other systems, such as bubbly flow, annular wavy flow with dispersed liquid, intermittent flow, stratified wavy flow, etc., are far more complex.

3. AVERAGING RELATIONS

3.1 Preliminaries

For convenience of the discussion to be presented in Part Two, we consider a general flow system occupying a region as illustrated in Fig. 2. The flow system coincides with the constant local averaging volume v, which is

Highly Dispersed Flow	Ideally Stratified Flow



- Gradient of volume fraction

  $\nabla\alpha \propto \nabla$(conc.) — $\nabla\alpha$ is of no significance

- Finite diffusivity, D
- Diffusion velocity, $D\nabla\alpha$

  } Concept of diffusivity is irrelevant

- Wave propagation
  Existence of speed of sound in the mixture — No common speed of sound
  Common characteristics — Individual characteristics of phases
  Transfer of inertia force across interface — Plane interface; no inertia force transferred across interface

- Bernoulli relationships for steady, incompressible, inviscid, one-dimensional flow

$$\rho_m = \sum_k \alpha_k \rho_k, \quad \alpha_k\rho_k \text{ is density of phase k based on } \underline{\text{mixture volume}}$$

$$\rho_m U_m = \sum_k \alpha_k \rho_k U_k, \qquad \boxed{\rho_m U_m^2 \neq \sum_k \alpha_k \rho_k U_k^2}$$

$$P_m = \sum_k \alpha_k P_k$$

Ideal Mixture

$$(1/2)\rho_m U_m^2 + P_m + \rho_m gz = \text{Constant} \qquad (2.1)$$

Individual Phase

$$(1/2)\rho_k U_k^2 + P_k + \rho_k gz = \text{Constant} \qquad (2.3)$$

Individual Phase

$$(1/2)\rho_k U_k^2 - (1/2)\rho_k (U_k - U_m)^2 + P_k + \rho_k gz = \text{Constant} \qquad (2.2)$$

Mixture

$$(1/2)\rho_m U_m^2 + (1/2)\sum_k \alpha_k \rho_k (U_k - U_m)^2 + P_m + \rho_m gz = \text{Constant} \qquad (2.4)$$

Fig. 1. Significance of Phase Configurations in Multiphase Flows

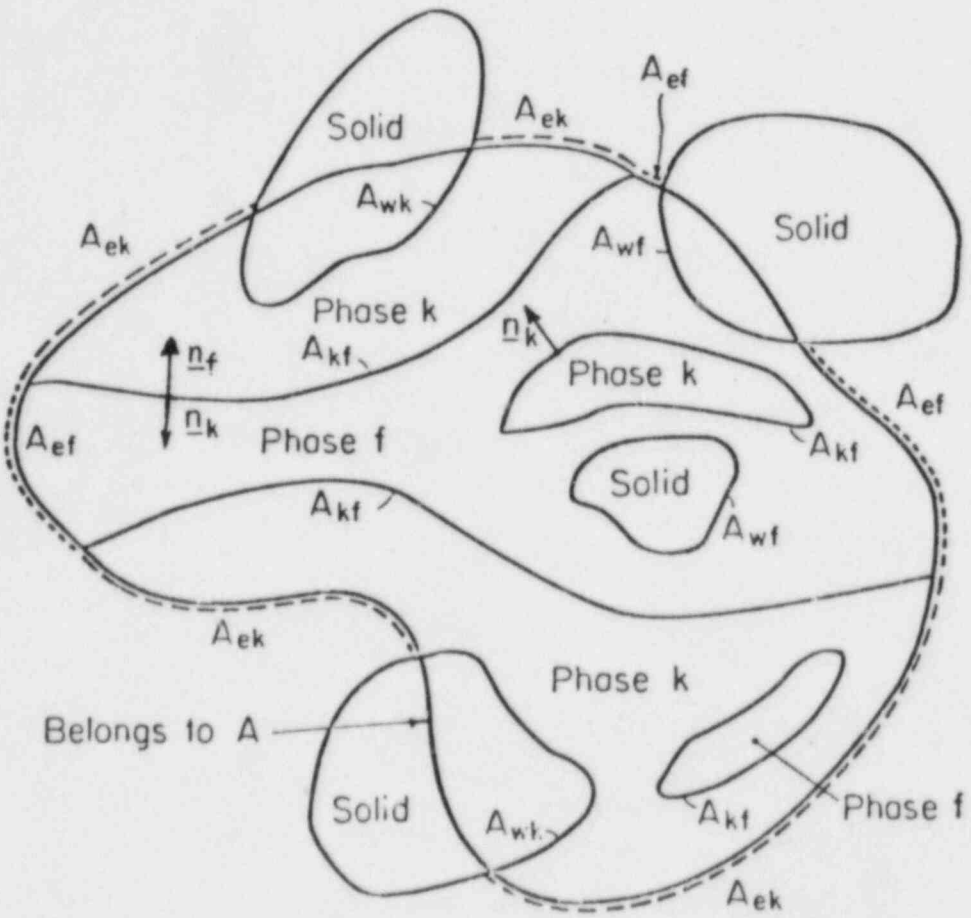


Fig. 2. Multiphase System with Dispersed, Stationary Solid Structures (Local Averaging Volume v with Enveloping Surface A)

invariant in both space and time, and its orientation relative to some inertial frame of reference is fixed. It has an enveloping surface of area A with unit outward normal $\underline{n}$. The region consists of a partially and/or totally immersed, fixed solid phase w and a fluid mixture with phase k and other phase or phases f flowing through the region. Phase k has a variable volume v_k with total interfacial area A_k in v. A portion of A_k is made of fluid-fluid interface A_{kf} and the remainder is fluid-solid interface A_{kw}. The unit normal vector $\underline{n}_k$ of A_k is always drawn outwardly from phase k, regardless whether it is associated with A_{kf} or A_{kw}. The local velocity of phase k is $\underline{U}_k$ and that of the interface $\underline{W}_k$. On A_{kw}, $\underline{W}_k$ vanishes except when there is a chemical reaction or if the solid is porous and fluid is passing through the pores. It is seen that the following relations hold:

a. Volume of fluid mixture:

$$v_m = \sum_k v_k \, . \tag{3.1.1}$$

b. Local averaging volume:

$$v = v_m + v_w, \tag{3.1.2}$$

where v_w is the total volume of the dispersed, stationary solid structures in v.

c. Volume porosity:

$$\gamma_v = v_m/v = 1 - (v_w/v), \tag{3.1.3}$$

which is a constant for a given v.

d. Volume fraction of phase k in fluid mixture:

$$\alpha_k = v_k/v_m, \tag{3.1.4}$$

which is a dependent variable.

3.2 Local Volume Average and Intrinsic Volume Average

For any intensive property ψ_k associated with phase k of the fluid mixture m, be it a scalar, vector, or tensor, the local volume average of ψ_k is defined by

$$^3\langle\psi_k\rangle = \frac{1}{v}\int_{v_k} \psi_k \, dv = \gamma_v \frac{1}{v_m}\int_{v_k} \psi_k \, dv \, . \tag{3.2.1}$$

The volume average over the fluid mixture is

$$^{3m}\langle\psi_k\rangle = \frac{1}{v_m}\int_{v_k} \psi_k \, dv = \alpha_k \frac{1}{v_k}\int_{v_k} \psi_k \, dv \, , \tag{3.2.2}$$

which has often been referred to as the phase average, and the intrinsic volume average for phase k is

$$^{3i}\langle\psi_k\rangle = \frac{1}{v_k}\int_{v_k} \psi_k \, dv \, . \tag{3.2.3}$$

These averages are related according to

$$^{3}\langle\psi_k\rangle = \gamma_v \, ^{3m}\langle\psi_k\rangle = \gamma_v\alpha_k \, ^{3i}\langle\psi_k\rangle \, . \tag{3.2.4}$$

In the absence of immersed solids, $v_m = v$, $\gamma_v = 1$, and hence,

$$^{3}\langle\psi_k\rangle = ^{3m}\langle\psi_k\rangle = \alpha_k \, ^{3i}\langle\psi_k\rangle \, . \tag{3.2.5}$$

The conservation equations presented in Part One pertain to this special case.

It should be noted that the volume averages, $^{3}\langle\psi_k\rangle$, $^{3m}\langle\psi_k\rangle$, and $^{3i}\langle\psi_k\rangle$, are defined <u>everywhere</u> in the space under consideration, not just in the space occupied by phase k. They are mathematically well behaved functions. It is easy to demonstrate that the following relations are valid:

$$^{3}\langle\,^{3}\langle\psi_k\rangle\rangle = \,^{3}\langle\psi_k\rangle, \quad ^{3m}\langle\,^{3m}\langle\psi_k\rangle\rangle = \,^{3m}\langle\psi_k\rangle, \quad ^{3i}\langle\,^{3i}\langle\psi_k\rangle\rangle = \,^{3i}\langle\psi_k\rangle \, ,$$

$$^{3m}\langle\,^{3i}\langle\psi_k\rangle\rangle = \alpha_k \, ^{3i}\langle\psi_k\rangle, \quad ^{3}\langle\,^{3i}\langle\psi_k\rangle\rangle = \gamma_v\alpha_k \, ^{3i}\langle\psi_k\rangle \, . \tag{3.2.6}$$

An example of the physical meanings of the three volume averages defined in Eqs. 3.2.1 through 3.2.3 can be obtained by applying them to a scalar such as fluid density ρ_k. Thus,

$$^{3i}\langle\rho_k\rangle = \frac{1}{v_k}\int_{v_k} \rho_k \, dv = \bar{\rho}_k \, , \tag{3.2.7a}$$

where $\bar{\rho}_k$ is the mean density of the material constituting phase k in v_k,

$$^{3m}\langle\rho_k\rangle = \alpha_k \, ^{3i}\langle\rho_k\rangle = \alpha_k \bar{\rho}_k \, , \tag{3.2.7b}$$

and

$$^{3}\langle\rho_k\rangle = \gamma_v\alpha_k\bar{\rho}_k \, . \tag{3.2.7c}$$

If we set $\psi_k = 1$, then ${}^{3i}\langle 1\rangle = 1$, ${}^{3m}\langle 1\rangle = \alpha_k$, and ${}^{3}\langle 1\rangle = \gamma_v \alpha_k$.

Similarly, for any property ψ associated with the fluid mixture, be it a scalar, vector, or tensor, the local volume average of ψ is defined by

$$ {}^{3}\langle \psi \rangle = \frac{1}{v} \int_{v_m} \psi \, dv = \gamma_v \frac{1}{v_m} \int_{v_m} \psi \, dv \, . \tag{3.2.8} $$

The volume average based on v_m is clearly the intrinsic average for the mixture; hence,

$$ {}^{3m}\langle \psi \rangle = \frac{1}{v_m} \int_{v_m} \psi \, dv = {}^{3i}\langle \psi \rangle \, , \tag{3.2.9} $$

and

$$ {}^{3}\langle \psi \rangle = \gamma_v \, {}^{3m}\langle \psi \rangle = \gamma_v \, {}^{3i}\langle \psi \rangle \, . \tag{3.2.10} $$

When $\gamma_v = 1$, the three averages ${}^{3}\langle \psi \rangle$, ${}^{3m}\langle \psi \rangle$, and ${}^{3i}\langle \psi \rangle$ are identical.

3.3 Local Area Average and Intrinsic Area Average

The various area averages of ψ_k are defined in a manner similar to the volume averages. The local area average of ψ_k is defined by

$$ {}^{2}\langle \psi_k \rangle = \frac{1}{A} \int_{A_{ek}} \psi_k \, dA = \gamma_A \frac{1}{A_e} \int_{A_{ek}} \psi_k \, dA \, , \tag{3.3.1} $$

in which A_e is the total free flow area available for the fluid mixture to enter or to exit from the averaging volume v, and A_{ek} is that allotted to phase k. The surface porosity γ_A is defined by

$$ \gamma_A = \frac{A_e}{A} \, , \tag{3.3.2} $$

i.e., the fraction of the enveloping surface A through which the fluid mixture flows.

The area average of ψ_k over the total free flow area A_e is

$$ {}^{2m}\langle \psi_k \rangle = \frac{1}{A_e} \int_{A_{ek}} \psi_k \, dA = \frac{A_{ek}}{A_e} \frac{1}{A_{ek}} \int_{A_{ek}} \psi_k \, dA \, , \tag{3.3.3} $$

and its intrinsic area average is

$$^{2i}\langle\psi_k\rangle = \frac{1}{A_{ek}} \int_{A_{ek}} \psi_k \, dA \, . \tag{3.3.4}$$

Clearly,

$$^{2}\langle\psi_k\rangle = \gamma_A \, ^{2m}\langle\psi_k\rangle = \gamma_A \frac{A_{ek}}{A_e} \, ^{2i}\langle\psi_k\rangle \, , \tag{3.3.5}$$

in which A_{ek}/A_e is the fraction of the free flow area allotted to phase k.

Similar expressions can be written for any property ψ associated with the fluid mixture. The results are:

$$^{2}\langle\psi\rangle = \frac{1}{A} \int_{A_e} \psi \, dA \, , \tag{3.3.6}$$

$$^{2m}\langle\psi\rangle = \frac{1}{A_e} \int_{A_e} \psi \, dA = \, ^{2i}\langle\psi\rangle \, , \tag{3.3.7}$$

and

$$^{2}\langle\psi\rangle = \gamma_A \, ^{2m}\langle\psi\rangle = \gamma_A \, ^{2i}\langle\psi\rangle \, . \tag{3.3.8}$$

The meaning of A_{ek}/A_e in Eq. 3.3.5 can be seen by examining the mass flux at a bounding surface of a local averaging volume in the form of a rectangular parallelopiped $\Delta x \Delta y \Delta z$. Consider, for example, the mass flow rate of a mixture of two phases k and f through area $\Delta A_{e,x}$ which may be a portion of $\Delta A_x (= \Delta y \Delta z)$. Thus, we write $\Delta A_{e,x} = \gamma_{Ax} \Delta A_x$. Clearly,

$$\rho_m U_{mx} \Delta A_{e,x} = \rho_k U_{kx} \Delta A_{ek,x} + \rho_f U_{fx} \Delta A_{ef,x} \, , \tag{3.3.9}$$

where U_{mx}, U_{kx}, etc. denote velocity components along the x-axis.

Since

$$\rho_m U_{mx} = \alpha_k \rho_k U_{kx} + \alpha_f \rho_f U_{fx} \, , \tag{3.3.10}$$

it follows that

$$\frac{\Delta A_{ek,x}}{\Delta A_{e,x}} = \alpha_k, \text{ and } \frac{\Delta A_{ef,x}}{\Delta A_{e,x}} = \alpha_f \, . \tag{3.3.11a}$$

Using the same reasoning, one may show that for flow in the y direction and through $\Delta A_{e,y}$,

$$\frac{\Delta A_{ek,y}}{\Delta A_{e,y}} = \alpha_k, \text{ and } \frac{\Delta A_{ef,y}}{\Delta A_{e,y}} = \alpha_f \ , \tag{3.3.11b}$$

and for flow through $\Delta A_{e,z}$,

$$\frac{\Delta A_{ef,z}}{\Delta A_{e,z}} = \alpha_k, \text{ and } \frac{\Delta A_{ef,z}}{\Delta A_{e,z}} = \alpha_f \ . \tag{3.3.11c}$$

The foregoing results are valid for approximating a homogeneous nonstructural medium, as has been pointed out by Whitaker[3]. Strictly speaking, they are applicable only to a highly dispersed system. The length scale restrictions of the local volume-average theorems developed by Whitaker[3] are consistent with these approximations.

3.4 Local Volume Averaging Theorems and their Length Scale Restrictions

The local volume averages of the spatial and time derivatives of a fluid property ψ_k, which may be a scalar, vector, or tensor, have been given by Whitaker[3,4], Slattery[5], Anderson and Jackson[6], Gray and Lee[7], and others. They are related to the corresponding derivatives of the averages and an interfacial area integral according to the following relations:

$$^3\langle \nabla \psi_k \rangle = \nabla \, ^3\langle \psi_k \rangle + v^{-1} \int_{A_k} \psi_k \, \underline{n}_k \, dA \ , \tag{3.4.1a}$$

$$^3\langle \nabla \cdot \underline{\psi}_k \rangle = \nabla \cdot \, ^3\langle \underline{\psi}_k \rangle + v^{-1} \int_{A_k} \underline{\psi}_k \cdot \underline{n}_k \, dA \ , \tag{3.4.1b}$$

and

$$^3\left\langle \frac{\partial \psi_k}{\partial t} \right\rangle = \frac{\partial \, ^3\langle \psi_k \rangle}{\partial t} - v^{-1} \int_{A_k} \psi_k \, \underline{W}_k \cdot \underline{n}_k \, dA \ . \tag{3.4.2}$$

In the foregoing equations, A_k denotes the sum of all interfacial areas associated with phase k inside the local averaging volume v. Thus, referring to Fig. 2, A_k consists of the fluid-fluid interface A_{kf} and the fluid-solid interface A_{kw}. For a stationary, nonporous and nonreacting solid, $\underline{W}_k$ vanishes on A_{kw}.

It is important to note that these averaging relations are subject to the following length scale restrictions, first given by Whitaker[3]:

$$\boxed{d \ll \ell \ll L} \tag{3.4.3}$$

where d is a characteristic length of the dispersed phase, ℓ is a characteristic length of v, and L is that of the physical system. Therefore, the averaging volume cannot be made arbitrarily small.

Whitaker[3], Slattery[4], Gray and Lee[7] also showed that

$$\nabla \cdot {}^{3}\langle \underline{\psi}_k \rangle = v^{-1} \int_{A_{ek}} \underline{\psi}_k \cdot \underline{n}_k \, dA \, . \tag{3.4.4}$$

In the Cartesian coordinate system, $\underline{\psi}_k = \underline{i}\,\psi_{kx} + \underline{j}\,\psi_{ky} + \underline{k}\,\psi_{kz}$, where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are unit vectors in the positive x, y, and z directions, respectively. For $\forall = \Delta x \Delta y \Delta z$ centered at the point (x,y,z), Eq. 3.4.4 can be written as

$$\begin{aligned}
&\nabla \cdot {}^{3}\langle \underline{\psi}_k \rangle \\
&\simeq \frac{1}{\Delta x}\left(\frac{\Delta A_{e,x+(\Delta x/2)}}{\Delta y \Delta z} \frac{\Delta A_{ek,x+(\Delta x/2)}}{\Delta A_{e,x+(\Delta x/2)}} \frac{1}{\Delta A_{ek,x+(\Delta x/2)}} \int_{\Delta A_{ek,x+(\Delta x/2)}} \psi_{kx} \, dA_x \right. \\
&\quad \left. - \frac{\Delta A_{e,x-(\Delta x/2)}}{\Delta y \Delta z} \frac{\Delta A_{ek,x-(\Delta x/2)}}{\Delta A_{e\ x-(\Delta x/2)}} \frac{1}{\Delta A_{ek,x-(\Delta x/2)}} \int_{\Delta A_{ek,x-(\Delta x/2)}} \psi_{kx} \, dA_x \right) \\
&\quad + \frac{1}{\Delta y}\left(\frac{\Delta A_{e,y+(\Delta y/2)}}{\Delta z \Delta x} \frac{\Delta A_{ek,y+(\Delta y/2)}}{\Delta A_{e,y+(\Delta y/2)}} \frac{1}{\Delta A_{ek,y+(\Delta y/2)}} \int_{\Delta A_{ek,y+(\Delta y/2)}} \psi_{ky} \, dA_y \right. \\
&\quad \left. - \frac{\Delta A_{e,y-(\Delta y/2)}}{\Delta z \Delta x} \frac{\Delta A_{ek,y-(\Delta y/2)}}{\Delta A_{e,y-(\Delta y/2)}} \frac{1}{\Delta A_{ek,y-(\Delta y/2)}} \int_{\Delta A_{ek,y-(\Delta y/2)}} \psi_{ky} \, dA_y \right) \\
&\quad + \frac{1}{\Delta z} (\text{analogous terms}) \\
&\simeq \frac{\partial}{\partial x} \left(\gamma_{Ax} \alpha_k^{2i} \langle \psi_{kx} \rangle \right) + \frac{\partial}{\partial y} \left(\gamma_{Ay} \alpha_k^{2i} \langle \psi_{ky} \rangle \right) \\
&\quad + \frac{\partial}{\partial z} \left(\gamma_{Az} \alpha_k^{2i} \langle \psi_{kz} \rangle \right) ,
\end{aligned} \tag{3.4.5}$$

in which γ_{Ax}, γ_{Ay}, and γ_{Az} are directional surface porosities defined by

$$\gamma_{Ax} = \begin{cases} \gamma_{A,x+(\Delta x/2)} = \dfrac{\Delta A_{e,x+(\Delta x/2)}}{\Delta y \Delta z} \\ \text{or} \\ \gamma_{A,x-(\Delta x/2)} = \dfrac{\Delta A_{e,x-(\Delta x/2)}}{\Delta y \Delta z} \, , \end{cases} \tag{3.4.6a}$$

$$\gamma_{Ay} = \begin{cases} \gamma_{A,y+(\Delta y/2)} = \dfrac{\Delta A_{e,y+(\Delta y/2)}}{\Delta z \Delta x} \\ \text{or} \\ \gamma_{A,y-(\Delta y/2)} = \dfrac{\Delta A_{e,y-(\Delta y/2)}}{\Delta z \Delta x} \end{cases} , \tag{3.4.6b}$$

and similar expressions for γ_{Az}. For compactness, we write Eq. 3.4.5 in vectorial form as

$$\nabla \cdot {}^{3}\langle \underline{\psi}_k \rangle = \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{\psi}_k \rangle . \tag{3.4.7}$$

Equation 3.4.7 shall be used for all flux-related quantities in the governing time and volume-averaged conservation equations presented in Part Two.

Upon setting $\psi_k = 1$ in Eq. 3.2.1, one obtains ${}^{3}\langle 1 \rangle = \gamma_v \alpha_k$ as has been noted previously; hence, Eq. 3.4.2 gives

$$\gamma_v \frac{\partial \alpha_k}{\partial t} = v^{-1} \int_{A_k} \underline{W}_k \cdot \underline{n}_k \, dA \tag{3.4.8}$$

since γ_v is time-independent. Furthermore, Eq. 3.4.1a gives

$$\nabla \gamma_v \alpha_k = - v^{-1} \int_{A_k} \underline{n}_k \, dA , \tag{3.4.9}$$

which, upon summing up for all k's, leads to

$$\nabla \gamma_v = - v^{-1} \sum_k \int_{A_{kf}} \underline{n}_k \, dA - v^{-1} \sum_k \int_{A_{kw}} \underline{n}_k \, dA \tag{3.4.10a}$$

since $\sum_k \alpha_k = 1$. The first term on the right-hand side of Eq. 3.4.10a vanishes since, on A_{kf}--be it a closed surface such as droplets or bubbles in dispersed flows or an open-ended surface such as that in stratified flows--$\underline{n}_k = - \underline{n}_f$ for any point on A_{kf}. Physically, this must be so because γ_v is totally unrelated to A_{kf}. For solids that are completely immersed in the fluids in v, the second term also vanishes. Accordingly,

$$\nabla \gamma_v = - v^{-1} \sum_k \int_{A_{kw,\text{boundary}}} \underline{n}_k \, dA , \tag{3.4.10b}$$

where $A_{kw,boundary}$ denotes fluid-solid interface for solids that are cut through by the bounding surface A of the local averaging volume v. It should be emphasized that the validity of Eqs. 3.4.8, 3.4.9, and 3.4.10b must necessarily be subject to the length scale restriction of Eq. 3.4.3. In fact, all results given in this report are, strictly speaking, subject to that restriction.

Equation 3.4.8 can be rearranged to read

$$\frac{\partial \alpha_k}{\partial t} = \frac{1}{v_m} \int_{A_k} \underline{W}_k \cdot \underline{n}_k \, dA \ , \tag{3.4.11}$$

for which a physical interpretation can be readily obtained. Consider, for instance, phase k to be expanding bubbles in a liquid. Then the surface integral in Eq. 3.4.11 simply means that the time rate of increase of volume of phase k in the mixture which, upon dividing by the mixture volume v_m, gives the time rate of increase of α_k. A physical interpretation of Eq. 3.4.9 for $\gamma_v = 1$ is given in Appendix A.

To conclude this section, we reiterate that for flux related to quantity $\underline{\psi}_k$, we have on one hand

$$\nabla \cdot {}^{3}\langle \underline{\psi}_k \rangle = \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{\psi}_k \rangle \ , \tag{3.4.12}$$

as has been demonstrated. On the other hand, for a vector $\underline{\psi}_k$ that is not flux related, we have

$$\nabla \cdot {}^{3}\langle \underline{\psi}_k \rangle = \nabla \cdot \gamma_v \alpha_k \, {}^{3i}\langle \underline{\psi}_k \rangle \ . \tag{3.4.13}$$

For any scalar intensive property ψ_k, the corresponding relation is

$$\nabla \, {}^{3}\langle \psi_k \rangle = \nabla \, \gamma_v \, \alpha_k \, {}^{3i}\langle \psi_k \rangle \ . \tag{3.4.14}$$

4. PHASIC CONSERVATION EQUATIONS AND INTERFACE BALANCE EQUATIONS

4.1 Phasic Conservation Equations

The equations of conservation for a pure phase are given by continuum mechanics. While a "pure" phase commonly refers to one physical phase, such as vapor, liquid, or solid, it also includes certain nonreactive mixtures, such as room atmosphere or an aqueous solution of glycerine. The identification of a multiphase system is best made in terms of its dynamic phases according to their different dynamic responses[8], despite the fact that they may be of the same material. For a pure phase k, the equations of continuity, momentum, and total energy are, respectively:

$$(\partial \rho_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k) = 0 \qquad (4.1.1)$$

$$(\partial \rho_k \underline{U}_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k \underline{U}_k) = - \nabla P_k + \nabla \cdot \underline{\underline{\tau}}_k + \rho_k \underline{f} \qquad (4.1.2)$$

$$(\partial \rho_k E_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k E_k) = - \nabla \cdot \underline{U}_k P_k + \nabla \cdot (\underline{U}_k \cdot \underline{\underline{\tau}}_k)$$

$$+ \rho_k \underline{U}_k \cdot \underline{f} - \nabla \cdot \underline{J}_{qk} + J_{Ek} , \qquad (4.1.3)$$

where ρ_k is the density of fluid in pure phase k, $\underline{U}_k$ is its velocity, P_k is the static pressure, $\underline{f}$ is the field force per unit mass which is taken to be a constant in the present study, $\underline{\underline{\tau}}_k$ is the viscous stress tensor, E_k is the total energy per unit mass, $\underline{J}_{qk}$ is the heat flux vector, and J_{Ek} is the heat source per unit volume inside phase k. By definition, $E_k = u_k + \underline{U}_k \cdot \underline{U}_k/2$, with u_k being the internal energy per unit mass. Alternatively, the energy equation may be expressed in terms of u_k or enthalpy per unit mass h_k:

$$\frac{\partial \rho_k u_k}{\partial t} + \nabla \cdot (\rho_k \underline{U}_k u_k) = - P_k \nabla \cdot \underline{U}_k - \nabla \cdot \underline{J}_{qk} + J_{Ek} + \underline{\underline{\tau}}_k : \nabla \underline{U}_k . \qquad (4.1.4)$$

The double dot in the last term denotes the scalar product of two second-order tensors and is usually represented as ϕ_k, the dissipation rate per unit volume of phase k.

$$\frac{\partial (\rho_k h_k)}{\partial t} + \nabla \cdot (\rho_k \underline{U}_k h_k) = \frac{dP_k}{dt} - \nabla \cdot \underline{J}_{qk} + J_{Ek} + \phi_k , \qquad (4.1.5)$$

in which the substantive derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{U}_k \cdot \nabla . \qquad (4.1.6)$$

4.2 Interfacial Balance Equations

The simplest case of the fluid-fluid interface is one of zero thickness. The mass, momentum, and total energy balances at the interface A_{kf} (between phases k and f, Fig. 2) are given by

Mass Balance:

$$\rho_k (\underline{U}_k - \underline{W}_{kf}) \cdot \underline{n}_k + \rho_f (\underline{U}_f - \underline{W}_{fk}) \cdot \underline{n}_f = 0 . \qquad (4.2.1)$$

Momentum Balance (Effect of changes in mean curvature ignored):

$$- \nabla_{kf} \sigma_{kf} + 2\sigma_{kf} H_{kf} \underline{n}_k - \rho_k \underline{U}_k (\underline{U}_k - \underline{W}_{kf}) \cdot \underline{n}_k - \rho_f \underline{U}_f (\underline{U}_f - \underline{W}_{fk}) \cdot \underline{n}_f$$

$$+ (- \underline{\underline{I}} P_k + \underline{\underline{\tau}}_k) \cdot \underline{n}_k + (- \underline{\underline{I}} P_f + \underline{\underline{\tau}}_f) \cdot \underline{n}_f = 0 . \qquad (4.2.2)$$

Total Energy Balance (Capillary energy ignored):

$$\rho_k E_k(\underline{U}_k - \underline{W}_{kf}) \cdot \underline{n}_k + \underline{J}_{qk} \cdot \underline{n}_k + \rho_f E_f(\underline{U}_f - \underline{W}_{fk}) \cdot \underline{n}_f + \underline{J}_{qf} \cdot \underline{n}_f$$

$$- \underline{U}_k \cdot (-\underline{\underline{I}} P_k + \underline{\underline{\tau}}_k) \cdot \underline{n}_k - \underline{U}_f \cdot (-\underline{\underline{I}} P_f + \underline{\underline{\tau}}_f) \cdot \underline{n}_f = 0 , \quad (4.2.3)$$

where $\underline{n}_k$ is the unit normal vector outward from phase k and directed along the mean curvature H_{kf}, σ_{kf} is the interfacial tension, ∇_{kf} is the surface gradient operator, and $\underline{\underline{I}}$ is the unitary tensor. The interfacial velocity $\underline{W}_{kf} = \underline{W}_{fk}$, and H_{kf} is positive when the associated radius is pointing outward. In Eq. 4.2.3, the energy associated with surface tension and the corresponding dissipation are neglected.

The internal energy and enthalpy balance equations for the interface A_{kf} are

$$\rho_k u_k(\underline{U}_k - \underline{W}_{kf}) \cdot \underline{n}_k + \underline{J}_{qk} \cdot \underline{n}_k + \rho_f u_f(\underline{U}_f - \underline{W}_{fk}) \cdot \underline{n}_f + \underline{J}_{qf} \cdot \underline{n}_f = 0 \quad (4.2.4)$$

and

$$\rho_k h_k(\underline{U}_k - \underline{W}_{kf}) \cdot \underline{n}_k - P_k(\underline{U}_k - \underline{W}_{kf}) \cdot \underline{n}_k + \underline{J}_{qk} \cdot \underline{n}_k$$

$$+ \rho_f h_f(\underline{U}_f - \underline{W}_{fk}) \cdot \underline{n}_f - P_f(\underline{U}_f - \underline{W}_{fk}) \cdot \underline{n}_f + \underline{J}_{qf} \cdot \underline{n}_f = 0 . \quad (4.2.5)$$

It may be noted that only one of Eqs. 4.2.3 through 4.2.5 is independent. Needless to say, all variables in Eqs. 4.2.1 through 4.2.5, such as density, velocity, pressure, viscous stress, total energy, internal energy, enthalpy, etc., refer to interface A_{kf}.

In principle, the coupled phasic equations should be solved for given initial conditions together with boundary conditions at the phase interfaces. Because the configuration and location of the fluid-fluid interfaces are not generally known, their detailed solutions are next to impossible. When the length scale over which the point variables undergo significant changes is small compared with that over which the knowledge of these variables is of practical interest, information of their volume-averages is all that is needed. A similar statement can be made regarding time scale considerations. To preserve the identity of the dynamic phases, local volume averaging is performed first; this is done in the following section. Time averaging of the volume-averaged equations is presented in Sec. 6.

5. LOCAL VOLUME-AVERAGED CONSERVATION EQUATIONS AND INTERFACE BALANCE EQUATIONS

In Part One, we consider the relatively simple case of a multiphase system without internal solid structures; hence, $\gamma_v = \gamma_A = 1$. Application of the local volume averaging theorems (Eqs. 3.4.1a and b, and Eq. 3.4.2) to the phasic conservation equations given in Sec. 4 leads to the following set of

local volume-averaged conservation equations for multiphase flow. These equations are rigorous and subject only to the length scale restriction, Eq. 3.4.3, which is inherent in the local volume averaging theorems. Since the details of the derivation of the local volume-averaged equations of conservation can be found in Ref. 1, only the results are listed here.

5.1 Local Volume Averaged Conservation Equations

Mass Conservation Equation

$$\frac{\partial}{\partial t}\alpha_k {}^{3i}\langle\rho_k\rangle + \nabla\cdot\alpha_k {}^{3i}\langle\rho_k \underline{U}_k\rangle = -v^{-1}\int_{A_k}\rho_k(\underline{U}_k - \underline{W}_k)\cdot\underline{n}_k\, dA\,. \qquad (5.1.1)$$

The integral on the right-hand side of Eq. 5.1 denotes the rate of total interfacial mass generation of phase k per unit volume of v. Denoting it by Γ_k, we have

$$\Gamma_k = -v^{-1}\int_{A_k}\rho_k(\underline{U}_k - \underline{W}_k)\cdot\underline{n}_k\, dA\,. \qquad (5.1.2)$$

In the absence of stationary, internal solid structures, the local averaging volume v and the volume of the fluid mixture v_m are identical.

Linear Momentum Conservation Equation

$$\frac{\partial}{\partial t}\alpha_k {}^{3i}\langle\rho_k\underline{U}_k\rangle + \nabla\cdot\alpha_k {}^{3i}\langle\rho_k\underline{U}_k\underline{U}_k\rangle = -\nabla\alpha_k {}^{3i}\langle P_k\rangle + \nabla\cdot\alpha_k {}^{3i}\langle\underline{\underline{\tau}}_k\rangle$$

$$+\alpha_k {}^{3i}\langle\rho_k\rangle\underline{f} + v^{-1}\int_{A_k}(-P_k\underline{\underline{I}}_k + \underline{\underline{\tau}}_k)\cdot\underline{n}_k dA - v^{-1}\int_{A_k}\rho_k\underline{U}_k(\underline{U}_k - \underline{W}_k)\cdot\underline{n}_k dA\,, \qquad (5.1.3)$$

in which the field force per unit mass $\underline{f}$ is taken to be constant.

Energy Conservation Equations

(a) In terms of total energy, $E_k = u_k + \frac{1}{2}\underline{U}_k\cdot\underline{U}_k$:

$$\frac{\partial}{\partial t}\alpha_k {}^{3i}\langle\rho_k E_k\rangle + \nabla\cdot\alpha_k {}^{3i}\langle\rho_k\underline{U}_k E_k\rangle = -\nabla\cdot\alpha_k {}^{3i}\langle\underline{U}_k P_k\rangle + \nabla\cdot\alpha_k {}^{3i}\langle\underline{U}_k\cdot\underline{\underline{\tau}}_k\rangle$$

$$-\nabla\cdot\alpha_k {}^{3i}\langle\underline{J}_{qk}\rangle + \alpha_k\left({}^{3i}\langle\rho_k\underline{U}_k\rangle\cdot\underline{f} + {}^{3i}\langle J_{Ek}\rangle\right) + \dot{Q}_k$$

$$+v^{-1}\int_{A_k}(-P_k\underline{U}_k + \underline{\underline{\tau}}_k\cdot\underline{U}_k)\cdot\underline{n}_k dA - v^{-1}\int_{A_k}\rho_k E_k(\underline{U}_k - \underline{W}_k)\cdot\underline{n}_k dA\,, \qquad (5.1.4)$$

where $\dot{Q}_k$ denotes the interfacial heat transfer to phase k per unit volume and time, i.e.,

$$\dot{Q}_k = - v^{-1} \int_{A_k} \underline{J}_{qk} \cdot \underline{n}_k \, dA \, . \tag{5.1.5}$$

(b) In terms of internal energy, u_k:

$$\frac{\partial}{\partial t} \alpha_k {}^{3i}\langle \rho_k u_k \rangle + \nabla \cdot \alpha_k {}^{3i}\langle \rho_k \underline{U}_k u_k \rangle = - \alpha_k {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle - \nabla \cdot \alpha_k {}^{3i}\langle \underline{J}_{qk} \rangle$$

$$+ \alpha_k \left({}^{3i}\langle J_{Ek} \rangle + {}^{3i}\langle \phi_k \rangle \right) + \dot{Q}_k - v^{-1} \int_{A_k} \rho_k u_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k dA \, , \tag{5.1.6}$$

where ϕ_k is the dissipation function given by

$$\phi_k = \underline{\underline{\tau}}_k : \nabla, \underline{U}_k \, , \tag{5.1.7}$$

in which the double dot denotes the scalar product of two second-order tensors and the comma denotes dyadic operation. ϕ_k gives the dissipation rate per unit volume of phase k due to the irreversible conversion of mechanical work into thermal energy.

(c) In terms of enthalpy, h_k:

$$\frac{\partial}{\partial t} \alpha_k {}^{3i}\langle \rho_k h_k \rangle + \nabla \cdot \alpha_k {}^{3i}\langle \rho_k \underline{U}_k h_k \rangle = \frac{\partial}{\partial t} \alpha_k {}^{3i}\langle P_k \rangle + \nabla \cdot \alpha_k {}^{3i}\langle \underline{U}_k P_k \rangle$$

$$- \alpha_k {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle - \nabla \cdot \alpha_k {}^{3i}\langle \underline{J}_{qk} \rangle + \alpha_k \left({}^{3i}\langle J_{Ek} \rangle + {}^{3i}\langle \phi_k \rangle \right) + \dot{Q}_k$$

$$+ v^{-1} \int_{A_k} P_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k dA - v^{-1} \int_{A_k} \rho_k h_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k dA \, . \tag{5.1.8}$$

We reiterate that A_k in all interfacial integrals denotes the sum of all interfacial areas associated with phase k in the local averaging volume v.

5.2 Local Volume-Averaged Interface Balance Equations

The volume-averaged interfacial balance relations can be readily obtained from Eqs. 4.2.1 through 4.2.5. They are:

Mass Balance

$$-v^{-1} \int_{A_k} \rho_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k dA = \Gamma_k = v^{-1} \int_{A_f} \rho_f (\underline{U}_f - \underline{W}_f) \cdot \underline{n}_f dA = - \Gamma_f \, , \tag{5.2.1}$$

where the interfacial velocity $\underline{W}_k$ implies $\underline{W}_{kf}$ and the interfacial velocity $\underline{W}_f$ implies $\underline{W}_{fk}$. Since the interface has zero thickness, $\underline{W}_k = \underline{W}_f$ at any location of the interface.

Linear Momentum Balance

$$v^{-1}\int_{A_k}\left(-P_k\,\underline{\underline{I}}+\underline{\underline{\tau}}_k\right)\cdot\underline{n}_k dA - v^{-1}\int_{A_k}\rho_k\underline{U}_k\left(\underline{U}_k-\underline{W}_k\right)\cdot\underline{n}_k dA$$

$$= - v^{-1}\int_{A_f}\left(-P_f\,\underline{\underline{I}}+\underline{\underline{\tau}}_f\right]\cdot\underline{n}_f dA + v^{-1}\int_{A_f}\rho_f\underline{U}_f\left(\underline{U}_f-\underline{W}_f\right)\cdot\underline{n}_f dA$$

$$-\int_{A_k}\left(-\nabla_{kf}\sigma_{kf}+2\sigma_{kf}H_{kf}\underline{n}_k\right)dA\,. \qquad (5.2.2)$$

For bubbles and droplets, the last integral in Eq. 5.2.2 can be expressed in terms of capillary pressure difference

$$\int_{A_k}\left(P_{ck}-P_{cf}\right)\underline{n}_k dA = \int_{A_k}\left(-\nabla_{kf}\sigma_{kf}+2\sigma_{kf}H_{kf}\underline{n}_k\right)dA\,. \qquad (5.2.3)$$

An equivalent expression can be written in terms of A_f, recognizing that for the interface between phase k and phase f, $A_k = A_f$, $\underline{n}_k = -\underline{n}_f$, and $H_{kf} = -H_{fk}$.

Total Energy Balance (capillary energy ignored)

$$v^{-1}\int_{A_k}\left(-P_k\underline{U}_k+\underline{\underline{\tau}}_k\cdot\underline{U}_k\right)\cdot\underline{n}_k dA - v^{-1}\int_{A_k}\rho_k E_k\left(\underline{U}_k-\underline{W}_k\right)\cdot\underline{n}_k dA$$

$$- v^{-1}\int_{A_k}\underline{J}_{qk}\cdot\underline{n}_k dA = - v^{-1}\int_{A_f}\left(-P_f\underline{U}_f+\underline{\underline{\tau}}_f\cdot\underline{U}_f\right)\cdot\underline{n}_f dA$$

$$+ v^{-1}\int_{A_f}\rho_f E_f\left(\underline{U}_f-\underline{W}_f\right)\cdot\underline{n}_f dA + v^{-1}\int_{A_f}\underline{J}_{qf}\cdot\underline{n}_f dA\,. \qquad (5.2.4)$$

Internal Energy Balance (dissipation and reversible work ignored)

$$- v^{-1}\int_{A_k}\rho_k u_k\left(\underline{U}_k-\underline{W}_k\right)\cdot\underline{n}_k dA - v^{-1}\int_{A_k}\underline{J}_{qk}\cdot\underline{n}_k dA$$

$$= v^{-1}\int_{A_f}\rho_f u_f\left(\underline{U}_f-\underline{W}_f\right)\cdot\underline{n}_f dA + v^{-1}\int_{A_f}\underline{J}_{qf}\cdot\underline{n}_f dA\,. \qquad (5.2.5)$$

Enthalpy Balance (capillary energy ignored)

$$v^{-1}\int_{A_k}P_k\left(\underline{U}_k-\underline{W}_k\right)\cdot\underline{n}_k dA - v^{-1}\int_{A_k}\rho_k h_k\left(\underline{U}_k-\underline{W}_k\right)\cdot\underline{n}_k dA$$

$$- v^{-1} \int_{A_k} \underline{J}_{qk} \cdot \underline{n}_k dA = - v^{-1} \int_{A_f} P_f(\underline{U}_f - \underline{W}_f) \cdot \underline{n}_f dA$$

$$+ v^{-1} \int_{A_f} \rho_f h_f(\underline{U}_f - \underline{W}_f) \cdot \underline{n}_f dA + v^{-1} \int_{A_f} \underline{J}_{qf} \cdot \underline{n}_f dA \ . \qquad (5.2.6)$$

6. TIME AVERAGING OF LOCAL VOLUME-AVERAGED CONSERVATION EQUATIONS

6.1 Basic Postulate

The local volume-averaged equations given in the preceding section are differential-integral equations. Before they can be used either for further analysis or for numerical computation, it is necessary (a) to express the volume averages of the product of the dependent variables in terms of the product of their volume averages, and (b) to evaluate the interfacial transport integrals which depend on the local values of the dependent variables at every point on the interface. To this end, we postulate that a point-dependent variable ψ_k for phase k can be expressed as the sum of its local intrinsic volume average $^{3i}\langle\psi_k\rangle$ and a deviation $\tilde{\psi}_k$. ψ_k can be a scalar, a vector, or a tensor. Both $^{3i}\langle\psi_k\rangle$ and $\tilde{\psi}_k$ have a low-frequency component to be denoted by the subscript LF and a high-frequency component to be denoted by a prime. Thus,

$$\psi_k = {}^{3i}\langle\psi_k\rangle + \tilde{\psi}_k \qquad (6.1.1a)$$

$$= {}^{3i}\langle\psi_k\rangle_{LF} + {}^{3i}\langle\psi_k\rangle' + \tilde{\psi}_{kLF} + \tilde{\psi}_k' = {}^{3i}\langle\psi_k\rangle_{LF} + \tilde{\psi}_{kLF} + {}^{c}\psi_k' \ , \qquad (6.1.1b)$$

where

$${}^{c}\psi_k' = {}^{3i}\langle\psi_k\rangle' + \tilde{\psi}_k' \ . \qquad (6.1.2)$$

The superscript c is a reminder that ${}^{c}\psi_k'$ is a composite of two high-frequency fluctuations. The low-frequency component refers to one that is a slowly varying function of time, including the time-independent limiting case. The high-frequency component varies rapidly with time.

The time that characterizes the low-frequency component is of the order of

$$\tau_{LF} = L_c/(\Delta U)_c = \text{(characteristic dimension of the physical system)/(characteristic low-frequency speed variation at a typical location).} \qquad (6.1.3a)$$

The characteristic time of the high-frequency component is of the order of

$\tau_{HF} = \Lambda/(\text{rms } U') =$ (characertistic length scale of high-frequency fluctuation)/(root mean square of the fluctuating velocity or turbulence intensity)
$= 1/(\text{characteristic spectral frequency}).$ (6.1.3b)

When time averaging is performed, the duration T over which the averaging is to be made must satisfy the following inequality:

$$\boxed{\tau_{HF} \ll T \ll \tau_{LF}} \tag{6.1.4}$$

The spatial decomposition of the form given by Eq. 6.1.1a was first suggested by Gray[9]. When the length scale inequalities (Eq. 3.4.3) are satisfied, the length scales associated with ${}^{3i}\langle\psi_k\rangle$ and $\tilde{\psi}_k$ are separable. The same is true for ${}^{3i}\langle\psi_k\rangle_{LF}$ and $\tilde{\psi}_{kLF}$. When Eq. 6.1.4 for the time scale inequalities is satisfied, quantities with subscript LF and those denoted by a prime also are separable in the time or frequency domain. When the two characteristic times τ_{LF} and τ_{HF} overlap, such separation will not be possible. However, in practical applications, distinctions are usually feasible. Examples are: duct flow with turbulence, a bubbly liquid in turbulent motion where the bubble phase configuration responds to low frequency pressure fluctuation, and the case of impulsive motion produced by sudden break (LOCA) where high-frequency wave motion might not be important[10].

If one adopts the Reynolds hypothesis used in elementary turbulence analysis, the point instantaneous variable ψ_k can be decomposed as

$$\psi_k = {}^{t}\langle\psi_k\rangle + \psi_k' = \psi_{kLF} + \psi_k' , \tag{6.1.5}$$

where ${}^{t}\langle\psi_k\rangle$ denotes the temporal mean or a low-frequency component ψ_{kLF} and ψ_k' denotes the high-frequency fluctuating component. The time average ${}^{t}\langle\psi_k\rangle$ is defined by

$${}^{t}\langle\psi_k\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \psi_k \, dt . \tag{6.1.6}$$

Comparing Eq. 6.1.1b with Eq. 6.1.5 leads to the conclusion that

$${}^{c}\psi_k' = \psi_k' , \tag{6.1.7}$$

as one would intuitively expect. Hence, while ${}^{3i}\langle\psi_k\rangle'$ and $\tilde{\psi}_k'$ are not local entities, their sum is a point quantity. Substituting Eq. 6.1.7 into Eq. 6.1.1b gives

$$\psi_k = {}^{3i}\langle\psi_k\rangle_{LF} + \tilde{\psi}_{kLF} + \psi_k' . \tag{6.1.8}$$

We shall soon demonstrate that ${}^{3i}\langle\psi_k\rangle_{LF}$ closely approximates ${}^{3i}\langle\psi_{kLF}\rangle$. The nature of the approximation will be made clear in Sec. 6.2.

Taking the intrinsic local volume averages of Eqs. 6.1.5 and 6.1.8, one obtains:

$$ {}^{3i}\langle\psi_k\rangle = {}^{3i}\langle\psi_{kLF}\rangle + {}^{3i}\langle\psi_k'\rangle \tag{6.1.9a} $$

and

$$ {}^{3i}\langle\psi_k\rangle = {}^{3i}\langle\psi_k\rangle_{LF} + {}^{3i}\langle\tilde{\psi}_{kLF}\rangle + {}^{3i}\langle\psi_k'\rangle \ . \tag{6.1.9b} $$

Hence,

$$ {}^{3i}\langle\tilde{\psi}_{kLF}\rangle = 0 \ . \tag{6.1.10} $$

While scalars ρ_k, E_k, u_k, and h_k, vectors $\underline{U}_k$ and $\underline{J}_{qk}$, and tensor $\underline{\underline{\tau}}_k$ are to be decomposed in accordance with Eq. 6.1.8, the local volume fraction α_k of phase k should be represented by

$$ \alpha_k = \alpha_{kLF} + \alpha_k' \ , \tag{6.1.11} $$

since α_k is inherently a volume averaged quantity. Clearly, ${}^{3i}\langle\alpha_k\rangle = \alpha_k$ and ${}^{3i}\langle\alpha_k'\rangle = \alpha_k'$.

In multiphase flows, the fluid-fluid interfaces would in general not only translate, but also fluctuate. Hence, the unit normal vector to the interface would also fluctuate. Under normal circumstances, sharp changes in surface curvature would not occur due to the existence of interfacial tension. The familiar shapes of oscillating bubbles and droplets are examples[11]. However, when breakup occurs, high-frequency oscillations[12] of the total interfacial area may exist. Accordingly, we write

$$ A_k = A_{kLF} + A_k' \tag{6.1.12} $$

and the associated unit outdrawn normal vectors are $\underline{n}_{kLF}$ and $\underline{n}_k'$, respectively. Although it is permissible to ignore A_k' under certain circumstances, A_k' in Eq. 6.1.12 cannot be deleted or retained arbitrarily without simultaneous consideration of the deletion or retention of α_k'. We shall return to this point in the next section.

For reasons just given, we may also write

$$ v_k = v_{kLF} + v_k' \ . \tag{6.1.13} $$

It follows then, that

$$\alpha_{kLF} = \frac{v_{kLF}}{v} , \quad \alpha'_k = \frac{v'_k}{v} . \tag{6.1.14a,b}$$

It is pertinent to note that v_{kLF} and v'_k are not physically separable regions in v_k. In the present analysis, both v'_k and α'_k are considered small perturbations. While v'_k and α'_k, like other high-frequency fluctuating quantities, take on positive and negative values, A'_k is always positive.

The interfacial velocity $\underline{W}_k$ appears only as a point variable in the interfacial transfer integrals of the governing differential-integral conservation equations. Thus, it needs only to be decomposed as

$$\underline{W}_k = \underline{W}_{kLF} + \underline{W}'_k . \tag{6.1.15}$$

It is pertinent to note that the unit vector $\underline{n}_k$ associated with A_k is <u>not</u> the sum of $\underline{n}_{kLF}$ and $\underline{n}'_k$, i.e.,

$$\underline{n}_k \neq \underline{n}_{kLF} + \underline{n}'_k . \tag{6.1.16}$$

6.2 Some Useful Observations

Equation 3.4.9 gives, for $\gamma_v = 1$,

$$\nabla \alpha_k = - v^{-1} \int_{A_k} \underline{n}_k \, dA ,$$

which, upon introducing Eqs. 6.1.11 and 6.1.12, leads to

$$\nabla \alpha_{kLF} + \nabla \alpha'_k = - v^{-1} \int_{A_{kLF}} \underline{n}_{kLF} \, dA - v^{-1} \int_{A'_k} \underline{n}'_k \, dA , \tag{6.2.1}$$

since $\underline{n}_{kLF}$ is coherent only with A_{kLF} and $\underline{n}'_k$ is coherent only with A'_k.

By separating the low- and high-frequency components, one obtains

$$\nabla \alpha_{kLF} = - v^{-1} \int_{A_{kLF}} \underline{n}_{kLF} \, dA \tag{6.2.2a}$$

and

$$\nabla \alpha'_k = - v^{-1} \int_{A'_k} \underline{n}'_k \, dA . \tag{6.2.2b}$$

Thus,

$$^t\left\langle \int_{A'_k} \underline{n}'_k \, dA \right\rangle = 0 , \tag{6.2.3}$$

since ${}^t\langle \alpha_k' \rangle = 0$.

Likewise, when $\gamma_v = 1$ and the low- and high-frequency components are separated, Eq. 3.4.8 leads to

$$\frac{\partial \alpha_{kLF}}{\partial t} = v^{-1} \int_{A_{kLF}} \underline{W}_{kLF} \cdot \underline{n}_{kLF} \, dA \tag{6.2.4a}$$

and

$$\frac{\partial \alpha_k'}{\partial t} = v^{-1} \int_{A_{kLF}} \underline{W}_k' \cdot \underline{n}_{kLF} \, dA + v^{-1} \int_{A_k'} \left(\underline{W}_{kLF} + \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \, . \tag{6.2.4b}$$

Since

$$\left\langle \int_{A_{kLF}} \underline{W}_k' \cdot \underline{n}_{kLF} \, dA \right\rangle = 0 \, , \tag{6.2.5a}$$

and

$$\left\langle \int_{A_k'} \underline{W}_{kLF} \cdot \underline{n}_k' \, dA \right\rangle = 0 \, , \tag{6.2.5b}$$

it follows that

$$\left\langle \int_{A_k'} \underline{W}_k' \cdot \underline{n}_k' \, dA \right\rangle = 0 \, . \tag{6.2.5c}$$

Equation 6.2.5b follows from Eq. 6.2.3 since time-averaging is carried out for the interval during which $\underline{W}_{kLF}$ changes little. Thus, for any low-frequency vector $\underline{\psi}_{kLF}$, we have

$$\left\langle \int_{A_k'} \underline{\psi}_{kLF} \cdot \underline{n}_k' \, dA \right\rangle = 0 \text{ and } \left\langle \int_{A_k'} \underline{\psi}_{kLF} , \underline{n}_k' \, dA \right\rangle = 0 \, , \tag{6.2.6a,b}$$

where the comma in the integrand denotes the dyadic product.

Likewise, for any low-frequency scalar ψ_{kLF},

$$\left\langle \int_{A_k'} \psi_{kLF} \, \underline{n}_k' \, dA \right\rangle = 0 \, . \tag{6.2.6c}$$

It is interesting to note that if $A_k' = 0$, then $\nabla \alpha_k' = 0$ according to Eq. 6.2.2b. Furthermore, $\underline{W}_k'$ must also vanish since it is physically impossible to have high-frequency interfacial velocity associated with A_{kLF}. Hence, in such a case, Eq. 6.2.4b gives $\frac{\partial \alpha_k'}{\partial t} = 0$. We thus conclude that if $A_k' = 0$, α_k' must not exist because the only other possibility, α_k' = constant, is contradictory to the definition that α_k' is a fluctuating quantity.

Next, we examine the difference between $^t\langle {}^3\langle \psi_k' \rangle \rangle$ and $^t\langle {}^{3i}\langle \psi_k' \rangle \rangle$. For convenience, we write $^{t3}\langle \cdot \rangle$ for $^t\langle {}^3\langle \cdot \rangle \rangle$, and $^{t3i}\langle \cdot \rangle$ for $^t\langle {}^{3i}\langle \cdot \rangle \rangle$. By definition

$$^{3i}\langle \psi_k' \rangle = \frac{1}{v_k} \int_{v_k} \psi_k' \, dv \, . \tag{6.2.7a}$$

Denoting $\frac{v_k'}{v_{kLF}}$ by ε, and for $|\varepsilon| < 1$, we may write Eq. 6.2.7a as

$$^{3i}\langle \psi_k' \rangle = \frac{1}{v_{kLF}} (1 - \varepsilon + \varepsilon^2 - \cdots) \left(\int_{v_{kLF}} \psi_k' \, dv + \int_{v_k'} \psi_k' \, dv \right)$$

$$= \left(\frac{1}{v_{kLF}} \int_{v_{kLF}} \psi_k' \, dv \right) [1 + O(\varepsilon) + O(\varepsilon^2) + \cdots] \, . \tag{6.2.7b}$$

Since, by hypothesis, $|\varepsilon| \ll 1$, it follows that

$$^{3i}\langle \psi_k' \rangle \tilde{=} \frac{1}{v_{kLF}} \int_{v_{kLF}} \psi_k' \, dv \, . \tag{6.2.7c}$$

Hence,

$$^{t3i}\langle \psi_k' \rangle \tilde{=} 0 \, . \tag{6.2.8a}$$

Likewise, we may demonstrate that for a vector $\underline{\psi}_k'$,

$$^{t3i}\langle \underline{\psi}_k' \rangle \tilde{=} 0 \, , \tag{6.2.8b}$$

and for a tensor $\underline{\psi}'_k$,

$$ {}^{t3i}\langle \underline{\psi}'_k \rangle \tilde{=} 0 \ . \qquad (6.2.8c) $$

Now, also by definition,

$$ {}^{3}\langle \psi'_k \rangle = \frac{1}{v}\int_{v_k} \psi'_k \, dv = \frac{1}{v}\int_{v_{kLF}} \psi'_k \, dv + \frac{1}{v}\int_{v'_k} \psi'_k \, dv $$

$$ = \frac{v_{kLF}}{v}\,\frac{1}{v_{kLF}}\int_{v_{kLF}} \psi'_k \, dv + \frac{v'_k}{v}\,\frac{1}{v'_k}\int_{v'_k} \psi'_k \, dv \ . \qquad (6.2.9a) $$

Using Eq. 6.2.7c, we obtain

$$ {}^{3}\langle \psi'_k \rangle \tilde{=} \alpha_{kLF}\ {}^{3i}\langle \psi'_k \rangle + \alpha'_k \left(\frac{1}{v'_k}\int_{v'_k} \psi'_k \, dv \right) \ . \qquad (6.2.9b) $$

Thus,

$$ {}^{t3}\langle \psi'_k \rangle \tilde{=} {}^{t}\left\langle \alpha'_k \left(\frac{1}{v'_k}\int_{v'_k} \psi'_k \, dv \right) \right\rangle \ , \qquad (6.2.10) $$

which is generally nonzero.

Alternatively, we may evaluate ${}^{t3}\langle \psi'_k \rangle$ as follows. We recall that, for $\gamma_v = 1$,

$$ {}^{3}\langle \psi'_k \rangle = \alpha_k\ {}^{3i}\langle \psi'_k \rangle = \left(\alpha_{kLF} + \alpha'_k \right)\ {}^{3i}\langle \psi'_k \rangle \ . \qquad (6.2.11) $$

Hence,

$$ {}^{t3}\langle \psi'_k \rangle = \alpha_{kLF}\ {}^{t3i}\langle \psi'_k \rangle + {}^{t}\left\langle \alpha'_k\ {}^{3i}\langle \psi'_k \rangle \right\rangle $$

$$ \tilde{=} {}^{t}\left\langle \alpha'_k\ {}^{3i}\langle \psi'_k \rangle \right\rangle \ , \qquad (6.2.12a) $$

since ${}^{t3i}\langle \psi'_k \rangle \tilde{=} 0$. It is thus seen that α'_k and ${}^{3i}\langle \psi'_k \rangle$ are correlated in time. Because of the fact that $\alpha'_k = {}^{3i}\langle \alpha'_k \rangle$, Eq. 6.2.12a may be written as

$$ {}^{t3}\langle\psi_k'\rangle = {}^{t3i}\langle\alpha_k'\psi_k'\rangle \ . \qquad (6.2.12b) $$

Comparing Eq. 6.2.9b and Eq. 6.2.11 gives

$$ {}^{3i}\langle\psi_k'\rangle = \frac{1}{v_k'}\int_{v_k'} \psi_k' \, dv \ . \qquad (6.2.13) $$

Accordingly, we obtain, by using Eq. 6.2.8a,

$$ {}^{t}\left\langle \frac{1}{v_k'}\int_{v_k'} \psi_k' \, dv \right\rangle \tilde{=} 0 \ . \qquad (6.2.14) $$

In view of the defining integral for ${}^{3i}\langle\psi_k'\rangle$ given by Eq. 6.2.7a, we also have the interesting result

$$ \frac{1}{v_k}\int_{v_k} \psi_k' \, dv = \frac{1}{v_k'}\int_{v_k'} \psi_k' \, dv \ . \qquad (6.2.15) $$

It is also of interest to compare ${}^{3i}\langle\psi_{kLF}\rangle$ and ${}^{3i}\langle\psi_k\rangle_{LF}$, the latter being the low-frequency component of ${}^{3i}\langle\psi_k\rangle$. Since

$$ \begin{aligned} {}^{3i}\langle\psi_{kLF}\rangle &= \frac{1}{v_k}\int_{v_k} \psi_{kLF} \, dv \\ &= \frac{1}{v_{kLF}}\left(1 - \varepsilon + \varepsilon^2 - \cdots\right)\left(\int_{v_{kLF}} \psi_{kLF} \, dv + \int_{v_k'} \psi_{kLF} \, dv\right) \\ &= \left(\frac{1}{v_{kLF}}\int_{v_{kLF}} \psi_{kLF} \, dv\right)\left[1 + O(\varepsilon) + O(\varepsilon^2) + \cdots\right] \ , \end{aligned} \qquad (6.2.16) $$

and

$$ \begin{aligned} {}^{3i}\langle\psi_k\rangle_{LF} &= \left(\frac{1}{v_k}\int_{v_k} \psi_k \, dv\right)_{LF} \\ &= \left[\frac{1}{v_{kLF}}\left(1 - \varepsilon + \varepsilon^2 - \cdots\right)\left(\int_{v_{kLF}} \psi_{kLF} \, dv + \int_{v_k'} \psi_{kLF} \, dv\right.\right. \end{aligned} $$

$$+ \int_{v_{kLF}} \psi_k' \, dv + \int_{v_k'} \psi_k' \, dv \Big) \Big]_{LF}$$

$$= \frac{1}{v_{kLF}} \int_{v_{kLF}} \psi_{kLF} \, dv \, , \qquad (6.2.17)$$

it follows that

$$^{3i}\langle \psi_{kLF} \rangle = {}^{3i}\langle \psi_k \rangle_{LF} \left[1 + 0(\varepsilon) + 0(\varepsilon^2) + \cdots \right] \tilde{=} {}^{3i}\langle \psi_k \rangle_{LF} \, , \qquad (6.2.18)$$

since $|\varepsilon| << 1$. In the present report, $^{3i}\langle \psi_{kLF} \rangle$ and $^{3i}\langle \psi_k \rangle_{LF}$ are considered identical.

We shall also have occasion to consider integrals of the type

$$v^{-1} \int_{A_k} \breve{\psi}_{kLF} \, \underline{n}_k \, dA \quad \text{and} \quad v^{-1} \int_{A_k} \breve{\underline{\psi}}_{kLF} \cdot \underline{n}_k \, dA \, .$$

Since

$$^{3}\langle \nabla \, \breve{\psi}_{kLF} \rangle = \nabla \, {}^{3}\langle \breve{\psi}_{kLF} \rangle + v^{-1} \int_{A_k} \breve{\psi}_{kLF} \, \underline{n}_k \, dA \, , \qquad (6.2.19)$$

and $^{3}\langle \breve{\psi}_{kLF} \rangle = 0$, one immediately has

$$v^{-1} \int_{A_k} \breve{\psi}_{kLF} \, \underline{n}_k \, dA = \alpha_k \, {}^{3i}\langle \nabla \, \breve{\psi}_{kLF} \rangle \, . \qquad (6.2.20a)$$

Likewise,

$$v^{-1} \int_{A_k} \breve{\underline{\psi}}_{kLF} \cdot \underline{n}_k \, dA = \alpha_k \, {}^{3i}\langle \nabla \cdot \breve{\underline{\psi}}_{kLF} \rangle \, . \qquad (6.2.20b)$$

The left-hand side of Eq. 6.2.20a can be replaced by

$$v^{-1} \int_{A_{kLF}} \breve{\psi}_{kLF} \, \underline{n}_{kLF} \, dA + v^{-1} \int_{A_k'} \breve{\psi}_{kLF} \, \underline{n}_k' \, d\lambda$$

because only $\underline{n}_{kLF}$ is coherent with A_{kLF}, and $\underline{n}'_k$ is coherent with A'_k. Consequently, time averaging of Eq. 6.2.20a leads to

$$v^{-1} \int_{A_{kLF}} \check{\psi}_{kLF}\, \underline{n}_{kLF}\, dA = \alpha_{kLF}\, {}^{3i}\langle \nabla\, \check{\psi}_{kLF} \rangle \,, \tag{6.2.21a}$$

since

$$\left\langle v^{-1} \int_{A'_k} \check{\psi}_{kLF}\, \underline{n}'_k\, dA \right\rangle^{t} = 0$$

according to Eq. 6.2.6c.

Likewise,

$$v^{-1} \int_{A_{kLF}} \check{\underline{\psi}}_{kLF} \cdot \underline{n}_{kLF}\, dA = \alpha_{kLF}\, {}^{3i}\langle \nabla \cdot \check{\underline{\psi}}_{kLF} \rangle \,. \tag{6.2.21b}$$

Also,

$$v^{-1} \int_{A_{kLF}} \check{\underline{\psi}}_{kLF}, \underline{n}_{kLF}\, dA = \alpha_{kLF}\, {}^{3i}\langle \nabla, \check{\underline{\psi}}_{kLF} \rangle \,. \tag{6.2.21c}$$

In Eq. 6.2.21c, the comma denotes dyadic product.

In deriving Eq. 6.2.21a, the approximation was again made that

$$\begin{aligned} {}^{3i}\langle \nabla\, \check{\psi}_{kLF} \rangle &= \left(\frac{1}{v_{kLF}} \int_{v_{kLF}} \nabla\, \check{\psi}_{kLF}\, dv \right) \left[1 + 0(\varepsilon) + 0(\varepsilon^2) + \ldots \right] \\ &\tilde{=} \frac{1}{v_{kLF}} \int_{v_{kLF}} \nabla\, \check{\psi}_{kLF}\, dv \end{aligned} \tag{2.2.22}$$

and similarly for ${}^{3i}\langle \nabla \cdot \check{\underline{\psi}}_{kLF} \rangle$ in Eq. 6.2.21b and for ${}^{3i}\langle \nabla, \check{\underline{\psi}}_{kLF} \rangle$ in Eq. 6.2.21c. It is seen that the approximation is not only consistent, but also necessary, since the left-hand sides of Eqs. 6.2.21a, b, and c are of low frequency only.

Finally, we demonstrate that

$$\left\langle \nabla\, {}^{3i}\langle \psi_k \rangle \right\rangle^{3i} = \nabla\, {}^{3i}\langle \psi_k \rangle \,. \tag{6.2.23a}$$

Since

$$\left\langle \nabla\ {}^{3i}\langle\psi_k\rangle \right\rangle = \nabla\ \left\langle {}^{3i}\langle\psi_k\rangle \right\rangle + v^{-1} \int_{A_k} {}^{3i}\langle\psi_k\rangle\ \underline{n}_k\ dA$$

$$= \nabla\ \alpha_k\ {}^{3i}\langle\psi_k\rangle - {}^{3i}\langle\psi_k\rangle\ \nabla\ \alpha_k = \alpha_k\ \nabla\ {}^{3i}\langle\psi_k\rangle\ ,$$

we have

$$\alpha_k\ {}^{3i}\left\langle \nabla\ {}^{3i}\langle\psi_k\rangle \right\rangle = \alpha_k\ \nabla\ {}^{3i}\langle\psi_k\rangle\ ,$$

which is Eq. 6.2.23a upon canceling the α_k's.

Likewise, it can be shown that

$$\left\langle \nabla \cdot\ {}^{3i}\langle\underline{\psi}_k\rangle \right\rangle = \nabla \cdot\ {}^{3i}\langle\underline{\psi}_k\rangle\ , \tag{6.2.23b}$$

$$\left\langle \nabla,\ {}^{3i}\langle\underline{\psi}_k\rangle \right\rangle = \nabla,\ {}^{3i}\langle\underline{\psi}_k\rangle\ , \tag{6.2.23c}$$

$$\left\langle \nabla \cdot\ {}^{3i}\langle\underline{\underline{\psi}}_k\rangle \right\rangle = \nabla \cdot\ {}^{3i}\langle\underline{\underline{\psi}}_k\rangle\ . \tag{6.2.23d}$$

6.3 Time-Volume-averaged Mass Conservation Equation $(\gamma_V = \gamma_A = 1)$

The local volume-averaged mass conservation is given by Eq. 5.1.1:

$$\frac{\partial}{\partial t}\ \alpha_k\ {}^{3i}\langle\rho_k\rangle + \nabla \cdot\ \alpha_k\ {}^{3i}\langle\rho_k \underline{U}_k\rangle = -\ v^{-1} \int_{A_k} \rho_k(\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k\ dA. \tag{5.1.1}$$

Time averaging requires consideration of:

- $\left\langle \alpha_k\ {}^{3i}\langle\rho_k\rangle \right\rangle^{t}$.

Since

$$\rho_k = {}^{3i}\langle\rho_k\rangle_{LF} + \tilde{\rho}_{kLF} + \rho'_k, \qquad {}^{3i}\langle\rho_k\rangle = {}^{3i}\langle\rho_k\rangle_{LF} + {}^{3i}\langle\rho'_k\rangle\ ,$$

and, by using Eq. 6.1.11, one obtains

$$\alpha_k \, {}^{3i}\langle\varphi_k\rangle = \alpha_{kLF} \, {}^{3i}\langle\varphi_k\rangle_{LF} + \alpha_{kLF} \, {}^{3i}\langle\varphi_k'\rangle + \alpha_k' \, {}^{3i}\langle\varphi_k\rangle_{LF} + \alpha_k' \, {}^{3i}\langle\varphi_k'\rangle \ ,$$

Hence,

$$\left\langle \alpha_k \, {}^{3i}\langle\varphi_k\rangle \right\rangle = \alpha_{kLF} \, {}^{3i}\langle\varphi_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'\rho_k'\rangle \ . \qquad (6.3.1)$$

$$\left\langle \alpha_k \, {}^{3i}\langle\varphi_k \underline{U}_k\rangle \right\rangle$$

$${}^{3i}\langle\varphi_k \underline{U}_k\rangle = {}^{3i}\langle\varphi_k\rangle_{LF} \, {}^{3i}\langle\underline{U}_k\rangle_{LF} + {}^{3i}\langle\varphi_k\rangle_{LF} \, {}^{3i}\langle\underline{U}_k'\rangle + {}^{3i}\langle\tilde{\varphi}_{kLF}\tilde{\underline{U}}_{kLF}\rangle$$

$$+ {}^{3i}\langle\tilde{\varphi}_{kLF}\underline{U}_k'\rangle + {}^{3i}\langle\rho_k'\rangle \, {}^{3i}\langle\underline{U}_k\rangle_{LF} + {}^{3i}\langle\rho_k'\tilde{\underline{U}}_{kLF}\rangle + {}^{3i}\langle\rho_k'\underline{U}_k'\rangle$$

since

$$\left\langle {}^{3i}\langle\varphi_k\rangle_{LF} \, \tilde{\underline{U}}_{kLF} \right\rangle = {}^{3i}\langle\varphi_k\rangle_{LF} \, {}^{3i}\langle\tilde{\underline{U}}_k\rangle_{LF} = 0,$$

$$\left\langle \rho_k' \, {}^{3i}\langle\underline{U}_k\rangle_{LF} \right\rangle = {}^{3i}\langle\varphi_k'\rangle \, {}^{3i}\langle\underline{U}_k\rangle_{LF} \ , \text{ etc.}$$

Hence,

$$\left\langle \alpha_k \, {}^{3i}\langle\varphi_k \underline{U}_k\rangle \right\rangle = \alpha_{kLF} \, {}^{3i}\langle\varphi_k\rangle_{LF} \, {}^{3i}\langle\underline{U}_k\rangle_{LF} + \alpha_{kLF} \, {}^{3i}\langle\tilde{\varphi}_{kLF}\tilde{\underline{U}}_{kLF}\rangle$$

$$+ \alpha_{kLF} \, {}^{t3i}\langle\varphi_k'\underline{U}_k'\rangle + {}^{3i}\langle\varphi_k\rangle_{LF} \, {}^{t3i}\langle\alpha_k'\underline{U}_k'\rangle$$

$$+ {}^{3i}\langle\underline{U}_k\rangle_{LF} \, {}^{t3i}\langle\alpha_k'\rho_k'\rangle + \left\langle \alpha_k' \, {}^{3i}\langle\tilde{\varphi}_{kLF}\underline{U}_k'\rangle \right\rangle$$

$$+ \left\langle \alpha_k' \, {}^{3i}\langle\varphi_k'\tilde{\underline{U}}_{kLF}\rangle \right\rangle + \left\langle \alpha_k' \, {}^{3i}\langle\varphi_k'\underline{U}_k'\rangle^{o} \right\rangle \ . \qquad (6.3.2a)$$

The last term is a time correlation of the third order. It is presumably small and will be deleted. We define

(a) Volume-averaged eddy diffusivity for mass transfer, D_{mk}^{T}, according to:

$$\alpha_{kLF}\,{}^{t3i}\langle\varphi'_k\underline{U}'_k\rangle + {}^{3i}\langle\varphi_k\rangle_{LF}\,{}^{t3i}\langle\alpha'_k\underline{U}'_k\rangle = -\,D^T_{mk}\,\nabla\,\alpha_{kLF}\,{}^{3i}\langle\varphi_k\rangle_{LF}\,. \qquad (6.3.3)$$

(b) Volume-averaged dispersive diffusivity for mass transfer, $\breve{D}^T_{mk}$, according to:

$$\left\langle\alpha'_k\,{}^{3i}\langle\breve{\varphi}_{kLF}\underline{U}'_k\rangle\right\rangle^t + \left\langle\alpha'_k\,{}^{3i}\langle\varphi'_k\breve{\underline{U}}_{kLF}\rangle\right\rangle^t = -\,\breve{D}^T_{mk}\,\nabla\,\alpha_{kLF}\,{}^{3i}\langle\varphi_k\rangle_{LF}\,, \qquad (6.3.4)$$

which also results from high-frequency fluctuations. When ρ_k = constant, $\breve{\rho}_{kLF} = \rho'_k = 0$, and $\breve{D}^T_{mk}$ vanishes. Lumping the two terms on the left-hand side of Eq. 6.3.4 is for convenience only. It may not be appropriate to express the second term in terms of the gradient of $\alpha_{kLF}\,{}^{3i}\langle\varphi_k\rangle_{LF}$.

By introducing Eqs. 6.3.3 and 6.3.4 into Eq. 6.3.2a, one obtains

$$\left\langle\alpha_k\,{}^{3i}\langle\varphi_k\underline{U}_k\rangle\right\rangle^t = \left(\alpha_{kLF}\,{}^{3i}\langle\varphi_k\rangle_{LF} + {}^{t3i}\langle\alpha'_k\rho'_k\rangle\right){}^{3i}\langle\underline{U}_k\rangle_{LF} + \underline{\Psi}_{mk}\,, \qquad (6.3.2b)$$

in which $\underline{\Psi}_{mk}$ is a mass flux vector defined by

$$\underline{\Psi}_{mk} = \alpha_{kLF}\,{}^{3i}\langle\breve{\varphi}_{kLF}\breve{\underline{U}}_{kLF}\rangle + \alpha_{kLF}\,{}^{t3i}\langle\varphi'_k\underline{U}'_k\rangle + {}^{3i}\langle\varphi_k\rangle_{LF}\,{}^{t3i}\langle\alpha'_k\underline{U}'_k\rangle$$

$$+ \left\langle\alpha'_k\,{}^{3i}\langle\breve{\varphi}_{kLF}\underline{U}'_k\rangle\right\rangle^t + \left\langle\alpha'_k\,{}^{3i}\langle\varphi'_k\breve{\underline{U}}_{kLF}\rangle\right\rangle^t$$

$$= \alpha_{kLF}\,{}^{3i}\langle\breve{\varphi}_{kLF}\breve{\underline{U}}_{kLF}\rangle - \left(D^T_{mk} + \breve{D}^T_{mk}\right)\nabla\,\alpha_{kLF}\,{}^{3i}\langle\varphi_k\rangle_{LF}\,. \qquad (6.3.5)$$

Consideration is now given to the evaluation of the time-average of the total interfacial mass generation integral for phase k within v.

$$\left\langle -\,v^{-1}\int_{A_k}\rho_k\left(\underline{U}_k - \underline{W}_k\right)\cdot\underline{n}_k\,dA\right\rangle^t = -\,v^{-1}\int_{A_{kLF}}{}^t\langle\rho_k\left(\underline{U}_k - \underline{W}_k\right)\rangle\cdot\underline{n}_{kLF}\,dA$$

$$-\,v^{-1}\left\langle\int_{A_k}\rho_k\left(\underline{U}_k - \underline{W}_k\right)\cdot\underline{n}'_k\,dA\right\rangle^t, \qquad (6.3.6)$$

since $\underline{n}_{kLF}$ is coherent with A_{kLF}, and $\underline{n}'_k$ is coherent with A'_k. It is straightforward to demonstrate that

$${}^t\langle\rho_k\left(\underline{U}_k - \underline{W}_k\right)\rangle = \left({}^{3i}\langle\rho_k\rangle_{LF} + \breve{\rho}_{kLF}\right)\left({}^{3i}\langle\underline{U}_k\rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF}\right)$$

$$+ {}^{t}\langle \varphi_k'(\underline{U}_k' - \underline{W}_k')\rangle \cdot$$

Consequently, the first integral on the right-hand side of Eq. 6.3.6 is

$$- v^{-1} \int_{A_{kLF}} {}^{t}\langle \varphi_k(\underline{U}_k - \underline{W}_k)\rangle \cdot \underline{n}_{kLF}\, dA$$

$$= {}^{3i}\langle \varphi_k\rangle_{LF} \left(\frac{\partial \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k\rangle_{LF} \cdot \nabla \alpha_{kLF} \right)$$

$$- v^{-1}\, {}^{3i}\langle \varphi_k\rangle_{LF} \int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF}\, dA$$

$$- v^{-1} \int_{A_{kLF}} \check{\rho}_{kLF} \left({}^{3i}\langle \underline{U}_k\rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF}\, dA$$

$$- v^{-1} \int_{A_{kLF}} {}^{t}\langle \varphi_k'(\underline{U}_k' - \underline{W}_k')\rangle \cdot \underline{n}_{kLF}\, dA \,. \qquad (6.3.7)$$

In deriving Eq. 6.3.7, use has been made of Eqs. 6.2.2a and 6.2.4a. It will be shown in Sec. 6.5 that for Newtonian fluids,

$$\int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF}\, dA = 0 \,. \qquad (6.3.8)$$

The second integral on the right-hand side of Eq. 6.3.6 is

$$- v^{-1} \left\langle \int_{A_k'} {}^{t}\rho_k(\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k'\, dA \right\rangle$$

$$= - v^{-1} \left\langle \int_{A_k'} \left({}^{3i}\langle \rho_k\rangle_{LF} + \check{\rho}_{kLF} \right) (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k'\, dA \right\rangle$$

$$- v^{-1} \left\langle \int_{A_k'} \rho_k' \left({}^{3i}\langle \underline{U}_k\rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k'\, dA \right\rangle$$

$$- v^{-1} \left\langle \int_{A_k'} \rho_k' (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle . \qquad (6.3.9)$$

Thus, the time average of the interfacial mass generation rate per unit volume is

$$\begin{aligned} {}^t\langle \Gamma_k \rangle &= \left\langle -v^{-1} \int_{A_k} \rho_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \right\rangle \\ &= {}^{3i}\langle \rho_k \rangle_{LF} \left(\frac{\partial \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} \right) + (MTI)_k , \end{aligned} \qquad (6.3.10)$$

in which $(MTI)_k$ stands for the interfacial mass transfer integral defined by

$$\begin{aligned} (MTI)_k = &- v^{-1} \, {}^{3i}\langle \rho_k \rangle_{LF} \int_{A_{kLF}} \breve{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA \\ &- v^{-1} \int_{A_{kLF}} \breve{\rho}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA \\ &- v^{-1} \int_{A_{kLF}} {}^t\langle \rho_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_{kLF} \, dA \\ &- v^{-1} \left\langle \int_{A_k'} \left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} \right) (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle \\ &- v^{-1} \left\langle \int_{A_k'} \rho_k' \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \right\rangle \\ &- v^{-1} \left\langle \int_{A_k'} \rho_k' (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle . \end{aligned} \qquad (6.3.11)$$

We reiterate that for Newtonian fluids the first term on the right-hand side of Eq. 6.3.11 vanishes. More work needs to be done to examine the relative importance of the various terms in Eq. 6.3.11. We further note that the first term on the right-hand side of Eq. 6.3.10 can be written as

$${}^{3i}\langle \rho_k \rangle_{LF} \frac{d\alpha_{kLF}}{dt_k} ,$$

where the substantive time derivative $\frac{d}{dt_k}$ is defined by

$$\frac{d}{dt_k} = \frac{\partial}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \ . \tag{6.3.12}$$

Performing the time averaging of Eq. 5.1.1, followed by introducing the results given in Eqs. 6.3.1, 6.3.2b, and 6.3.10, leads to the desired time-volume-averaged mass conservation equation:

$$\frac{\partial}{\partial t}\left(\alpha_{kLF}\ {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) + \nabla \cdot \left(\alpha_{kLF}\ {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot \underline{\Psi}_{mk} = {}^{t}\langle \Gamma_k \rangle \ . \tag{6.3.13}$$

When ρ_k = constant, ${}^{3i}\langle \rho_k \rangle = \rho_k$, $\breve{\rho}_{kLF} = \rho_k' = 0$, $\breve{D}^T_{mk} = 0$, and D^T_{mk} becomes ${}^{o}D^T_{mk}$, which is defined by

$$-\ {}^{o}D^T_{mk}\ \nabla\ \alpha_{kLF} = {}^{t3i}\langle \alpha_k' \underline{U}_k' \rangle \ , \tag{6.3.14}$$

and $\underline{\Psi}_{mk}$ becomes ${}^{o}\underline{\Psi}_{mk}$ defined by

$${}^{o}\underline{\Psi}_{mk} = \rho_k\ {}^{t3i}\langle \alpha_k' \underline{U}_k' \rangle = -\ \rho_k\ {}^{o}D^T_{mk}\ \nabla\ \alpha_{kLF} \ . \tag{6.3.15}$$

In this case, the time-volume-averaged mass conservation equation simplifies to

$$\nabla \cdot \alpha_{kLF}\ {}^{3i}\langle \underline{U}_k \rangle_{LF} - \nabla \cdot {}^{o}D^T_{mk}\ \nabla\ \alpha_{kLF}$$

$$= {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla\ \alpha_{kLF} - v^{-1} \int_{A_{kLF}} \breve{\underline{U}}_{kLF} \cdot \underline{n}_{kLF}\ dA$$

$$- v^{-1} \left\langle {}^{t}\left(\int_{A_k'} \underline{U}_k' \cdot \underline{n}_k'\ dA \right) \right\rangle \ , \tag{6.3.16}$$

for which the relation given in Eq. 6.2.5c has been used. Equation 6.3.16 can be derived directly from

$$\nabla \cdot \underline{U}_k = 0 \ , \tag{6.3.17}$$

which is valid for constant ρ_k. Application of local volume averaging of Eq. 6.3.17 leads to

$$\nabla \cdot \alpha_k \, {}^{3i}\langle \underline{U}_k \rangle = - v^{-1} \int_{A_k} \underline{U}_k \cdot \underline{n}_k \, dA \, , \tag{6.3.18}$$

which, upon time averaging, gives

$$\nabla \cdot \alpha_{kLF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot {}^{t3i}\langle \alpha_k' \underline{U}_k' \rangle$$

$$= {}^{3i}\langle \underline{U}_k \rangle_{LF} \nabla \alpha_{kLF} - v^{-1} \int_{A_{kLF}} \breve{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \, {}^{t}\left\langle \int_{A_k'} \underline{U}_k' \cdot \underline{n}_k' \, dA \right\rangle . \tag{6.3.19}$$

Clearly, Eqs. 6.3.16 and 6.3.19 are equivalent.

6.4 Time-Volume-averaged Interfacial Mass Balance Equation $(\gamma_v = \gamma_A = 1)$

The local volume-averaged mass balance equation for interface A_{kf} is given by Eq. 5.2.1. Using Eq. 6.3.10, we readily have

$${}^{t}\langle \Gamma_k \rangle + {}^{t}\langle \Gamma_f \rangle = 0 \, , \tag{6.4.1a}$$

or, equivalently,

$${}^{3i}\langle \rho_k \rangle_{LF} \left(\frac{\partial \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} \right) + (MTI)_k$$

$$+ {}^{3i}\langle \rho_f \rangle_{LF} \left(\frac{\partial \alpha_{fLF}}{\partial t} + {}^{3i}\langle \underline{U}_f \rangle_{LF} \cdot \nabla \alpha_{fLF} \right) + (MTI)_f$$

$$= 0 \, , \tag{6.4.1b}$$

where $(MTI)_f$ is given by Eq. 6.3.11 with subscript k replaced by f. We note that in $(MTI)_k$ and $(MTI)_f$, $A_{kLF} = A_{fLF}$, $A_k' = A_f'$, $\underline{W}_{kLF} = \underline{W}_{fLF}$, $\underline{W}_k' = \underline{W}_f'$, $\underline{n}_{kLF} = - \underline{n}_{fLF}$, and $\underline{n}_k' = - \underline{n}_f'$.

When ρ_k = constant, Eq. 6.4.1b reduces to

$$\frac{\partial \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} - v^{-1} \int_{A_{kLF}} \breve{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \, {}^{t}\left\langle \int_{A_k'} \underline{U}_k' \cdot \underline{n}_k' \, dA \right\rangle + \frac{\partial \alpha_{fLF}}{\partial t} + {}^{3i}\langle \underline{U}_f \rangle_{LF} \cdot \nabla \alpha_{fLF}$$

$$- v^{-1} \int_{A_{fLF}} \breve{\underline{U}}_{fLF} \cdot \underline{n}_{fLF} \, dA - v^{-1} \, t\left\langle \int_{A'_f} \underline{U}'_f \cdot \underline{n}'_f \, dA \right\rangle$$

$$= 0 \, . \qquad (6.4.2)$$

6.5 Time-Volume-averaged Linear Momentum Conservation Equation ($\gamma_v = \gamma_A = 1$)

The local volume-averaged linear momentum conservation equation for constant field force is given by Eq. 5.1.3:

$$\frac{\partial}{\partial t} \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k \rangle + \nabla \cdot \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k \underline{U}_k \rangle$$

$$= - \nabla \alpha_k \, {}^{3i}\langle P_k \rangle + \nabla \cdot \alpha_k \, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle + \alpha_k \, {}^{3i}\langle \rho_K \rangle \, \underline{f}$$

$$+ v^{-1} \int_{A_k} \left(- P_k \underline{\underline{I}}_k + \underline{\underline{\tau}}_k \right) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_k} \rho_k \underline{U}_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}_k \, dA \, . \qquad (5.1.3)$$

Time averaging of Eq. 5.1.3 requires the consideration of

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k \rangle \right\rangle = \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \, , \qquad (6.5.1)$$

which is Eq. 6.3.2b.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k \underline{U}_k \rangle \right\rangle$$

$$= \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + 2 \, \underline{\Psi}_{mk} \, {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$- \alpha_{kLF} \left({}^{3i}\langle \underline{\underline{\tau}}^{T}_k \rangle + {}^{3i}\langle \underline{\underline{\tilde{\tau}}}_k \rangle + {}^{3i}\langle \underline{\underline{\tilde{\tau}}}^{T}_k \rangle \right) , \qquad (6.5.2)$$

in which

(a) The volume-averaged Reynolds stress tensor ${}^{3i}\langle \underline{\underline{\tau}}^{T}_k \rangle$ is defined by

$$- {}^{3i}\langle \rho_k \rangle_{LF} \, {}^{t3i}\langle \underline{U}'_k \underline{U}'_k \rangle - {}^{t3i}\langle \tilde{\rho}_{kLF} \underline{U}'_k \underline{U}'_k \rangle = {}^{3i}\langle \underline{\underline{\tau}}^{T}_k \rangle \, , \qquad (6.5.3)$$

(b) The volume-averaged dispersive stress tensor ${}^{3i}\langle \underline{\underline{\tilde{\tau}}}_k \rangle$ is defined by

$$- {}^{3i}\langle \rho_k \rangle_{LF} \, {}^{3i}\langle \breve{\underline{U}}_{kLF} \breve{\underline{U}}_{kLF} \rangle - {}^{t3i}\langle \tilde{\rho}_{kLF} \breve{\underline{U}}_{kLF} \breve{\underline{U}}_{kLF} \rangle = {}^{3i}\langle \underline{\underline{\tilde{\tau}}}_k \rangle \, , \text{ and} \qquad (6.5.4)$$

(c) The volume-averaged turbulent, dispersive stress tensor ${}^{3i}\langle \check{\underline{\underline{T}}}_k \rangle$ is defined by:

$$
\begin{aligned}
&- \rightsquigarrow_{kLF} {}^{t3i}\langle \check{\underline{U}}_{kLF} \rho_k' \underline{U}_k' \rangle - 2\, {}^{3i}\langle \rho_k \rangle_{LF} \left\langle \alpha_k' \, {}^{3i}\langle \check{\underline{U}}_{kLF} \underline{U}_k' \rangle \right\rangle \\
&- 2 \left\langle \alpha_k' \, {}^{3i}\langle \tilde{\rho}_{kLF} \check{\underline{U}}_{kLF} \underline{U}_k' \rangle \right\rangle - \left\langle \alpha_k' \, {}^{3i}\langle \rho_k' \check{\underline{U}}_{kLF} \check{\underline{U}}_{kLF} \rangle \right\rangle \\
&= \alpha_{kLF} \, {}^{3i}\langle \check{\underline{\underline{T}}}_k \rangle \, . \qquad (6.5.5)
\end{aligned}
$$

$$
\left\langle \alpha_k \, {}^{3i}\langle P_k \rangle \right\rangle = \alpha_{kLF} \, {}^{3i}\langle P_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' P_k' \rangle \qquad (6.5.6)
$$

$$
\left\langle \alpha_k \, {}^{3i}\langle \underline{\underline{T}}_k \rangle \right\rangle = \alpha_{kLF} \, {}^{3i}\langle \underline{\underline{T}}_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \underline{\underline{T}}_k' \rangle \qquad (6.5.7)
$$

Equation 6.5.7 is merely the consequence of a mathematical operation when the point variable $\underline{\underline{T}}_k$ (a second-order tensor) is decomposed according to Eq. 6.1.8. Since $\underline{\underline{T}}_k$ is related to fluid viscosity and velocity gradients, the physical meaning of ${}^{3i}\langle \underline{\underline{T}}_k \rangle_{LF}$ and $\underline{\underline{T}}_k'$ requires more careful consideration. For Newtonian fluids

$$
\underline{\underline{T}}_k = \left(\lambda_k - \frac{2}{3} \mu_k \right) \left(\nabla \cdot \underline{U}_k \right) \underline{\underline{I}} + \mu_k \left[\nabla, \underline{U}_k + \left(\nabla, \underline{U}_k \right)_c \right] , \qquad (6.5.8a)
$$

where λ_k is the bulk viscosity, $\nabla, \underline{U}_k$ is a dyad, and subscript c denotes conjugate. Since λ_k and μ_k are independent of velocity gradients, Eq. 6.5.8a gives, following substituting the relation $\underline{U}_k = {}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} + \underline{U}_k'$,

$$
\begin{aligned}
\underline{\underline{T}}_k = {} & \left(\lambda_k - \frac{2}{3} \mu_k \right) \left(\nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \underline{\underline{I}} + \mu_k \left[\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \left(\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} \right)_c \right] \\
& + \left(\lambda_k - \frac{2}{3} \mu_k \right) \left(\nabla \cdot \check{\underline{U}}_{kLF} \right) \underline{\underline{I}} + \mu_k \left[\nabla, \check{\underline{U}}_{kLF} + \left(\nabla, \check{\underline{U}}_{kLF} \right)_c \right] \\
& + \left(\lambda_k - \frac{2}{3} \mu_k \right) \left(\nabla \cdot \underline{U}_k' \right) \underline{\underline{I}} + \mu_k \left[\nabla, \underline{U}_k' + \left(\nabla, \underline{U}_k' \right)_c \right] , \qquad (6.5.8b)
\end{aligned}
$$

which, upon comparing with

$$
\underline{\underline{T}}_k = {}^{3i}\langle \underline{\underline{T}}_k \rangle_{LF} + \check{\underline{\underline{T}}}_{kLF} + \underline{\underline{T}}_k' , \qquad (6.5.8c)
$$

leads to the following defining relations:

$$^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} = \left(\lambda_k - \frac{2}{3}\mu_k\right)\left(\nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF}\right) \underline{\underline{I}} + \mu_k \left[\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \left(\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF}\right)_c\right] \quad (6.5.8d)$$

$$\check{\underline{\underline{\tau}}}_{kLF} = \left(\lambda_k - \frac{2}{3}\mu_k\right)\left(\nabla \cdot \check{\underline{U}}_{kLF}\right) \underline{\underline{I}} + \mu_k \left[\nabla, \check{\underline{U}}_{kLF} + \left(\nabla, \check{\underline{U}}_{kLF}\right)_c\right] \quad (6.5.8e)$$

$$\underline{\underline{\tau}}_k' = \left(\lambda_k - \frac{2}{3}\mu_k\right)\left(\nabla \cdot \underline{U}_k'\right) \underline{\underline{I}} + \mu_k \left[\nabla, \underline{U}_k' + \left(\nabla, \underline{U}_k'\right)_c\right] . \quad (6.5.8f)$$

By taking the intrinsic volume average of Eq. 6.5.8c, followed by multiplying by α_k, one obtains

$$\alpha_k \, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle = \alpha_{kLF} \, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \alpha_{kLF} \, {}^{3i}\langle \underline{\underline{\tau}}_k' \rangle + \alpha_k' \, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \alpha_k' \, {}^{3i}\langle \underline{\underline{\tau}}_k' \rangle ,$$

which, upon time averaging, leads immediately to Eq. 6.5.7, since ${}^{t3i}\langle \underline{\underline{\tau}}_k' \rangle = 0$ according to Eq. 6.2.8c. Clearly, in Eq. 6.5.7, ${}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF}$ is defined by Eq. 6.5.8d and

$${}^{t3i}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle = \left(\lambda_k - \frac{2}{3}\mu_k\right) {}^{t3i}\langle \alpha_k' \nabla \cdot \underline{U}_k' \rangle \underline{\underline{I}} + \mu_k \, {}^{t3i}\left\langle \alpha_k' \left[\nabla, \underline{U}_k' + \left(\nabla, \underline{U}_k'\right)_c\right]\right\rangle . \quad (6.5.8g)$$

It is noted that $\underline{\underline{\tau}}_k'$ is <u>not</u> related to Reynolds stress which is independent of viscosity.

Since ${}^{3i}\langle \check{\underline{\underline{\tau}}}_{kLF} \rangle = 0$, we obtain from Eq. 6.5.8e

$${}^{3i}\langle \nabla \cdot \check{\underline{U}}_{kLF} \rangle = 0, \quad \text{and hence} \quad \int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA = 0 , \quad (6.5.9a,b)$$

according to Eq. 6.2.21b. Also,

$${}^{3i}\langle \nabla, \check{\underline{U}}_{kLF} \rangle = 0, \quad \text{and hence} \quad \int_{A_{kLF}} \check{\underline{U}}_{kLF} , \underline{n}_{kLF} \, dA = 0 \quad (6.5.9c,d)$$

and

$${}^{3i}\langle \left(\nabla, \check{\underline{U}}_{kLF}\right)_c \rangle = 0, \quad \text{and hence} \quad \int_{A_{kLF}} \left(\check{\underline{U}}_{kLF} , \underline{n}_{kLF}\right)_c \, dA = 0 . \quad (6.5.9e,f)$$

It should be emphasized that the group of relations given by Eqs. 6.5.9a through f hold not only for Newtonian fluids, but also for others that obey linear stress-strain rate relations. They will not be valid if the stress-strain rate relation is nonlinear.

When the viscosities in Eq. 6.5.8a are dependent on strain rates, the resulting expression for $\left\langle \alpha_k \, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle \right\rangle^t$ is complicated. Details are given in Appendix B.

$$\left\langle \alpha_k \, {}^{3i}\langle \rho_k \rangle \right\rangle^t = \alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \, , \tag{6.5.10}$$

which is Eq. 6.3.1.

$$\left\langle v^{-1} \int_{A_k} \left(- P_k \underline{\underline{I}}_k + \underline{\underline{\tau}}_k \right) \cdot \underline{n}_k \, dA \right\rangle^t = v^{-1} \int_{A_{kLF}} {}^t\langle -P_k \underline{\underline{n}}_{kLF} + \underline{\underline{\tau}}_k \cdot \underline{n}_{kLF} \rangle \, dA$$

$$+ v^{-1} \left\langle \int_{A_k'} \left(-P_k \underline{\underline{n}}_k' + \underline{\underline{\tau}}_k \cdot \underline{n}_k' \right) dA \right\rangle^t \, . \tag{6.5.11}$$

Now

$$v^{-1} \int_{A_{kLF}} {}^t\langle -P_k \underline{\underline{n}}_{kLF} + \underline{\underline{\tau}}_k \cdot \underline{n}_{kLF} \rangle \, dA$$

$$= {}^{3i}\langle P_k \rangle_{LF} \nabla \alpha_{kLF} - v^{-1} \int_{A_{kLF}} \breve{P}_{kLF} \underline{\underline{n}}_{kLF} \, dA$$

$$- {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} + v^{-1} \int_{A_{kLF}} \breve{\underline{\underline{\tau}}}_{kLF} \cdot \underline{n}_{kLF} \, dA \tag{6.5.12}$$

and

$$v^{-1} \left\langle \int_{A_k'} \left(-P_k \underline{\underline{n}}_k' + \underline{\underline{\tau}}_k \cdot \underline{n}_k' \right) dA \right\rangle^t = v^{-1} \left\langle \int_{A_k'} \left(-P_k' \underline{\underline{n}}_k' + \underline{\underline{\tau}}_k' \cdot \underline{n}_k' \right) dA \right\rangle^t \, , \tag{6.5.13}$$

for which use has been made of Eqs. 6.2.3 and 6.2.6a, b, and c. Hence,

$$\left\langle v^{-1} \int_{A_k} \left(-P_k \underline{\underline{n}}_k + \underline{\underline{\tau}}_k \cdot \underline{n}_k \right) dA \right\rangle^t$$

$$= {}^{3i}\langle P_k \rangle_{LF} \nabla \alpha_{kLF} - {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} + (PTI)_k - (VSTI)_k \, , \tag{6.5.14}$$

in which $(PTI)_k$ stands for the interfacial pressure transfer integral defined by

$$(PTI)_k = - v^{-1} \int_{A_{kLF}} \tilde{P}_{kLF} \underline{n}_{kLF} \, dA - v^{-1} \left\langle \int_{A'_k} P'_k \, \underline{n}'_k \, dA \right\rangle \qquad (6.5.15)$$

and $(VSTI)_k$ stands for the interfacial viscous stress transfer integral defined by

$$(VSTI)_k = - v^{-1} \int_{A_{kLF}} \tilde{\underline{\underline{\tau}}}_{kLF} \cdot \underline{n}_{kLF} \, dA - v^{-1} \left\langle \int_{A'_k} \underline{\underline{\tau}}'_k \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.5.16)$$

$$\bullet \quad \left\langle - v^{-1} \int_{A_k} \rho_k \underline{U}_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \right\rangle$$

$$= - v^{-1} \int_{A_{kLF}} \langle \rho_k \underline{U}_k (\underline{U}_k - \underline{W}_k) \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \left\langle \int_{A'_k} \rho_k \underline{U}_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.5.17)$$

Now

$$- v^{-1} \int_{A_{kLF}} \langle \rho_k \underline{U}_k (\underline{U}_k - \underline{W}_k) \rangle \cdot \underline{n}_{kLF} \, dA$$

$$= {}^{3i}\langle \rho_k \rangle_{LF} \; {}^{3i}\langle \underline{U}_k \rangle_{LF} \left(\frac{\partial \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} \right)$$

$$- v^{-1} \; {}^{3i}\langle \rho_k \rangle_{LF} \; {}^{3i}\langle \underline{U}_k \rangle_{LF} \int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \; {}^{3i}\langle \underline{U}_k \rangle_{LF} \int_{A_{kLF}} \check{\rho}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \check{\rho}_{kLF} \right) \check{\underline{U}}_{kLF} + {}^{t}\langle \rho'_k \underline{U}'_k \rangle \right]$$

$$\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \rho_k \rangle_{LF} + \check{\rho}_{kLF} \right) {}^{t}\langle \underline{U}'_k (\underline{U}'_k - \underline{W}'_k) \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} \right) {}^{t}\langle \rho_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_{kLF} \, dA \qquad (6.5.18)$$

and

$$- v^{-1} \left\langle {}^{t/} \int_{A_k'} \rho_k \underline{U}_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k' \, dA \right\rangle$$

$$= - v^{-1} \left\langle {}^{t/} \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \check{\rho}_{kLF} + \rho_k' \right) \underline{U}_k' + \rho_k' \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} \right) \right] \right.$$

$$\left. \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \right\rangle$$

$$- v^{-1} \left\langle {}^{t/} \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \check{\rho}_{kLF} \right) \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} + \underline{U}_k' \right) \right. \right.$$

$$\left. \left. + \rho_k' \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} \right) \right] (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle$$

$$- v^{-1} \left\langle {}^{t/} \int_{A_k'} \rho_k' \underline{U}_k' (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle . \qquad (6.5.19)$$

Multiplying Eq. 6.3.10 by ${}^{3i}\langle \underline{U}_k \rangle_{LF}$, followed by introducing the result into Eqs. 6.5.17, 6.5.18, and 6.5.19 leads to

$$- v^{-1} \left\langle {}^{t/} \int_{A_k} \rho_k \underline{U}_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \right\rangle$$

$$= {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + (MMTI)_k , \qquad (6.5.20)$$

in which the interfacial momentum transfer integral is defined by

$$(MMTI)_k = - v^{-1} \int_{A_{kLF}} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \check{\rho}_{kLF} \right) \check{\underline{U}}_{kLF} + {}^{t}\langle \rho_k' \underline{U}_k' \rangle \right]$$

$$\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \rho_k \rangle_{LF} + \tilde{\rho}_{kLF} \right) {}^{t}\langle \underline{U}'_k (\underline{U}'_k - \underline{W}'_k) \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \underline{\tilde{U}}_{kLF} \, {}^{t}\langle \rho'_k (\underline{U}'_k - \underline{W}'_k) \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} {}^{t}\left\langle \int_{A'_k} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \tilde{\rho}_{kLF} + \rho'_k \right) \underline{U}'_k + \rho'_k \underline{\tilde{U}}_{kLF} \right] \right.$$

$$\left. \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\tilde{J}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}'_k \, dA \right\rangle$$

$$- v^{-1} {}^{t}\left\langle \int_{A'_k} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \tilde{\rho}_{kLF} \right) \left(\underline{\tilde{U}}_{kLF} + \underline{U}'_k \right) + \rho'_k \underline{\tilde{U}}_{kLF} \right] \right.$$

$$\left. (\underline{U}'_k - \underline{W}'_k) \cdot \underline{n}'_k \, dA \right\rangle$$

$$- v^{-1} {}^{t}\left\langle \int_{A'_k} \rho'_k \underline{U}'_k (\underline{U}'_k - \underline{W}'_k) \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.5.21)$$

It is seen from Eq. 6.5.20a that the time and volume-averaged interfacial momentum transfer rate consists of two parts: (1) transfer that is directly related to interfacial mass generation, and (2) extraneous transfer due to spatial deviation of velocity and due to time-correlations of density, velocities, and interfacial area fluctuations. The time-averaged momentum source per unit volume, ${}^{t}\langle \underline{M}_k \rangle$, resulting from interfacial pressure and viscous stresses and from interfacial momentum transfer is

$${}^{t}\langle \underline{M}_k \rangle = \left({}^{3i}\langle P_k \rangle_{LF} \, \underline{\underline{I}} - {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \right) \cdot \nabla \alpha_{kLF} + {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$+ (PTI)_k - (VSTI)_k + (MMTI)_k . \qquad (6.5.22)$$

Using the results given in Eqs. 6.5.1, 6.5.2, 6.5.6, 6.5.7, 6.5.10, 6.5.14, and 6.5.20, one obtains the time-volume averaged linear momentum conservation equation:

$$\frac{\partial}{\partial t} \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$+ \nabla \cdot \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$+ \frac{\partial \underline{\Psi}_{mk}}{\partial t} + 2\,\nabla \cdot \underline{\Psi}_{mk}\; {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$= -\nabla \left(\alpha_{kLF}\; {}^{3i}\langle P_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' P_k' \rangle \right) + \nabla \cdot \left(\alpha_{kLF}\; {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + {}^{t}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle \right)$$

$$+ \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{\underline{\tau}}_k^{T} \rangle + {}^{3i}\langle \tilde{\underline{\underline{\tau}}}_k \rangle + {}^{3i}\langle \tilde{\underline{\underline{\tau}}}_k^{T} \rangle \right)$$

$$+ \left(\alpha_{kLF}\; {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) \underline{f} + {}^{t}\langle \underline{M}_k \rangle \,. \qquad (6.5.23a)$$

Equivalently,

$$\frac{\partial}{\partial t}\; \alpha_{kLF}\; {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \;\; {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$+ \nabla \cdot \left(\alpha_{kLF}\; {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF}\; {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$+ \frac{\partial \underline{\Psi}_{mk}}{\partial t} + 2\,\nabla \cdot \underline{\Psi}_{mk}\; {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$= -\alpha_{kLF}\, \nabla\; {}^{3i}\langle P_k \rangle_{LF} - \nabla\; {}^{t3i}\langle \alpha_k' P_k' \rangle$$

$$+ \alpha_{kLF}\, \nabla \cdot\; {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \nabla \cdot\; {}^{t3i}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle$$

$$+ \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{\underline{\tau}}_k^{T} \rangle + {}^{3i}\langle \tilde{\underline{\underline{\tau}}}_k \rangle + {}^{3i}\langle \tilde{\underline{\underline{\tau}}}_k^{T} \rangle \right)$$

$$+ \left(\alpha_{kLF}\; {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) \underline{f}$$

$$+ {}^{t}\langle T_k \rangle\; {}^{3i}\langle \underline{U}_k \rangle_{LF} + (\mathrm{PTI})_k - (\mathrm{VSTI})_k + (\mathrm{MMTI})_k \,, \qquad (6.5.23b)$$

in which $\underline{\Psi}_{mk}$ is defined in Eq. 6.3.5, ${}^{3i}\langle \underline{\underline{\tau}}_k^{T} \rangle$, ${}^{3i}\langle \tilde{\underline{\underline{\tau}}}_k \rangle$, and ${}^{3i}\langle \tilde{\underline{\underline{\tau}}}_k^{T} \rangle$ are defined in Eqs. 6.5.3, 6.5.4, and 6.5.5, respectively. ${}^{t}\langle T_k \rangle$ is defined in Eq. 6.3.10 and $(\mathrm{MMTI})_k$ is defined in Eq. 6.5.21.

For Newtonian fluids, ${}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF}$ is given by Eq. 6.5.8d and ${}^{t3i}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle$ is given by Eq. 6.5.8g.

Multiplying Eq. 6.3.13 by ${}^{3i}\langle \underline{U}_k \rangle_{LF}$ and introducing the result into Eq. 6.5.23b, one obtains, after combination of certain terms and rearrangement, an alternative form of the time-volume averaged linear momentum conservation equation:

$$\left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right)\left(\frac{\partial\, {}^{3i}\langle \underline{U}_k \rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla\, {}^{3i}\langle \underline{U}_k \rangle_{LF}\right)$$

$$+ \frac{\partial\, \underline{\Psi}_{mk}}{\partial t} + \nabla \cdot \underline{\Psi}_{mk}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \cdot \nabla\, {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$= -\alpha_{kLF} \nabla\, {}^{3i}\langle P_k \rangle_{LF} - \nabla\, {}^{t3i}\langle \alpha_k' P_k' \rangle$$

$$+ \alpha_{kLF} \nabla \cdot {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \nabla \cdot {}^{t3i}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle$$

$$+ \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{\underline{\tau}}_k^{T} \rangle + {}^{3i}\langle \underline{\underline{\tilde{\tau}}}_k \rangle + {}^{3i}\langle \underline{\underline{\tilde{\tau}}}_k^{T} \rangle\right)$$

$$+ \left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) \underline{f}$$

$$+ \left(PTI\right)_k - \left(VSTI\right)_k + \left(MMTI\right)_k . \qquad (6.5.23c)$$

The relative importance of various terms in Eqs. 6.5.23a, b, or c remains to be assessed. In highly turbulent flows, the Reynolds stress probably dominates all viscosity-related stresses.

For any single-phase system, $\alpha_k = 1$ and $A_{kf} = 0$. Therefore, all interfacial integrals vanish. If the system is at rest, all quantities associated with $\underline{U}_k$ also vanish. Accordingly, Eq. 6.5.23a, b, or c reduces to

$$-\nabla\, {}^{3i}\langle P_k \rangle + {}^{3i}\langle \rho_k \rangle\, \underline{g} = 0 , \qquad (6.5.24a)$$

with $\underline{f} = \underline{g}$, $\underline{g}$ being the gravitational acceleration vector. The subscript LF for ${}^{3i}\langle P_k \rangle$ and ${}^{3i}\langle \rho_k \rangle$ has been dropped, since the fluid is everywhere at rest. The characteristic length scale d in Eq. 3.4.3 is zero for a single-phase system; hence, ℓ can be made as small as desired. Thus, in the limit, ${}^{3i}\langle P_k \rangle \to P_k$ and ${}^{3i}\langle \rho_k \rangle \to \rho_k$ and Eq. 6.5.24a becomes

$$-\nabla P_k + \rho_k \underline{g} = 0 , \qquad (6.5.24b)$$

thus satisfying the basic relation of hydrostatics.

When ρ_k = constant, Eq. 6.5.23b simplifies to

$$\rho_k \left(\frac{\partial}{\partial t} \alpha_{kLF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot \alpha_{kLF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \right)$$

$$+ \frac{\partial \, {}^{o}\underline{\Psi}_{mk}}{\partial t} + 2 \nabla \cdot {}^{o}\underline{\Psi}_{mk} \, {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$= - \alpha_{kLF} \nabla \, {}^{3i}\langle P_k \rangle_{LF} - \nabla \, {}^{t3i}\langle \alpha_k' P_k' \rangle$$

$$+ \alpha_{kLF} \nabla \cdot {}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k \rangle_{LF} + \nabla \cdot {}^{t3i}\langle \alpha_k' \, {}^{o}\underline{\underline{\tau}}_k' \rangle$$

$$+ \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k^{T} \rangle + {}^{3i}\langle {}^{o\sim}\underline{\underline{\tau}}_k \rangle + {}^{3i}\langle {}^{o\sim T}\underline{\underline{\tau}}_k \rangle \right) + \alpha_{kLF} \rho_k \underline{f}$$

$$+ {}^{ot}\langle \mathcal{T}_k \rangle \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + (PTI)_k - {}^{o}(VSTI)_k + {}^{o}(MMTI)_k \, , \qquad (6.5.25a)$$

in which ${}^{o}\underline{\Psi}_{mk}$ is given by Eq. 6.3.15 and

$${}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k \rangle_{LF} = \mu_k \left[\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \left(\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} \right)_c \right] , \qquad (6.5.26a)$$

$${}^{t3i}\langle \alpha_k' \, {}^{o}\underline{\underline{\tau}}_k' \rangle = \mu_k \, {}^{t3i}\langle \alpha_k' \left[\nabla \cdot \underline{U}_k' + \left(\nabla, \underline{U}_k' \right)_c \right] \rangle \, , \qquad (6.5.26b)$$

$${}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k^{T} \rangle = - \rho_k \, {}^{t3i}\langle \underline{U}_k' \underline{U}_k' \rangle \, , \qquad (6.5.27)$$

$${}^{3i}\langle {}^{o\sim}\underline{\underline{\tau}}_k \rangle = - \rho_k \, {}^{3i}\langle \tilde{\underline{U}}_{kLF} \tilde{\underline{U}}_{kLF} \rangle \, , \qquad (6.5.28)$$

$$\alpha_{kLF} \, {}^{3i}\langle {}^{o\sim T}\underline{\underline{\tau}}_k \rangle = - 2 \rho_k \, {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \tilde{\underline{U}}_{kLF} \underline{U}_k' \rangle \right\rangle \qquad (6.5.29)$$

$${}^{o}(VSTI)_k = - v^{-1} \, {}^{t}\left\langle \int_{A_k'} \underline{\underline{\tau}}_k' \cdot \underline{n}_k' \, dA \right\rangle \qquad (6.5.30)$$

$${}^{ot}\langle \mathcal{T}_k \rangle = \rho_k \left(\frac{\partial \, \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \, \alpha_{kLF} \right) + {}^{o}(MTI)_k \, , \qquad (6.5.31)$$

with

$$ {}^{o}(MTI)_k = - v^{-1} \rho_k \left\langle {}^{t}\int_{A'_k} \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle \qquad (6.5.32) $$

and

$$ \begin{aligned} {}^{o}(MMTI)_k = & - v^{-1} \rho_k \int_{A_{kLF}} \underline{\breve{U}}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\breve{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA \\ & - v^{-1} \rho_k \int_{A_{kLF}} {}^{t}\langle \underline{U}'_k (\underline{U}'_k - \underline{W}'_k) \rangle \cdot \underline{n}_{kLF} \, dA \\ & - v^{-1} \rho_k \left\langle {}^{t}\int_{A'_k} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\breve{U}}_{kLF} - \underline{W}_{kLF} \right) \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle \\ & - v^{-1} \rho_k \left\langle {}^{t}\int_{A'_k} \left(\underline{\breve{U}}_{kLF} + \underline{U}'_k \right) \left(\underline{U}'_k - \underline{W}'_k \right) \cdot \underline{n}'_k \, dA \right\rangle . \end{aligned} \qquad (6.5.33) $$

The interfacial pressure transfer integral remains unchanged and is defined by Eq. 6.5.15.

An alternative form of Eq. 6.5.25a is

$$ \begin{aligned} & \alpha_{kLF} \rho_k \left(\frac{\partial \, {}^{3i}\langle \underline{U}_k \rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \\ & + \frac{\partial \, {}^{o}\underline{\Psi}_{mk}}{\partial t} + \nabla \cdot {}^{o}\underline{\Psi}_{mk} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\underline{\Psi}_{mk} \cdot \nabla \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \\ & = - \alpha_{kLF} \nabla \, {}^{3i}\langle P_k \rangle_{LF} - \nabla \, {}^{t3i}\langle \alpha'_k P'_k \rangle \\ & + \alpha_{kLF} \nabla \cdot {}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k \rangle_{LF} + \nabla \cdot {}^{t3i}\langle \alpha'_k \underline{\underline{\tau}}'_k \rangle \\ & + \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle {}^{o}\underline{\underline{\tau}}^{T}_{k} \rangle + {}^{3i}\langle {}^{o\sim}\underline{\underline{\tau}}_{k} \rangle + {}^{3i}\langle {}^{o\sim}\underline{\underline{\tau}}^{T}_{k} \rangle \right) + \alpha_{kLF} \rho_k \underline{f} \\ & + (PTI)_k - {}^{o}(VSTI)_k + {}^{o}(MMTI)_k . \end{aligned} \qquad (6.5.25b) $$

6.6 Time-Volume-averaged Interfacial Linear Momentum Balance Equation ($\gamma_v = \gamma_A = 1$)

The local volume-averaged linear momentum balance equation for interface A_{kf} is given by Eq. 5.2.2:

$$v^{-1} \int_{A_k} \left(- \rho_k \underline{\underline{I}} + \underline{\underline{\tau}}_k \right) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_k} \rho_k \underline{U}_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}_k \, dA$$

$$= v^{-1} \int_{A_f} \left(- P_f \underline{\underline{I}} + \underline{\underline{\tau}}_f \right) \cdot \underline{n}_f \, dA + v^{-1} \int_{A_f} \rho_f \underline{U}_f \left(\underline{U}_f - \underline{W}_f \right) \cdot n_f \, dA$$

$$- \int_{A_k} \left(- \nabla_{kf} \sigma_{kf} + 2 \sigma_{kf} H_{kf} \underline{n}_k \right) dA \, . \qquad (6.6.1)$$

Using Eqs. 6.5.14, 6.5.20, and 6.5.22, one has

$$\int_{A_{kLF}} \left(- \nabla_{kf} \sigma_{kfLF} + 2\sigma_{kfLF} H_{kfLF} \underline{n}_{kLF} \right) dA$$

$$+ \left\langle {}^{t}\!\left(\int_{A'_k} \left(- \nabla'_{kf} \sigma'_{kf} + 2 \sigma'_{kf} H'_{kf} \underline{n}'_k \right) dA \right) \right\rangle$$

$$+ {}^{t}\langle \underline{M}_k \rangle + {}^{t}\langle \underline{M}_f \rangle = 0 \, , \qquad (6.6.2)$$

where ∇'_{kf} is the surface gradient operator associated with A'_k.

Equivalently, Eq. 6.6.2 can be written as

$$^{3i}\langle P_k \rangle_{LF} \nabla \alpha_{kLF} - {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} + (PTI)_k - (VSTI)_k$$

$$+ {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + (MMTI)_k$$

$$= - {}^{3i}\langle P_f \rangle_{LF} \nabla \alpha_{fLF} + {}^{3i}\langle \underline{\underline{\tau}}_f \rangle_{LF} \cdot \nabla \alpha_{fLF} - (PTI)_f + (VSTI)_f$$

$$- {}^{t}\langle \Gamma_f \rangle \, {}^{3i}\langle \underline{U}_f \rangle_{LF} - (MMTI)_f$$

$$- \int_{A_{kLF}} \left(- \nabla_{kf} \sigma_{kfLF} + 2\sigma_{kfLF} \, H_{kfLF} \, \underline{n}_{kLF} \right) dA$$

$$- \left\langle \int_{A'_k} \left(- \nabla'_{kf} \sigma'_{kf} + 2\sigma'_{kf} \, H'_{kf} \, \underline{n}'_k \right) dA \right\rangle^t . \qquad (6.6.3)$$

When $\underline{\rho_k = \text{constant}}$, one needs to replace ${}^{3i}\langle \rho_k \rangle_{LF}$ by ρ_k, $(MTI)_k$ by ${}^{o}(MTI)_k$ (which is given by Eq. 6.5.32), and $(MMTI)_k$ by ${}^{o}(MMTI)_k$ (which is defined in Eq. 6.5.33), $\underline{\Psi}_{mk}$ by ${}^{o}\underline{\Psi}_{mk}$; and finally in Eq. 6.6.3b, ${}^{t3i}\langle \alpha'_k \rho'_k \rangle$ should be set to zero.

6.7 Time-Volume-averaged Total Energy Conservation Equation $(\gamma_v = \gamma_A = 1)$

The local volume-averaged total energy conservation equation is given by Eq. 5.1.4:

$$\frac{\partial}{\partial t} \alpha_k \, {}^{3i}\langle \rho_k E_k \rangle + \nabla \cdot \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k E_k \rangle$$

$$= - \nabla \cdot \alpha_k \, {}^{3i}\langle P_k \underline{U}_k \rangle + \nabla \cdot \alpha_k \, {}^{3i}\langle \underline{\tau}_k \cdot \underline{U}_k \rangle$$

$$- \nabla \cdot \alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle + \alpha_k \left({}^{3i}\langle \rho_k \underline{U}_k \rangle \cdot \underline{f} + {}^{3i}\langle J_{Ek} \rangle \right) + \dot{Q}_k$$

$$+ v^{-1} \int_{A_k} \left(-P_k \underline{U}_k + \underline{\tau}_k \cdot \underline{U}_k \right) \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_k} \rho_k E_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}_k \, dA , \qquad (5.1.4)$$

in which

$$\dot{Q}_k = - v^{-1} \int_{A_k} \underline{J}_{qk} \cdot \underline{n}_k \, dA , \qquad (5.1.5)$$

denoting the interfacial heat transfer rate per unit volume of the fluid mixture.

Since $E_k = u_k + \frac{1}{2} \underline{U}_k \cdot \underline{U}_k$, and if we write

$$E_k = {}^{3i}\langle E_k \rangle_{LF} + \breve{E}_{kLF} + E'_k \,, \tag{6.7.1a}$$

it is easy to demonstrate that

$${}^{3i}\langle E_k \rangle_{LF} = {}^{3i}\langle u_k \rangle_{LF} + \frac{1}{2}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \tag{6.7.1b}$$

$$\breve{E}_{kLF} = \breve{u}_{kLF} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \breve{\underline{U}}_{kLF} + \frac{1}{2}\, \breve{\underline{U}}_{kLF} \cdot \breve{\underline{U}}_{kLF} \tag{6.7.1c}$$

$$E'_k = u'_k + \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF}\right) \cdot \underline{U}'_k + \frac{1}{2}\, \underline{U}'_k \cdot \underline{U}'_k \,. \tag{6.7.1d}$$

Time averaging of Eqs. 5.1.4 and 5.1.5 requires consideration of

$$\bullet \quad {}^{t}\left\langle \alpha_k\, {}^{3i}\langle \rho_k E_k \rangle \right\rangle = \left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle\right) {}^{3i}\langle E_k \rangle_{LF} + \Phi_{EK} \,, \tag{6.7.2}$$

in which Φ_{Ek} is a scalar total energy function defined by

$$\Phi_{Ek} = \alpha_{kLF}\, {}^{3i}\langle \breve{\rho}_{kLF} \breve{E}_{kLF} \rangle + \alpha_{kLF}\, {}^{t3i}\langle \rho'_k E'_k \rangle + {}^{3i}\langle \rho_k \rangle_{LF}\, {}^{t3i}\langle \alpha'_k E'_k \rangle$$

$$+ {}^{t}\left\langle \alpha'_k\, {}^{3i}\langle \tilde{\rho}_{kLF} E'_k \rangle \right\rangle + {}^{t}\left\langle \alpha'_k\, {}^{3i}\langle \rho' \breve{E}_{kLF} \rangle \right\rangle . \tag{6.7.3}$$

In Eq. 6.7.3, a term involving triple time correlation is neglected.

$$\bullet \quad {}^{t}\left\langle \alpha_k\, {}^{3i}\langle \rho_k \underline{U}_k E_k \rangle \right\rangle = \left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle\right) {}^{3i}\langle \underline{U}_K \rangle_{LF}\, {}^{3i}\langle E_k \rangle_{LF}$$

$$+ {}^{3i}\langle \underline{U}_k \rangle_{LF}\, \Phi_{Ek} + \underline{\Psi}_{mk}\, {}^{3i}\langle E_k \rangle_{LF}$$

$$+ \alpha_{kLF} \left({}^{3i}\langle \underline{E}^{T}_k \rangle + {}^{3i}\langle \breve{\underline{E}}_k \rangle + {}^{3i}\langle \tilde{\underline{E}}^{T}_k \rangle\right) , \tag{6.7.4}$$

in which

(a) The volume-averaged turbulent total energy flux vector ${}^{3i}\langle \underline{E}^{T}_k \rangle$ is defined by

$${}^{3i}\langle \phi_\kappa \rangle_{LF}\, {}^{t3i}\langle \underline{U}'_k E'_k \rangle + {}^{t3i}\langle \tilde{\rho}_{kLF} \underline{U}'_k E'_k \rangle = {}^{3i}\langle \underline{E}^{T}_k \rangle \,, \tag{6.7.5}$$

(b) The volume-averaged dispersive total energy flux vector ${}^{3i}\langle \breve{\underline{E}}_k \rangle$ is defined by

$$ {}^{3i}\langle \rho_k \rangle_{LF} \; {}^{3i}\langle \check{\underline{U}}_{kLF} \check{E}_{kLF} \rangle + {}^{3i}\langle \check{\rho}_{kLF} \check{\underline{U}}_{kLF} \check{E}_{kLF} \rangle = {}^{3i}\langle \underline{\check{E}}_k \rangle \text{ , and} \qquad (6.7.6) $$

(c) The volume-averaged turbulent, dispersive total energy flux vector ${}^{3i}\langle \underline{\check{E}}^T_k \rangle$ is defined by

$$ \begin{aligned} & \alpha_{kLF} \left({}^{t3i}\langle \check{\underline{U}}_{kLF} \rho'_k E'_k \rangle + {}^{t3i}\langle \check{E}_{kLF} \rho'_k \underline{U}'_k \rangle \right) \\ & + {}^{3i}\langle \rho_k \rangle_{LF} \left({}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \check{\underline{U}}_{kLF} E'_k \rangle \right\rangle + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \underline{U}'_k \check{E}_{kLF} \rangle \right\rangle \right) \\ & + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \check{\rho}_{kLF} \check{\underline{U}}_{kLF} E'_k \rangle \right\rangle + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \check{\rho}_{kLF} \check{E}_{kLF} \underline{U}'_k \rangle \right\rangle \\ & + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \rho'_k \check{\underline{U}}_{kLF} \check{E}_{kLF} \rangle \right\rangle \\ & = \alpha_{kLF} \; {}^{3i}\langle \underline{\check{E}}^T_k \rangle \text{ .} \end{aligned} \qquad (6.7.7) $$

In Eq. 6.7.7, terms involving triple and quadruple time correlations have been neglected.

$$ \bullet \quad {}^{t}\left\langle \alpha_k \; {}^{3i}\langle P_k \underline{U}_k \rangle \right\rangle = \left(\alpha_{kLF} \; {}^{3i}\langle P_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k P'_k \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{Pk} \text{ ,} \qquad (6.7.8) $$

where $\underline{\Psi}_{Pk}$ is a vector pressure work function defined by

$$ \begin{aligned} \underline{\Psi}_{Pk} = {} & \alpha_{kLF} \; {}^{3i}\langle \check{P}_{kLF} \check{\underline{U}}_{kLF} \rangle + \alpha_{kLF} \; {}^{t3i}\langle P'_k \underline{U}'_k \rangle + {}^{3i}\langle P_k \rangle_{LF} \; {}^{t3i}\langle \alpha'_k \underline{U}'_k \rangle \\ & + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \check{P}_{kLF} \underline{U}'_k \rangle \right\rangle + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle P'_k \check{\underline{U}}_{kLF} \rangle \right\rangle \text{ .} \end{aligned} \qquad (6.7.9)^* $$

$$ \bullet \quad {}^{t}\left\langle \alpha_k \; {}^{3i}\langle \underline{\underline{\tau}}_k \cdot \underline{U}_k \rangle \right\rangle = \left(\alpha_{kLF} \; {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \underline{\underline{\tau}}'_k \rangle \right) \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{\tau k} \text{ ,} \qquad (6.7.10) $$

where $\underline{\Psi}_{\tau k}$ is a vector viscous stress work function defined by

$$ \begin{aligned} \underline{\Psi}_{\tau k} = {} & \alpha_{kLF} \; {}^{3i}\langle \check{\underline{\underline{\tau}}}_{kLF} \cdot \check{\underline{U}}_{kLF} \rangle + \alpha_{kLF} \; {}^{t3i}\langle \underline{\underline{\tau}}'_k \cdot \underline{U}'_k \rangle + {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{t3i}\langle \alpha'_k \underline{U}'_k \rangle \\ & + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \check{\underline{\underline{\tau}}}_{kLF} \cdot \underline{U}'_k \rangle \right\rangle + {}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \underline{\underline{\tau}}'_k \cdot \check{\underline{U}}_{kLF} \rangle \right\rangle \text{ .} \end{aligned} \qquad (6.7.11)^* $$

---

* ${}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \check{P}_{kLF} \underline{U}'_k \rangle \right\rangle = {}^{t3i}\langle \alpha'_k \check{P}_{kLF} \underline{U}'_k \rangle$, ${}^{t}\left\langle \alpha'_k \; {}^{3i}\langle P'_k \check{\underline{U}}_{kLF} \rangle \right\rangle = {}^{t3i}\langle \alpha'_k P'_k \check{\underline{U}}_{kLF} \rangle$

${}^{t}\left\langle \alpha'_k \; {}^{3i}\langle \check{\underline{\underline{\tau}}}_{kLF} \cdot \underline{U}'_k \rangle \right\rangle = {}^{t3i}\langle \alpha'_k \check{\underline{\underline{\tau}}}_{kLF} \cdot \underline{U}'_k \rangle$, etc.

For Newtonian fluids, ${}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF}$, $\underline{\underline{\tilde{\tau}}}_{kLF}$, and $\underline{\underline{\tau}}'_k$ are given by Eqs. 6.5.8d, 6.5.8e, and 6.5.8f, respectively.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle \right\rangle = \alpha_{kLF} \, {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} + {}^{t3i}\langle \alpha'_k \underline{J}'_{qk} \rangle \tag{6.7.12}$$

The Fourier law of isotropic conduction states that

$$\underline{J}_{qk} = - \kappa_k \nabla T_k \, . \tag{6.7.13a}$$

If the thermal conductivity κ_k is independent of T_k, then

$${}^{3i}\langle \underline{J}_{qk} \rangle_{LF} = - \kappa_k \nabla \, {}^{3i}\langle T_k \rangle_{LF} \tag{6.7.13b}$$

and

$${}^{t3i}\langle \alpha'_k \underline{J}'_{qk} \rangle = - \kappa_k \, {}^{t3i}\langle \alpha'_k \nabla T'_k \rangle \, . \tag{6.7.13c}$$

The derivations of Eq. 6.7.13b and c are analogous to those of Eqs. 6.5.8d and 6.5.8g.

When expressed in terms of internal energy u_k, Eqs. 6.7.13b and c become for constant specific heat c_{vk}:

$${}^{3i}\langle \underline{J}_{qk} \rangle_{LF} = - \frac{\kappa_k}{c_{vk}} \nabla \, {}^{3i}\langle u_k \rangle_{LF} \tag{6.7.13d}$$

and

$${}^{t3i}\langle \alpha'_k \underline{J}'_{qk} \rangle = - \frac{\kappa_k}{c_{vk}} \, {}^{t3i}\langle \alpha'_k \nabla u'_k \rangle \, . \tag{6.7.13e}$$

The case of variable conductivity and specific heat is treated in Appendix C.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k \rangle \right\rangle = \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \, , \tag{6.7.14}$$

which is Eq. 6.3.2.b. The vector function $\underline{\Psi}_{mk}$ is defined in Eq. 6.3.5.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle J_{Ek} \rangle \right\rangle = \alpha_{kLF} \, {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha'_k J'_{EK} \rangle \tag{6.7.15}$$

$$\bullet \quad {}^{t}\langle \dot{Q}_k \rangle = - v^{-1} \left\langle {}^{t}\!\int_{A_k} \underline{J}_{qk} \cdot \underline{n}_k \, dA \right\rangle$$

$$= - v^{-1} \int_{A_{kLF}} {}^{t}\langle \underline{J}_{qk} \cdot \underline{n}_{kLF} \rangle \, dA - v^{-1} \left\langle {}^{t}\!\int_{A'_k} \underline{J}_{qk} \cdot \underline{n}'_k \, dA \right\rangle \qquad (6.7.16)$$

Now

$$- v^{-1} \int_{A_{kLF}} {}^{t}\langle \underline{J}_{qk} \cdot \underline{n}_{kLF} \rangle \, dA = {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} \cdot \nabla \alpha_{kLF}$$

$$- v^{-1} \int_{A_{kLF}} \underline{\tilde{J}}_{qkLF} \cdot \underline{n}_{kLF} \, dA \, , \qquad (6.7.17)$$

and

$$- v^{-1} \left\langle {}^{t}\!\int_{A'_k} \underline{J}_{qk} \cdot \underline{n}'_k \, dA \right\rangle = - v^{-1} \left\langle {}^{t}\!\int_{A'_k} \underline{J}'_{qk} \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.7.18)$$

Hence,

$$ {}^{t}\langle \dot{Q}_k \rangle = {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} \cdot \nabla \alpha_{kLF} + (\mathrm{HTI})_k \, , \qquad (6.7.19)$$

where $(\mathrm{HTI})_k$ is the interfacial heat transfer integral defined by

$$(\mathrm{HTI})_k = - v^{-1} \int_{A_{kLF}} \underline{\tilde{J}}_{qkLF} \cdot \underline{n}_{kLF} \, dA - v^{-1} \left\langle {}^{t}\!\int_{A'_k} \underline{J}'_{qk} \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.7.20)$$

The interfacial heat transfer rate per unit volume of the mixture, ${}^{t}\langle \dot{Q}_k \rangle$, can be correlated by experimental data.

$$\bullet \quad v^{-1} \left\langle {}^{t}\!\int_{A_k} \left(-P_k \underline{\underline{U}}_k + \underline{\underline{\tau}}_k \cdot \underline{U}_k \right) \cdot \underline{n}_k \, dA \right\rangle$$

$$= v^{-1} \int_{A_{kLF}} {}^{t}\langle -P_k \underline{\underline{U}}_k + \underline{\underline{\tau}}_k \cdot \underline{U}_k \rangle \cdot \underline{n}_{kLF} \, dA$$

$$+ v^{-1} \left\langle {}^{t}\!\int_{A'_k} \left(-P_k \underline{\underline{U}}_k + \underline{\underline{\tau}}_k \cdot \underline{U}_k \right) \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.7.21)$$

Now

$$v^{-1} \int_{A_{kLF}} {}^{t}\langle -P_k \underline{\underline{U}}_k + \underline{\underline{\tau}}_k \cdot \underline{U}_k \rangle \cdot \underline{n}_{kLF} \, dA$$

$$= {}^{3i}\langle P_k \rangle_{LF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} - v^{-1} \, {}^{3i}\langle P_k \rangle_{LF} \int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \int_{A_{kLF}} \check{P}_{kLF} \, \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \check{P}_{kLF} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA \quad - v^{-1} \int_{A_{kLF}} {}^{t}\langle P'_k \underline{U}'_k \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- \left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \cdot \nabla \alpha_{kLF}$$

$$+ v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot \check{\underline{U}}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$+ v^{-1} \int_{A_{kLF}} \left(\check{\underline{\underline{\tau}}}_{kLF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$+ v^{-1} \int_{A_{kLF}} \left(\check{\underline{\underline{\tau}}}_{kLF} \cdot \check{\underline{U}}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$+ v^{-1} \int_{A_{kLF}} {}^{t}\langle \underline{\underline{\tau}}'_k \cdot \underline{U}'_k \rangle \cdot \underline{n}_{kLF} \, dA \, . \qquad (6.7.22)$$

Again, we note that for Newtonian fluids, the term involving the integral $\int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$ vanishes. The second term on the right-hand side of Eq. 6.7.21 is

$$v^{-1} \left\langle \int_{A'_k} {}^{t}\left(-P_k \underline{U}_k + \underline{\underline{\tau}}_k \cdot \underline{U}_k \right) \cdot \underline{n}'_k \, dA \right\rangle$$

$$= v^{-1} \left\langle \int_{A'_k} {}^{t} - \left({}^{3i}\langle P_k \rangle_{LF} + \check{P}_{kLF} \right) \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle$$

$$+ v^{-1} \left\langle \int_{A'_k} {}^{t} - P'_k \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} \right) \cdot \underline{n}'_k \, dA \right\rangle$$

$$+ v^{-1} \left\langle \int_{A'_k} {}^{t} -P'_k \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle$$

$$+ v^{-1} \left\langle \int_{A'_k} \left[\left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \tilde{\underline{\underline{\tau}}}_{kLF} \right) \cdot \underline{U}'_k \right] \cdot \underline{n}'_k \, dA \right\rangle$$

$$+ v^{-1} \left\langle \int_{A'_k} \left[\underline{\underline{\tau}}'_k \cdot \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \tilde{\underline{U}}_{kLF} \right) \right] \cdot \underline{n}'_k \, dA \right\rangle$$

$$+ v^{-1} \left\langle \int_{A'_k} \underline{\underline{\tau}}'_k \cdot \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.7.23)$$

Thus,

$$v^{-1} \left\langle \int_{A_k} \left(-P_k \underline{\underline{U}}_k + \underline{\underline{\tau}}_k \cdot \underline{U}_k \right) \cdot \underline{n}_k \, dA \right\rangle$$

$$= {}^{3i}\langle P_k \rangle_{LF} \; {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} - \left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \cdot \nabla \alpha_{kLF}$$

$$+ (PWI)_k - (VWI)_k , \qquad (6.7.24)$$

in which the interfacial pressure work integral $(PWI)_k$ is defined by:

$$(PWI)_k = - v^{-1} \, {}^{3i}\langle P_k \rangle_{LF} \int_{A_{kLF}} \tilde{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \int_{A_{kLF}} \tilde{P}_{kLF} \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left(\tilde{P}_{kLF} \tilde{\underline{U}}_{kLF} + {}^{t}\langle P'_k \underline{U}'_k \rangle \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \left\langle \int_{A'_k} \left({}^{3i}\langle P_k \rangle_{LF} + \tilde{P}_{kLF} \right) \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle$$

$$- v^{-1} \left\langle \int_{A'_k} P'_k \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \tilde{\underline{U}}_{kLF} \right) \cdot \underline{n}'_k \, dA \right\rangle$$

$$- v^{-1} \left\langle \int_{A'_k} P'_k \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle \qquad (6.7.25)$$

and the interfacial viscous stress work integral $(VWI)_k$ is defined by

$$(\mathrm{VWI})_k = - v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot \check{\underline{U}}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left(\check{\underline{\underline{\tau}}}_{kLF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left(\check{\underline{\underline{\tau}}}_{kLF} \cdot \check{\underline{U}}_{kLF} + {}^{t}\langle \underline{\underline{\tau}}'_k \cdot \underline{U}'_k \rangle \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \, {}^{t}\left\langle \int_{A'_k} \left[\left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \check{\underline{\underline{\tau}}}_{kLF} \right) \cdot \underline{U}'_k \right] \cdot \underline{n}'_k \, dA \right\rangle$$

$$- v^{-1} \, {}^{t}\left\langle \int_{A'_k} \left[\underline{\underline{\tau}}'_k \cdot \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} \right) \right] \cdot \underline{n}'_k \, dA \right\rangle$$

$$- v^{-1} \, {}^{t}\left\langle \int_{A'_k} \left(\underline{\underline{\tau}}'_k \cdot \underline{U}'_k \right) \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.7.26)$$

$$\bullet \quad - v^{-1} \, {}^{t}\left\langle \int_{A_k} \rho_k E_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}_k \, dA \right\rangle$$

$$= - v^{-1} \int_{A_{kLF}} {}^{t}\langle \rho_k E_k \left(\underline{U}_k - \underline{W}_k \right) \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \, {}^{t}\left\langle \int_{A'_k} \rho_k E_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}'_k \, dA \right\rangle . \qquad (6.7.27)$$

Now

$$- v^{-1} \int_{A_{kLF}} {}^{t}\langle \rho_k E_k \left(\underline{U}_k - \underline{W}_k \right) \rangle \cdot \underline{n}_{kLF} \, dA$$

$$= {}^{3i}\langle \rho_k \rangle_{LF} \; {}^{3i}\langle E_k \rangle_{LF} \left(\frac{\partial \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} \right)$$

$$- v^{-1} \; {}^{3i}\langle \rho_k \rangle_{LF} \; {}^{3i}\langle E_k \rangle_{LF} \int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \; {}^{3i}\langle E_k \rangle_{LF} \int_{A_{kLF}} \check{\rho}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left[\left({}^{3i}\langle \varphi_k \rangle_{LF} + \breve{\rho}_{kLF} \right) \breve{E}_{kLF} + {}^{t}\langle \rho_k' E_k' \rangle \right]$$

$$\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \varphi_k \rangle_{LF} + \breve{\rho}_{kLF} \right) {}^{t}\langle E_k' \, (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle E_k \rangle_{LF} + \breve{E}_{kLF} \right) {}^{t}\langle \rho_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_{kLF} \, dA \qquad (6.7.28)$$

and

$$- v^{-1} \left\langle {}^{t} \int_{A_k'} \rho_k E_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k' \, dA \right\rangle$$

$$= - v^{-1} \left\langle {}^{t} \int_{A_k'} \left[\left({}^{3i}\langle \varphi_k \rangle_{LF} + \breve{\rho}_{kLF} + \rho_k' \right) E_k' + \rho_k' \left({}^{3i}\langle E_k \rangle_{LF} + \breve{E}_{kLF} \right) \right] \right.$$

$$\left. \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \right\rangle$$

$$- v^{-1} \left\langle {}^{t} \int_{A_k'} \left[\left({}^{3i}\langle \varphi_k \rangle_{LF} + \breve{\rho}_{kLF} \right) \left({}^{3i}\langle E_k \rangle_{LF} + \breve{E}_{kLF} + E_k' \right) \right. \right.$$

$$\left. \left. + \rho_k' \left({}^{3i}\langle E_k \rangle_{LF} + \breve{E}_{kLF} \right) \right] (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle$$

$$- v^{-1} \left\langle {}^{t} \int_{A_k'} \rho_k' E_k' (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle . \qquad (6.7.29)$$

Multiplying Eq. 6.3.10 by ${}^{3i}\langle E_k \rangle_{LF}$, followed by introducing the result into Eqs. 6.7.27, 6.7.28, and 6.7.29, leads to

$$- v^{-1} \left\langle {}^{t} \int_{A_k} \rho_k E_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \right\rangle$$

$$= {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle E_k \rangle_{LF} + (\mathrm{TETI})_k , \qquad (6.7.30)$$

in which the interfacial total energy transfer integral $(\mathrm{TETI})_k$ is defined by

$$(\mathrm{TETI})_k = - v^{-1} \int_{A_{kLF}} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \tilde{\rho}_{kLF} \right) \tilde{E}_{kLF} + {}^{t}\langle \rho_k' E_k' \rangle \right]$$

$$\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \tilde{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \rho_k \rangle_{LF} + \tilde{\rho}_{kLF} \right) {}^{t}\langle E_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \int_{A_{kLF}} \tilde{E}_{kLF} \, {}^{t}\langle \rho_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \left\langle \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \tilde{\rho}_{kLF} + \rho_k' \right) E_k' + \rho_k' \tilde{E}_{kLF} \right] \right.$$

$$\left. \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \tilde{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \right\rangle^{t}$$

$$- v^{-1} \left\langle \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \tilde{\rho}_{kLF} \right) \left(\tilde{E}_{kLF} + E_k' \right) + \rho_k' \tilde{E}_{kLF} \right] \right.$$

$$\left. (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle^{t}$$

$$- v^{-1} \left\langle \int_{A_k'} \rho_k' E_k' (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \right\rangle^{t} . \qquad (6.7.31)$$

The similarity between Eqs. 6.7.30 and 6.5.20 is noted, as is that between Eqs. 6.7.31 and 6.5.21.

The time-averaged interfacial total energy source per unit volume, ${}^{t}\langle \mathcal{E}_k \rangle$, is seen to consist of three parts: (1) interfacial heat transfer ${}^{t}\langle \dot{Q}_k \rangle$, (2) interfacial work done by pressure and viscous forces as given by Eq. 6.7.24, and (3) interfacial total energy transfer as given by Eq. 6.7.30. Thus,

$${}^{t}\langle \mathcal{E}_k \rangle = {}^{t}\langle \dot{Q}_k \rangle + {}^{3i}\langle P_k \rangle_{LF} \; {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF}$$

$$- \left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \cdot \nabla \alpha_{kLF}$$

$$+ {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle E_k \rangle_{LF} + (\mathrm{PWI})_k - (\mathrm{VWI})_k + (\mathrm{TETI})_k . \qquad (6.7.32)$$

Performing the time averaging of Eq. 5.1.4, followed by introducing the results given by Eqs. 6.7.2, 6.7.4, 6.7.8, 6.7.10, 6.7.12, 6.7.14, 6.7.15, 6.7.19, 6.7.24, 6.7.30, and 6.7.32, one obtains, after combining and rearranging terms, the time-volume averaged total energy conservation equation:

$$\frac{\partial}{\partial t}\left(\alpha_{kLF}\ {}^{3i}\langle\varphi_k\rangle_{LF} + {}^{t3i}\langle\alpha'_k\rho'_k\rangle\right)\ {}^{3i}\langle E_k\rangle_{LF}$$

$$+\nabla\cdot\left(\alpha_{kLF}\ {}^{3i}\langle\varphi_k\rangle_{LF} + {}^{t3i}\langle\alpha'_k\rho'_k\rangle\right)\ {}^{3i}\langle\underline{U}_k\rangle_{LF}\ {}^{3i}\langle E_k\rangle_{LF}$$

$$+\frac{\partial\ \Phi_{Ek}}{\partial t} + \nabla\cdot\Phi_{Ek}\ {}^{3i}\langle\underline{U}_k\rangle_{LF} + \nabla\cdot\underline{\Psi}_{mk}\ {}^{3i}\langle E_k\rangle_{LF}$$

$$= -\nabla\cdot\left(\alpha_{kLF}\ {}^{3i}\langle P_k\rangle_{LF} + {}^{t3i}\langle\alpha'_k P'_k\rangle\right)\ {}^{3i}\langle\underline{U}_k\rangle_{LF} - \nabla\cdot\underline{\Psi}_{pk}$$

$$+\nabla\cdot\left[\left(\alpha_{kLF}\ {}^{3i}\langle\underline{\underline{\tau}}_k\rangle + {}^{t3i}\langle\alpha'_k\underline{\underline{\tau}}'_k\rangle\right)\cdot\ {}^{3i}\langle\underline{U}_k\rangle_{LF}\right] + \nabla\cdot\underline{\Psi}_{\tau k}$$

$$-\nabla\cdot\alpha_{kLF}\left({}^{3i}\langle\underline{\underline{E}}^{T}_k\rangle + {}^{3i}\langle\underline{\underline{\tilde{E}}}_k\rangle\ {}^{3i}\langle\underline{\underline{\tilde{E}}}^{T}_k\rangle\right)$$

$$-\nabla\cdot\left(\alpha_{kLF}\ {}^{3i}\langle\underline{J}_{qk}\rangle_{LF} + {}^{t3i}\langle\alpha'_k\underline{J}'_{qk}\rangle\right)$$

$$+\left(\alpha_{kLF}\ {}^{3i}\langle\varphi_k\rangle_{LF} + {}^{t3i}\langle\alpha'_k\rho'_k\rangle\right)\ {}^{3i}\langle\underline{U}_k\rangle_{LF}\cdot\underline{f} + \underline{\Psi}_{mk}\cdot\underline{f}$$

$$+\alpha_{kLF}\ {}^{3i}\langle J_{Ek}\rangle_{LF} + {}^{t3i}\langle\alpha'_k J'_{Ek}\rangle + {}^{t}\langle\mathcal{E}_k\rangle\ , \qquad (6.7.33a)$$

or, equivalently,

$$\frac{\partial}{\partial t}\left(\alpha_{kLF}\ {}^{3i}\langle\varphi_k\rangle_{LF} + {}^{t3i}\langle\alpha'_k\rho'_k\rangle\right)\ {}^{3i}\langle E_k\rangle_{LF}$$

$$+\nabla\cdot\left(\alpha_{kLF}\ {}^{3i}\langle\varphi_k\rangle_{LF} + {}^{t3i}\langle\alpha'_k\rho'_k\rangle\right)\ {}^{3i}\langle\underline{U}_k\rangle_{LF}\ {}^{3i}\langle E_k\rangle_{LF}$$

$$+ \frac{\partial \Phi_{Ek}}{\partial t} + \nabla \cdot \Phi_{Ek}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot \underline{\Psi}_{mk}\, {}^{3i}\langle E_k \rangle_{LF}$$

$$= - \nabla \cdot \alpha_{kLF}\, {}^{3i}\langle P_k \rangle_{LF}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} - \nabla \cdot \left({}^{t3i}\langle \alpha_k' P_k' \rangle\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{pk} \right)$$

$$+ \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{\tau}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) + \nabla \cdot \left({}^{t3i}\langle \alpha_k' \underline{\tau}_k' \rangle \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{\tau k} \right)$$

$$- \nabla \cdot \left(\alpha_{kLF}\, {}^{3i}\langle \underline{E}_k^T \rangle + {}^{t3i}\langle \underline{\tilde{E}}_k \rangle + {}^{3i}\langle \underline{\tilde{E}}_k^T \rangle \right)$$

$$- \nabla \cdot \alpha_{kLF}\, {}^{3i}\langle \underline{J}_{qk} \rangle - \nabla \cdot {}^{t3i}\langle \alpha_k' \underline{J}_{qk}' \rangle$$

$$+ \left[\left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \right] \cdot \underline{f}$$

$$+ \alpha_{kLF}\, {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha_k' J_{Ek}' \rangle$$

$$+ {}^{t}\langle \mathcal{T}_k \rangle\, {}^{3i}\langle E_k \rangle_{LF} + {}^{t}\langle \dot{Q}_k \rangle + {}^{3i}\langle P_k \rangle_{LF}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF}$$

$$- \left({}^{3i}\langle \underline{\tau}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \cdot \nabla \alpha_{kLF}$$

$$+ (PWI)^k - (VWI)_k + (TETI)^k \,. \qquad (6.7.33b)$$

An alternative form of the time-volume averaged total energy conservation equation is obtained if ${}^{t}\langle \mathcal{T}_k \rangle$ in Eq. 6.7.33b is replaced by the relation given by Eq. 6.3.13:

$$\left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) \left(\frac{\partial\, {}^{3i}\langle E_k \rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla\, {}^{3i}\langle E_k \rangle_{LF} \right)$$

$$+ \frac{\partial \Phi_{Ek}}{\partial t} + \nabla \cdot \Phi_{Ek}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \cdot \nabla\, {}^{3i}\langle E_k \rangle_{LF}$$

$$= - \alpha_{kLF} \nabla \cdot {}^{3i}\langle P_k \rangle_{LF}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} - \nabla \cdot \left({}^{t3i}\langle \alpha_k' P_k' \rangle\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{pk} \right)$$

$$+ \alpha_{kLF} \nabla \cdot \left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) + \nabla \cdot \left({}^{t3i}\langle \alpha'_k \underline{\underline{\tau}}'_k \rangle \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{\tau k} \right)$$

$$- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{E}^T_k \rangle + {}^{3i}\langle \underline{\tilde{E}}_k \rangle + {}^{3i}\langle \underline{\tilde{E}}^T_k \rangle \right)$$

$$- \nabla \cdot \alpha_{kLF} \, {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} - \nabla \cdot {}^{t3i}\langle \alpha'_k \underline{J}'_{qk} \rangle$$

$$+ \left[\left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \rho'_k \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \right] \cdot \underline{f}$$

$$+ \alpha_{kLF} \, {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha'_k J'_{Ek} \rangle$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + (PWI)_k - (VWI)_k + (TETI)_k \, . \qquad (6.7.33c)$$

When $\underline{\rho_k = \text{constant}}$, Eq. 6.7.33b reduces to

$$\rho_k \left(\frac{\partial}{\partial t} \alpha_{kLF} \, {}^{3i}\langle E_k \rangle_{LF} + \nabla \cdot \alpha_{kLF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} \, {}^{3i}\langle E_k \rangle_{LF} \right)$$

$$+ \frac{\partial \, {}^{o}\Phi_{Ek}}{\partial t} + \nabla \cdot {}^{o}\Phi_{Ek} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot {}^{o}\underline{\Psi}_{mk} \, {}^{3i}\langle E_k \rangle_{LF}$$

$$= - \nabla \cdot \alpha_{kLF} \, {}^{3i}\langle P_k \rangle_{LF} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} - \nabla \cdot \left({}^{t3i}\langle \alpha'_k P'_k \rangle \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{pk} \right)$$

$$+ \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) + \nabla \cdot \left({}^{t3i}\langle \alpha'_k \, {}^{o}\underline{\underline{\tau}}'_k \rangle \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\underline{\Psi}_{\tau k} \right)$$

$$- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle {}^{o}\underline{E}^T_k \rangle + {}^{3i}\langle {}^{o}\underline{\tilde{E}}_k \rangle + {}^{3i}\langle {}^{o}\underline{\tilde{E}}^T_k \rangle \right)$$

$$- \nabla \cdot \alpha_{kLF} \, {}^{3i}\langle \underline{J}_{qk} \rangle - \nabla \cdot {}^{t3i}\langle \alpha'_k \underline{J}'_{qk} \rangle$$

$$+ \left(\alpha_{kLF} \rho_k \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\underline{\Psi}_{mk} \right) \cdot \underline{f} + \alpha_{kLF} \, {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha'_k J'_{Ek} \rangle$$

$$+ {}^{ot}\langle \Gamma_k \rangle \, {}^{3i}\langle E_k \rangle_{LF} + {}^{t}\langle \dot{Q}_k \rangle + {}^{3i}\langle P_k \rangle_{LF} \, {}^{3i}\langle \underline{U}_k \rangle \cdot \nabla \alpha_{kLF}$$

$$- \left({}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k \rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right) \cdot \nabla \alpha_{kLF}$$

$$+ \left(PWI \right)_k - {}^{o}\left(VWI \right)_k + {}^{o}\left(TETI \right)_k \, , \qquad (6.7.34a)$$

in which

$${}^{o}\Phi_{Ek} = \rho_k \, {}^{t3i}\langle \alpha_k' E_k' \rangle \qquad (6.7.35)$$

$${}^{3i}\langle {}^{o}\underline{E}_k^{T} \rangle = \rho_k \, {}^{t3i}\langle \underline{U}_k' E_k' \rangle \qquad (6.7.36)$$

$${}^{3i}\langle {}^{o}\check{\underline{E}}_k \rangle = \rho_k \, {}^{3i}\langle \check{\underline{U}}_{kLF} \check{E}_{kLF} \rangle \qquad (6.7.37)$$

$$\alpha_{kLF} \, {}^{3i}\langle {}^{o}\check{\underline{E}}_k^{T} \rangle = \rho_k \left({}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \check{\underline{U}}_{kLF} E_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \underline{U}_k' \check{E}_{kLF} \rangle \right\rangle \right) \qquad (6.7.38)$$

and

$$\begin{aligned} {}^{o}\left(TETI \right)_k = & - v^{-1} \rho_k \int_{A_{kLF}} \check{E}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA \\ & - v^{-1} \rho_k \int_{A_{kLF}} {}^{t}\langle E_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_{kLF} \, dA \\ & - v^{-1} \rho_k \left\langle {}^{t}\int_{A_k'} E_k' \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \right\rangle \\ & - v^{-1} \rho_k \left\langle {}^{t}\int_{A_k'} \left(\check{E}_{kLF} + \check{E}_k' \right) \left(\underline{U}_k' - \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \right\rangle . \end{aligned} \qquad (6.7.39)$$

The functions ${}^{o}\underline{\Psi}_{mk}$, ${}^{3i}\langle {}^{o}\underline{\underline{L}}_k \rangle_{LF}$, ${}^{t3i}\langle \alpha_k' \, {}^{o}\underline{\underline{L}}_k' \rangle$, and ${}^{ot}\langle \Gamma_k \rangle$ are given by Eqs. 6.3.15, 6.5.26a, 6.5.26b, and 6.5.31, respectively.

The vector function $\underline{\Psi}_{Pk}$ and the interfacial pressure work integral $(PWI)_k$ remain unaltered. They are given by Eqs. 6.7.9 and 6.7.25, respectively. The vector function ${}^{o}\underline{\Psi}_{\tau k}$ and the interfacial work integral due to viscous stress ${}^{o}(VWI)_k$ are given by Eqs. 6.7.11 and 6.7.26, but with the viscous stress tensor simplified for constant density.

An alternative form of Eq. 6.7.34a is

$$
\begin{aligned}
&\alpha_{kLF}\,\rho_k\left(\frac{\partial\,{}^{3i}\langle E_k\rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k\rangle_{LF}\cdot\nabla\,{}^{3i}\langle E_k\rangle_{LF}\right)\\
&+\frac{\partial\,{}^{o}\Phi_{Ek}}{\partial t} + \nabla\cdot{}^{o}\Phi_{Ek}\,{}^{3i}\langle \underline{U}_k\rangle_{LF} + {}^{o}\underline{\Psi}_{mk}\cdot\nabla\,{}^{3i}\langle E_k\rangle_{LF}\\
&= -\alpha_{kLF}\,\nabla\cdot{}^{3i}\langle P_k\rangle_{LF}\,{}^{3i}\langle \underline{U}_k\rangle_{LF} - \nabla\cdot\left({}^{t3i}\langle\alpha'_k\underline{P}'_k\rangle\,{}^{3i}\langle \underline{U}_k\rangle_{LF} + \underline{\Psi}_{Pk}\right)\\
&+\alpha_{kLF}\,\nabla\cdot{}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k\rangle_{LF}\cdot{}^{3i}\langle \underline{U}_k\rangle_{LF} + \nabla\cdot\left({}^{t3i}\langle\alpha'_k\,{}^{o}\underline{\underline{\tau}}'_k\rangle\cdot{}^{3i}\langle \underline{U}_k\rangle_{LF} + {}^{o}\underline{\Psi}_{\tau k}\right)\\
&-\nabla\cdot\alpha_{kLF}\left({}^{3i}\langle {}^{o}\underline{E}^{T}_{k}\rangle_{LF} + {}^{3i}\langle {}^{o}\underline{\tilde{E}}_k\rangle\;{}^{3i}\langle {}^{o}\underline{\tilde{E}}^{T}_{k}\rangle\right)\\
&-\nabla\cdot\alpha_{kLF}\,{}^{3i}\langle \underline{J}_{qk}\rangle - \nabla\cdot{}^{t3i}\langle\alpha'_k\underline{J}'_{qk}\rangle\\
&+\left(\alpha_{kLF}\,\rho_k\,{}^{3i}\langle \underline{U}_k\rangle_{LF} + {}^{o}\underline{\Psi}_{mk}\right)\cdot\underline{f} + \alpha_{kLF}\,{}^{3i}\langle J_{Ek}\rangle_{LF} + {}^{t3i}\langle\alpha'_k J'_{Ek}\rangle\\
&+{}^{t}\langle\dot{Q}_k\rangle + (PWI)_k - {}^{o}(VWI)_k + {}^{o}(TETI)_k\,,
\end{aligned}
\tag{6.7.34b}
$$

which can be readily written down from Eq. 6.7.33c.

6.8 Time-Volume-averaged Interfacial Total Energy Balance Equation $(Y_v = Y_A = 1)$

The local volume-averaged interfacial total energy balance equation is given by Eq. 5.24 when the capillary energy is ignored. By performing time averaging and making use of Eqs. 6.7.16, 6.7.24, 6.7.30, and 6.7.32, one obtains the following two alternative forms:

$$
{}^{t}\langle\mathcal{E}_k\rangle + {}^{t}\langle\mathcal{E}_f\rangle = 0 \tag{6.8.1a}
$$

or, equivalently,

$$\left({}^{3i}\langle P_k\rangle_{LF}\ {}^{3i}\langle \underline{U}_k\rangle_{LF} - {}^{3i}\langle \underline{\underline{\tau}}_k\rangle_{LF} \cdot {}^{3i}\langle \underline{U}_k\rangle_{LF}\right) \cdot \nabla\, \alpha_{kLF} + {}^{t}\langle \dot{Q}_k\rangle$$

$$+ {}^{t}\langle \Gamma_k\rangle\ {}^{3i}\langle E_k\rangle_{LF} + (PWI)_k - (VWI)_k + (TETI)_k$$

$$+ \left({}^{3i}\langle P_f\rangle_{LF}\ {}^{3i}\langle \underline{U}_f\rangle_{LF} - {}^{3i}\langle \underline{\underline{\tau}}_f\rangle_{LF} \cdot {}^{3i}\langle \underline{U}_f\rangle_{LF}\right) \cdot \nabla\, \alpha_{fLF} + {}^{t}\langle \dot{Q}_f\rangle$$

$$+ {}^{t}\langle \Gamma_f\rangle\ {}^{3i}\langle E_f\rangle_{LF} + (PWI)_f - (VWI)_f + (TETI)_f = 0 \ . \qquad (6.8.1b)$$

When ρ_k = constant, ${}^{3i}\langle \underline{\underline{\tau}}_k\rangle_{LF}$ should be replaced by ${}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k\rangle_{LF}$, ${}^{t}\langle \Gamma_k\rangle$ by ${}^{ot}\langle \Gamma_k\rangle$, $(VWI)_k$ by ${}^{o}(VWI)_k$, $(TETI)_k$ by ${}^{o}(TETI)_k$, etc. Likewise, the same simplifications should be introduced for all analogous quantities with subscript f.

6.9 Time-Volume-averaged Internal Energy Conservation Equation ($\gamma_V = \gamma_A = 1$)

The local volume-averaged internal energy conservation equation is given by Eq. 5.1.6:

$$\frac{\partial}{\partial t}\alpha_k\ {}^{3i}\langle \rho_k u_k\rangle + \nabla \cdot \alpha_k\ {}^{3i}\langle \rho_k \underline{U}_k u_k\rangle = -\alpha_k\ {}^{3i}\langle P_k \nabla \cdot \underline{U}_k\rangle - \nabla \cdot \alpha_k\ {}^{3i}\langle \underline{J}_{qk}\rangle$$

$$+ \alpha_k \left({}^{3i}\langle J_{Ek}\rangle + {}^{3i}\langle \phi_k\rangle\right) + \dot{Q}_k - v^{-1} \int_{A_k} \rho_k u_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \ , \qquad (5.1.6)$$

in which

$$\dot{Q}_k = -v^{-1} \int_{A_k} \underline{J}_{qk} \cdot \underline{n}_k \, dA \ , \qquad (5.1.5)$$

and ϕ_k is the dissipation function given by

$$\phi_k = \underline{\underline{\tau}}_k : \nabla, \underline{U}_k \ . \qquad (5.1.7)$$

The time average of $\dot{Q}_k$ is given by

$${}^{t}\langle \dot{Q}_k\rangle = {}^{3i}\langle \underline{J}_{qk}\rangle_{LF} \cdot \nabla \alpha_{kLF} + (HTI)_k \ , \qquad (6.7.19)$$

where the interfacial heat transfer integral $(HTI)_k$ is defined in Eq. 6.7.20. Also, the time averages of $\alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle$ and of $\alpha_k \, {}^{3i}\langle \underline{J}_{Ek} \rangle$ have been presented. They are given by Eqs. 6.7.12 and 6.7.15, respectively. The time averages of the remaining terms in Eq. 5.1.6 are as follows:

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \rho_k u_k \rangle \right\rangle = \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) {}^{3i}\langle u_k \rangle_{LF} + \Phi_{uk} \, , \tag{6.9.1}$$

where Φ_{uk} is a scalar internal energy function defined by

$$\Phi_{uk} = \alpha_{kLF} \, {}^{3i}\langle \tilde{\rho}_{kLF} \tilde{u}_{kLF} \rangle + \alpha_{kLF} \, {}^{t3i}\langle \rho_k' u_k' \rangle + {}^{3i}\langle \rho_k \rangle_{LF} \, {}^{t3i}\langle \alpha_k' u_k' \rangle$$

$$+ {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \tilde{\rho}_{kLF} u_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \rho_k' \tilde{u}_k \rangle \right\rangle \, , \tag{6.9.2}$$

in which a term involving triple time correlation has been deleted.

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k u_k \rangle \right\rangle = \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} \, {}^{3i}\langle u_k \rangle_{LF}$$

$$+ {}^{3i}\langle \underline{U}_k \rangle_{LF} \, \Phi_{uk} + \underline{\Psi}_{mk} \, {}^{3i}\langle u_k \rangle_{LF}$$

$$+ \alpha_{kLF} \left({}^{3i}\langle \underline{u}_k^T \rangle + {}^{3i}\langle \tilde{\underline{u}}_k \rangle + {}^{3i}\langle \tilde{\underline{u}}_k^T \rangle \right) \, , \tag{6.9.3}$$

in which

(a) The volume-averaged turbulent internal energy flux ${}^{3i}\langle \underline{u}_k^T \rangle$ is defined by

$${}^{3i}\langle \rho_k \rangle_{LF} \, {}^{t3i}\langle \underline{U}_k' u_k' \rangle + {}^{t3i}\langle \tilde{\rho}_{kLF} \underline{U}_k' u_k' \rangle = {}^{3i}\langle \underline{u}_k^T \rangle \, . \tag{6.9.4}$$

${}^{3i}\langle \underline{u}_k^T \rangle$ can be expressed in terms of eddy diffusivity for internal energy transfer D_{uk}^T according to

$${}^{3i}\langle \underline{u}_k^T \rangle = - \, {}^{3i}\langle \rho_k \rangle_{LF} \, D_{uk}^T \, \nabla \, {}^{3i}\langle u_k \rangle_{LF} \, . \tag{6.9.5a}$$

Since $\nabla \, {}^{3i}\langle u_k \rangle_{LF} = c_{vk} \nabla \, {}^{3i}\langle T_k \rangle_{LF}$, c_{vk} being the specific heat at constant volume (assumed constant in the present analysis), Eq. 6.9.5a can be written as

$$ {}^{3i}\langle \underline{u}_k^T \rangle = - \kappa_k^T \nabla \, {}^{3i}\langle T_k \rangle_{LF} \, , \tag{6.9.5b} $$

where κ_k^T is the turbulent conductivity related to D_{uk}^T according to

$$ \kappa_k^T = {}^{3i}\langle \rho_k \rangle_{LF} \, c_{vk} \, D_{uk}^T \, . \tag{6.9.6} $$

(b) The volume-averaged dispersive internal energy flux ${}^{3i}\langle \check{\underline{u}}_k \rangle$ is defined by

$$ {}^{3i}\langle \rho_k \rangle_{LF} \, {}^{3i}\langle \check{\underline{U}}_{kLF} \check{u}_{kLF} \rangle + {}^{3i}\langle \check{\rho}_{kLF} \check{\underline{U}}_{kLF} \check{u}_{kLF} \rangle = {}^{3i}\langle \check{\underline{u}}_k \rangle \, . \tag{6.9.7} $$

(c) The volume-averaged turbulent, dispersive internal energy flux ${}^{3i}\langle \check{\underline{u}}_k^T \rangle$ is defined by

$$ \begin{aligned} & \alpha_{kLF} \left({}^{t3i}\langle \check{\underline{U}}_{kLF} \rho_k' u_k' \rangle + {}^{t3i}\langle \check{u}_{kLF} \rho_k' \underline{U}_k' \rangle \right) \\ & + {}^{3i}\langle \rho_k \rangle_{LF} \left({}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \check{\underline{U}}_{kLF} u_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \underline{U}_k' \check{u}_{kLF} \rangle \right\rangle \right) \\ & + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \check{\rho}_{kLF} \check{\underline{U}}_{kLF} u_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \check{\rho}_{kLF} \underline{U}_k' \check{u}_{kLF} \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \rho_k' \check{\underline{U}}_{kLF} \check{u}_{kLF} \rangle \right\rangle \\ & = \alpha_{kLF} \, {}^{3i}\langle \check{\underline{u}}_k^T \rangle \, . \end{aligned} \tag{6.9.8} $$

In Eq. 6.9.8, terms involving triple and quadruple time correlations have been deleted.

- $$ {}^{t}\left\langle \alpha_k \, {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle \right\rangle $$

$$ \begin{aligned} & = \alpha_{kLF} \left({}^{3i}\langle P_k \rangle_{LF} \nabla \cdot \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{3i}\langle \check{P}_{kLF} \nabla \cdot \check{\underline{U}}_{kLF} \rangle + {}^{t3i}\langle P_k' \nabla \cdot \underline{U}_k' \rangle \right) \\ & + {}^{3i}\langle P_k \rangle_{LF} \, {}^{t}\left\langle \nabla \cdot \, {}^{3i}\langle \alpha_k' \underline{U}_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \check{P}_{kLF} \nabla \cdot \underline{U}_k' \rangle \right\rangle \\ & + {}^{t3i}\langle \alpha_k' P_k' \rangle \nabla \cdot \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle P_k' \nabla \cdot \check{\underline{U}}_{kLF} \rangle \right\rangle \\ & + v^{-1} \, {}^{3i}\langle P_k \rangle_{LF} \left(\int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA + {}^{t}\left\langle \int_{A_k'} \underline{U}_k' \cdot \underline{n}_k' \, dA \right\rangle \right) . \end{aligned} \tag{6.9.9a} $$

Once again, we note that for Newtonian fluids, $\int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF}\, dA = 0$. In deriving Eq. 6.9.9a, use was made of Eqs. 3.4.1b, 6.2.23b, and that $\left\langle {}^{3i}\langle \check{\underline{P}}_{kLF} \nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} \right\rangle = 0$.

Equation 6.9.9a can be rearranged to read:

$$\left\langle \alpha_k {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle \right\rangle = \left(\alpha_{kLF} {}^{3i}\langle P_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' P_k' \rangle \right) \nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$+ \Phi_{Pk} - (PWI)_k^{(u)} , \tag{6.9.9b}$$

where Φ_{Pk} is a scalar pressure work function defined by

$$\Phi_{Pk} = \alpha_{kLF} {}^{3i}\langle \check{P}_{kLF} \nabla \cdot \check{\underline{U}}_{kLF} \rangle + \alpha_{kLF} {}^{t3i}\langle P_k' \nabla \cdot \underline{U}_k' \rangle + {}^{3i}\langle P_k \rangle_{LF} \left\langle \nabla \cdot {}^{3i}\langle \alpha_k' \underline{U}_k' \rangle \right\rangle$$

$$+ \left\langle \alpha_k' {}^{3i}\langle \check{P}_{kLF} \nabla \cdot \underline{U}_k' \rangle \right\rangle + \left\langle \alpha_k' {}^{3i}\langle P_k' \nabla \cdot \check{\underline{U}}_{kLF} \rangle \right\rangle \tag{6.9.10}$$

and $(PWI)_k^{(u)}$ denotes a portion of the interfacial pressure work integral $(PWI)_k$ defined in Eq. 6.7.25:

$$(PWI)_k^{(u)} = - v^{-1} {}^{3i}\langle P_k \rangle_{LF} \left(\int_{A_{kLF}} \check{\underline{U}}_{kLF} \cdot \underline{n}_{kLF}\, dA + \left\langle \int_{A_k'} \underline{U}_k' \cdot \underline{n}_k'\, dA \right\rangle \right) . \tag{6.9.11}$$

The superscript (u) is a reminder that it is associated with internal energy.

- $\left\langle \alpha_k {}^{3i}\langle \phi_k \rangle \right\rangle$

$$= \alpha_{kLF} \left({}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} : \nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{3i}\langle \check{\underline{\underline{\tau}}}_{kLF} : \nabla, \check{\underline{U}}_{kLF} \rangle + {}^{t3i}\langle \check{\underline{\underline{\tau}}}_k' : \nabla, \underline{U}_k' \rangle \right)$$

$$+ {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} : \left\langle \nabla, {}^{3i}\langle \alpha_k' \underline{U}_k' \rangle \right\rangle + \left\langle \alpha_k' {}^{3i}\langle \check{\underline{\underline{\tau}}}_{kLF} : \nabla, \underline{U}_k' \rangle \right\rangle$$

$$+ {}^{t3i}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle : \nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \left\langle \alpha_k' {}^{3i}\langle \underline{\underline{\tau}}_k' : \nabla, \check{\underline{U}}_{kLF} \rangle \right\rangle$$

$$+ v^{-1} {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} : \left(\int_{A_{kLF}} \underline{U}_{kLF}, \underline{n}_{kLF}\, dA + \left\langle \int_{A_k'} \underline{U}_k', \underline{n}_k'\, dA \right\rangle \right) . \tag{6.9.12a}$$

The similarity between Eq. 6.9.9a and 6.9.12a is understandable, if one recalls that $\phi_k = \underline{\underline{\tau}}_k : \nabla, \underline{U}_k$. Physically, Eq. 6.9.9a gives the reversible conversion of mechanical work into thermal energy, which may be either positive or negative. Equation 6.9.12a gives the irreversible conversion of mechanical work into thermal energy and it is always positive.

Equation 6.9.12a can likewise be arranged to read:

$$^{t}\left\langle \alpha_k \, ^{3i}\langle \phi_k \rangle \right\rangle = \left(\alpha_{kLF} \, ^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + \, ^{t3i}\langle \alpha'_k \underline{\underline{\tau}}'_k \rangle \right) : \nabla, \, ^{3i}\langle \underline{U}_k \rangle_{LF}$$

$$+ \Phi_{\tau k} + \left(VDI\right)_k , \tag{6.9.12b}$$

where $\Phi_{\tau k}$ is a scalar viscous dissipation function defined by

$$\Phi_{\tau k} = \alpha_{kLF} \, ^{3i}\langle \tilde{\underline{\underline{\tau}}}_{kLF} : \nabla, \tilde{\underline{U}}_{kLF} \rangle + \alpha_{kLF} \, ^{t3i}\langle \underline{\underline{\tau}}'_k : \nabla, \underline{U}'_k \rangle$$

$$+ \, ^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} : {}^{c}\left\langle \nabla, \, ^{3i}\langle \alpha'_k \underline{U}'_k \rangle \right\rangle$$

$$+ \, ^{t}\left\langle \alpha'_k \, ^{3i}\langle \tilde{\underline{\underline{\tau}}}_{kLF} : \nabla, \underline{U}'_k \rangle \right\rangle + \, ^{t}\left\langle \alpha'_k \, ^{3i}\langle \underline{\underline{\tau}}'_k : \nabla, \tilde{\underline{U}}_{kLF} \right\rangle \tag{6.9.13}$$

and $\left(VDI\right)_k$ is the interfacial viscous dissipation integral defined by

$$\left(VDI\right)_k = + v^{-1} \, ^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} : \left(\int_{A_{kLF}} \tilde{\underline{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA + \, ^{t}\left\langle \int_{A'_k} \underline{U}'_k \cdot \underline{n}'_k \, dA \right\rangle \right) . \tag{6.9.14}$$

For Newtonian fluids, the firt term in the parentheses of Eq. 6.9.14 vanishes.

$$- v^{-1} \left\langle \int_{A_k} \rho_k u_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \right\rangle = - v^{-1} \int_{A_{kLF}} \left\langle \rho_k u_k (\underline{U}_k - \underline{W}_k) \right\rangle \cdot \underline{n}_{kLF} dA$$

$$- v^{-1} \left\langle \int_{A'_k} \rho_k u_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}'_k \, dA \right\rangle . \tag{6.9.15}$$

The result can be readily written by using Eqs. 6.7.28 and 6.7.29:

$$- v^{-1} \left\langle \int_{A_k} \rho_k u_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \right\rangle$$

$$= {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle u_k \rangle_{LF} + (IETI)_k \, , \tag{6.9.16}$$

in which the interfacial internal energy transfer integral $(IETI)_k$ is defined by

$$\begin{aligned}
(IETI)_k = &- v^{-1} \int_{A_{kLF}} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} \right) \breve{u}_{kLF} + {}^{t}\langle \rho_k' u_k' \rangle \right] \\
&\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA \\
&- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} \right) {}^{t}\left\langle u_k' \left(\underline{U}_k' - \underline{W}_k' \right) \right\rangle \cdot \underline{n}_{kLF} \, dA \\
&- v^{-1} \int_{A_{kLF}} \breve{u}_{kLF} \, {}^{t}\left\langle \rho_k' \left(\underline{U}_k' - \underline{W}_k' \right) \right\rangle \cdot \underline{n}_{kLF} \, dA \\
&- v^{-1} \, {}^{t}\Bigg\langle \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} + \rho_k' \right) u_k' + \rho_k' \breve{u}_{kLF} \right] \\
&\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \Bigg\rangle \\
&- v^{-1} \, {}^{t}\Bigg\langle \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} \right) \left(\breve{u}_{kLF} + u_k' \right) + \rho_k' \breve{u}_{kLF} \right] \\
&\left(\underline{U}_k' - \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \Bigg\rangle \\
&- v^{-1} \, {}^{t}\Bigg\langle \int_{A_k'} \rho_k' u_k' \left(\underline{U}_k' - \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \Bigg\rangle \, .
\end{aligned} \tag{6.9.17}$$

An examination of the foregoing results shows that the time-averaged interfacial internal energy source per unit volume, ${}^{t}\langle \mathcal{Q}_k \rangle$, consists of: (1) a portion of the interfacial pressure work $(PWI)_k^{(u)}$ defined by Eq. 6.9.11, (2) interfacial dissipation $(VDI)_k$ defined by Eq. 6.9.14, and (3) interfacial heat transfer ${}^{t}\langle \dot{Q}_k \rangle$ and interfacial internal energy transfer given by Eq. 6.9.16. Hence,

$${}^{t}\langle \mathcal{Q}_k \rangle = {}^{t}\langle \dot{Q}_k \rangle + {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle u_k \rangle_{LF} + (PWI)_k^{(u)} + (VDI)_k + (IETI)_k \, . \tag{6.9.18}$$

It should be noted that the "extraneous" interfacial thermal energy sources, $(PWI)_k^{(u)}$ and $(VDI)_k$, arise, respectively, from the two volume averages, $^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle$ and $^{3i}\langle \phi_k \rangle$, the latter being the equivalent of $^{3i}\langle \underline{\underline{\tau}}_k : \nabla, \underline{U}_k \rangle$. For Newtonian fluids, both can be neglected since A_k' is assumed to be a small perturbation of A_k.

By using the foregoing results, the time-volume averaged internal energy conservation equation can be obtained in a manner similar to that for the total energy conservation equation. The result is:

$$\frac{\partial}{\partial t}\left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) {}^{3i}\langle u_k \rangle_{LF}$$

$$+ \nabla \cdot \left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) {}^{3i}\langle \underline{U}_k \rangle_{LF}\, {}^{3i}\langle u_k \rangle_{LF}$$

$$+ \frac{\partial \Phi_{uk}}{\partial t} + \nabla \cdot \Phi_{uk}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot \underline{\psi}_{mk}\, {}^{3i}\langle u_k \rangle_{LF}$$

$$= -\left(\alpha_{kLF}\, {}^{3i}\langle P_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' P_k' \rangle\right) \nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} - \Phi_{Pk}$$

$$+ \left(\alpha_{kLF}\, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle\right) : \nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \Phi_{\tau k}$$

$$- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{u}_k^T \rangle + {}^{3i}\langle \tilde{\underline{u}}_k \rangle + {}^{3i}\langle \tilde{\underline{u}}_k^T \rangle\right)$$

$$- \nabla \cdot \left(\alpha_{kLF}\, {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} + {}^{t3i}\langle \alpha_k' \underline{J}_{qk}' \rangle\right)$$

$$+ \alpha_{kLF}\, {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha_k' J_{Ek}' \rangle + {}^{t}\langle Q_k \rangle \,. \qquad (6.9.19a)$$

We note that the third and fourth terms on the right-hand side, taken collectively, are simply the time-volume averaged dissipation function for the bulk fluid (see Eq. 6.9.12b). It is written in the indicated form in order to bring out its similarity to the pressure work term.

An alternative form of the time-volume averaged internal energy conservation equation can be obtained by using Eq. 6.3.13 to eliminate $^{t}\langle \Gamma_k \rangle$ in Eq. 6.9.18 and introducing the result into Eq. 6.9.19a. After combining and rearranging terms, one obtains

$$\left(\alpha_{kLF}\, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) \left(\frac{\partial\, {}^{3i}\langle u_k \rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla\, {}^{3i}\langle u_k \rangle_{LF}\right)$$

$$+ \frac{\partial \Phi_{uk}}{\partial t} + \nabla \cdot \Phi_{uk}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \cdot \nabla\, {}^{3i}\langle u_k \rangle_{LF}$$

$$= -\left(\alpha_{kLF}\, {}^{3i}\langle P_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k P'_k \rangle\right) \nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF} - \Phi_{Pk}$$

$$+ \left(\alpha_{kLF}\, {}^{3i}\langle \underline{\tau}_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k \underline{\tau}'_k \rangle\right) : \nabla\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \Phi_{\tau k}$$

$$- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{u}_k^T \rangle + {}^{3i}\langle \underline{\tilde{u}}_k \rangle + {}^{3i}\langle \underline{\tilde{u}}_k^T \rangle\right)$$

$$- \nabla \cdot \left(\alpha_{kLF}\, {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} + {}^{t3i}\langle \alpha'_k \underline{J}'_{qk} \rangle\right)$$

$$+ \alpha_{kLF}\, {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha'_k J'_{Ek} \rangle$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + \left(PWI\right)_k^{(u)} + \left(VDI\right)_k + \left(IETI\right)_k . \qquad (6.9.19b)$$

We note that ${}^{3i}\langle \underline{J}_{qk} \rangle_{LF}$ can be expressed in terms of a molecular thermal diffusivity D_{uk} which, for constant κ_k and c_{vk}, is defined by

$$D_{uk} = \frac{\kappa_k}{c_{vk}\, {}^{3i}\langle \rho_k \rangle_{LF}} . \qquad (6.9.20)$$

It follows then, that

$${}^{3i}\langle \underline{J}_{qk} \rangle_{LF} = - {}^{3i}\langle \rho_k \rangle_{LF}\, D_{uk}\, \nabla\, {}^{3i}\langle u_k \rangle_{LF} , \qquad (6.9.21)$$

which may be compared with

$${}^{3i}\langle \underline{u}_k^T \rangle = - {}^{3i}\langle \rho_k \rangle_{LF}\, D_{uk}^T\, \nabla\, {}^{3i}\langle u_k \rangle_{LF} . \qquad (6.9.5a)$$

When ρ_k = constant, $\nabla \cdot \underline{U}_k = 0$, Eq. 6.9.19a simplifies to

$$\rho_k \left(\frac{\partial}{\partial t} \alpha_{kLF}\, {}^{3i}\langle u_k \rangle_{LF} + \nabla \cdot \alpha_{kLF}\, {}^{3i}\langle \underline{U}_k \rangle_{LF}\, {}^{3i}\langle u_k \rangle_{LF}\right)$$

$$+ \frac{\partial {}^{o}\Phi_{uk}}{\partial t} + \nabla \cdot {}^{o}\Phi_{uk}\, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot {}^{o}\underline{\Psi}_{mk}\, {}^{3i}\langle u_k \rangle_{LF}$$

$$= -\nabla \cdot \left(\alpha_{kLF}\, {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} + {}^{t3i}\langle \alpha_k' \underline{J}_k' \rangle \right)$$

$$- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle {}^{o}\underline{u}_k^{T} \rangle + {}^{3i}\langle {}^{o\sim}\underline{u}_k \rangle + {}^{3i}\langle {}^{o\sim T}\underline{u}_k \rangle \right)$$

$$+ \alpha_{kLF}\, {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha_k' J_{Ek}' \rangle$$

$$+ \left(\alpha_{kLF}\, {}^{3i}\langle {}^{o}\underline{\tau}_k \rangle_{LF} + {}^{t3i}\langle \alpha_k', {}^{o}\underline{\tau}_k' \rangle \right) : \nabla, \; {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\Phi_{\tau k}$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + {}^{o}(\mathrm{VDI})_k + {}^{o}(\mathrm{IETI})_k \,, \tag{6.9.22a}$$

in which

$${}^{o}\Phi_{uk} = \rho_k\, {}^{t3i}\langle \alpha_k' u_k' \rangle \tag{6.9.23}$$

$${}^{3i}\langle {}^{o}\underline{u}_k^{T} \rangle = \rho_k\, {}^{t3i}\langle \underline{U}_k' u_k' \rangle = -\rho_k\, {}^{o}D_{uk}^{T} \nabla\, {}^{3i}\langle u_k \rangle_{LF} \,, \tag{6.9.24}$$

where ${}^{o}D_{uk}^{T} = \kappa_k^{T}/(c_{vk}\rho_k)$, with κ_k^{T} being the turbulent conductivity, (6.9.25)

$${}^{3i}\langle {}^{o\sim}\underline{u}_k \rangle = \rho_k\, {}^{3i}\langle \breve{\underline{U}}_{kLF} \breve{u}_{kLF} \rangle \tag{6.9.26}$$

$$\alpha_{kLF}\, {}^{3i}\langle {}^{o\sim T}\underline{u}_k \rangle = \rho_k \left({}^{t}\left\langle \alpha_k'\, {}^{3i}\langle \breve{\underline{U}}_{kLF} u_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k'\, {}^{3i}\langle \underline{U}_k' \breve{u}_{kLF} \right\rangle \right) \tag{6.9.27}$$

$${}^{o}\Phi_{\tau k} = \alpha_{kLF}\, {}^{3i}\langle {}^{o\sim}\underline{\tau}_{kLF} : \nabla, \breve{\underline{U}}_{kLF} \rangle + \alpha_{kLF}\, {}^{t3i}\langle {}^{o}\underline{\tau}_k' : \nabla, \underline{U}_k' \rangle$$

$$+ {}^{3i}\langle {}^{o}\underline{\tau}_k \rangle_{LF} : {}^{t}\left\langle \nabla, {}^{3i}\langle \alpha_k' \underline{U}_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k'\, {}^{3i}\langle {}^{o\sim}\underline{\tau}_{kLF} : \nabla, \underline{U}_k' \right\rangle \right\rangle$$

$$+ {}^{t}\left\langle \alpha_k'\, {}^{3i}\langle {}^{o}\underline{\tau}_k' : \nabla, \breve{\underline{U}}_{kLF} \rangle \right\rangle \tag{6.9.28}$$

$$ {}^{o}(\mathrm{VDI})_k = + v^{-1}\ {}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k \rangle : \left(\int_{A_{kLF}} \underline{\check{U}}_{kLF}, \underline{n}_{kLF}\ dA + {}^{t}\left\langle \int_{A'_k} \underline{U}'_k, \underline{n}'_k\ dA \right\rangle \right) \qquad (6.9.29) $$

and

$$ \begin{aligned} {}^{o}(\mathrm{IETI})_k = &- v^{-1} \rho_k \int_{A_{kLF}} \check{u}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\check{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF}\ dA \\ &- v^{-1} \rho_k \int_{A_{kLF}} {}^{t}\left\langle u'_k \left(\underline{U}'_k - \underline{W}'_k \right) \right\rangle \cdot \underline{n}_{kLF}\ dA \\ &- v^{-1} \rho_k {}^{t}\left\langle \int_{A'_k} u'_k \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\check{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}'_k\ dA \right\rangle \\ &- v^{-1} \rho_k {}^{t}\left\langle \int_{A'_k} \left(\check{u}_{kLF} + u'_k \right) \left(\underline{U}'_k - \underline{W}'_k \right) \cdot \underline{n}'_k\ dA \right\rangle . \end{aligned} \qquad (6.9.30) $$

The two remaining functions, ${}^{o}\underline{\Psi}_{mk}$ and ${}^{ot}\langle \Gamma_k \rangle$ are given by Eqs. 6.3.15 and 6.5.31, respectively. In many problems of practical interest, the viscous dissipation effect can be ignored.

An alternative form of Eq. 6.9.22a can be deduced from Eq. 6.9.19b. The result is

$$ \begin{aligned} &\alpha_{kLF} \rho_k \left(\frac{\partial\ {}^{3i}\langle u_k \rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla\ {}^{3i}\langle u_k \rangle_{LF} \right) \\ &+ \frac{\partial\ {}^{o}\Phi_{uk}}{\partial t} + \nabla \cdot {}^{o}\Phi_{uk}\ {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\underline{\Psi}_{mk} \cdot \nabla\ {}^{3i}\langle u_k \rangle_{LF} \\ = &- \nabla \cdot \left(\alpha_{kLF}\ {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} + {}^{t3i}\langle \alpha'_k \underline{J}'_{qk} \rangle \right) \\ &- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle {}^{o}\underline{u}^{T}_k \rangle + {}^{3i}\langle {}^{o\sim}\underline{u}_k \rangle + {}^{3i}\langle {}^{o\sim T}\underline{u}_k \rangle \right) \\ &+ \alpha_{kLF}\ {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha'_k J'_{EK} \rangle \\ &+ \left(\alpha_{kLF}\ {}^{3i}\langle {}^{o}\underline{\underline{\tau}}_k \rangle_{LF} + {}^{t3i}\langle \alpha'_k\ {}^{o}\underline{\underline{\tau}}_k \rangle \right) : \nabla,\ {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\Phi_{\tau k} \end{aligned} $$

$$+ {}^{t}\langle \dot{Q}_k \rangle + {}^{o}(\mathrm{VDI})_k + {}^{o}(\mathrm{IETI})_k \ . \qquad (6.9.22b)$$

6.10 Time-Volume-averaged Interfacial Internal Energy Balance Equation $(\gamma_v = \gamma_A = 1)$

The local volume-averaged internal energy balance relation is given by Eq. 5.2.5. Upon performing time averaging and making use of Eqs. 6.7.16 and 6.9.16 and, in addition, taking into account the extraneous interfacial pressure work $(\mathrm{PWI})_k^{(u)}$ and interfacial dissipation $(\mathrm{VDI})_k$, we obtain for interface A_{kf}:

$$ {}^{t}\langle \mathcal{Q}_k \rangle + {}^{t}\langle \mathcal{Q}_f \rangle = 0 \qquad (6.10.1a)$$

or, equivalently,

$$ {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle u_k \rangle_{LF} + {}^{t}\langle \dot{Q}_k \rangle + (\mathrm{PWI})_k^{(u)} + (\mathrm{VDI})_k + (\mathrm{IETI})_k $$

$$ + {}^{t}\langle \Gamma_f \rangle \, {}^{3i}\langle u_f \rangle_{LF} + {}^{t}\langle \dot{Q}_f \rangle + (\mathrm{PWI})_f^{(u)} + (\mathrm{VDI})_f + (\mathrm{IETI})_f = 0 \ . \qquad (6.10.1b)$$

For reasons given previously, both $(\mathrm{PWI})^{(u)}$ and (VDI) can be neglected for Newtonian fluids.

When ρ_k = constant, ${}^{3i}\langle \rho_k \rangle_{LF} = \rho_k$, and $\tilde{\rho}_{kLF} = \rho_k' = 0$. Hence, $(\mathrm{PWI})_k^{(u)} = 0$, ${}^{t}\langle \Gamma_k \rangle$ should be replaced by ${}^{ot}\langle \Gamma_k \rangle$, $(\mathrm{VDI})_k$ by ${}^{o}(\mathrm{VDI})_k$, and $(\mathrm{IETI})_k$ by ${}^{o}(\mathrm{IETI})_k$; ${}^{ot}\langle \Gamma_k \rangle$, ${}^{o}(\mathrm{VDI})_k$, and ${}^{o}(\mathrm{IETI})_k$ are defined in Eqs. 6.5.31, 6.9.29, and 6.9.30, respectively. Similar reductions should be made for the corresponding quantities with subscript f.

6.11 Time-Volume-averaged Enthalpy Conservation Equation $(\gamma_v = \gamma_A = 1)$

The local volume-averaged enthalpy conservation equation is given by Eq. 5.1.8:

$$\frac{\partial}{\partial t} \alpha_k \, {}^{3i}\langle \rho_k h_k \rangle + \nabla \cdot \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k h_k \rangle = \frac{\partial}{\partial t} \alpha_k \, {}^{3i}\langle P_k \rangle + \nabla \cdot \alpha_k \, {}^{3i}\langle P_k \underline{U}_k \rangle$$

$$- \alpha_k \, {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle - \nabla \cdot \alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle + \alpha_k \left({}^{3i}\langle J_{Ek} \rangle + {}^{3i}\langle \phi_k \rangle \right) + \dot{Q}_k$$

$$+ v^{-1} \int_{A_k} P_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_k} \rho_k h_k (\underline{U}_k - \underline{W}_k) \cdot \underline{n}_k \, dA \ . \qquad (5.8)$$

The time-averaged results of all individual terms in Eq. 5.1.8 have been given earlier, except the following:

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \rho_k h_k \rangle \right\rangle = \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) {}^{3i}\langle h_k \rangle_{LF} + \Phi_{hk} \, , \qquad (6.11.1)$$

where Φ_{hk} is a scalar enthalpy function defined by

$$\Phi_{hk} = \alpha_{kLF} \, {}^{3i}\langle \tilde{\rho}_{kLF} \tilde{h}_{kLF} \rangle + \alpha_{kLF} \, {}^{t3i}\langle \rho_k' h_k' \rangle + {}^{3i}\langle \rho_k \rangle_{LF} \, {}^{t3i}\langle \alpha_k' h_k' \rangle$$
$$+ {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \tilde{\rho}_{kLF} h_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \rho_k' \tilde{h}_k \rangle \right\rangle \, . \qquad (6.11.2)$$

$$\bullet \quad {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k h_k \rangle \right\rangle = \left(\alpha_{kLF} \, {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) {}^{3i}\langle \underline{U}_k \rangle_{LF} \, {}^{3i}\langle h_k \rangle_{LF}$$
$$+ {}^{3i}\langle \underline{U}_k \rangle_{LF} \, \Phi_{hk} + \underline{\Psi}_{mk} \, {}^{3i}\langle h_k \rangle_{LF}$$
$$+ \alpha_{kLF} \left({}^{3i}\langle \underline{h}_k^T \rangle + {}^{3i}\langle \tilde{\underline{h}}_k \rangle + {}^{3i}\langle \tilde{\underline{h}}_k^T \rangle \right) \, , \qquad (6.11.3)$$

in which

(a) The volume-averaged turbulent enthalpy flux ${}^{3i}\langle \underline{h}_k^T \rangle$ is defined by

$${}^{3i}\langle \rho_k \rangle_{LF} \, {}^{t3i}\langle \underline{U}_k' h_k' \rangle + {}^{t3i}\langle \tilde{\rho}_{kLF} \underline{U}_k' h_k' \rangle = {}^{3i}\langle \underline{h}_k^T \rangle \, , \qquad (6.11.4)$$

(b) The volume-averaged dispersive enthalpy flux ${}^{3i}\langle \tilde{\underline{h}}_k \rangle$ is defined by

$${}^{3i}\langle \rho_k \rangle_{LF} \, {}^{3i}\langle \tilde{\underline{U}}_{kLF} \tilde{h}_{kLF} \rangle + {}^{3i}\langle \tilde{\rho}_{kLF} \tilde{\underline{U}}_{kLF} \tilde{h}_{kLF} \rangle = {}^{3i}\langle \tilde{\underline{h}}_k \rangle \, , \text{ and} \qquad (6.11.5)$$

(c) The volume-averaged turbulent, dispersive enthalpy flux ${}^{3i}\langle \tilde{h}_k^T \rangle$ is defined by

$$\alpha_{kLF} \left({}^{t3i}\langle \tilde{\underline{U}}_{kLF} \rho_k' h_k' \rangle + {}^{t3i}\langle \tilde{h}_{kLF} \rho_k' \underline{U}_k' \rangle \right)$$
$$+ {}^{3i}\langle \rho_k \rangle_{LF} \left({}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \tilde{\underline{U}}_{kLF} h_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \underline{U}_k' \tilde{h}_{kLF} \rangle \right\rangle \right)$$
$$+ {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \tilde{\rho}_{kLF} \tilde{\underline{U}}_{kLF} h_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \tilde{\rho}_{kLF} \underline{U}_k' \tilde{h}_{kLF} \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \rho_k' \tilde{\underline{U}}_{kLF} \tilde{h}_{kLF} \rangle \right\rangle$$

$$= \alpha_{kLF}\ {}^{3i}\langle \underline{\tilde{h}}_k^T \rangle \tag{6.11.6}$$

$$\bullet \quad {}^{t}\left\langle \alpha_k\ {}^{3i}\langle P_k \rangle \right\rangle = \alpha_{kLF}\ {}^{3i}\langle P_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' P_k' \rangle \tag{6.11.7}$$

$$\bullet \quad v^{-1} \left\langle \int_{A_k} P_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}_k \, dA \right\rangle = v^{-1} \int_{A_{kLF}} \left\langle P_k \left(\underline{U}_k - \underline{W}_k \right) \right\rangle \cdot \underline{n}_{kLF} \, dA$$

$$+ v^{-1} \left\langle \int_{A_k'} P_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}_k' \, dA \right\rangle , \tag{6.11.8}$$

in which

$$v^{-1} \int_{A_{kLF}} \left\langle P_k \left(\underline{U}_k - \underline{W}_k \right) \right\rangle \cdot \underline{n}_{kLF} \, dA$$

$$= - {}^{3i}\langle P_k \rangle_{LF}\ {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \alpha_{kLF} - {}^{3i}\langle P_k \rangle_{LF} \frac{\partial \alpha_{kLF}}{\partial t}$$

$$+ v^{-1}\ {}^{3i}\langle P_k \rangle_{LF} \int_{A_{kLF}} \underline{\tilde{U}}_{kLF} \cdot \underline{n}_{kLF} \, dA$$

$$+ v^{-1} \int_{A_{kLF}} \tilde{P}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\tilde{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA$$

$$+ v^{-1} \int_{A_{kLF}} \left\langle P_k' \left(\underline{U}_k' - \underline{W}_k' \right) \right\rangle \cdot \underline{n}_{kLF} \, dA \tag{6.11.9}$$

and

$$v^{-1} \left\langle \int_{A_k'} P_k \left(\underline{U}_k - \underline{W}_k \right) \cdot \underline{n}_k' \, dA \right\rangle$$

$$= v^{-1} \left\langle \int_{A_k'} \left({}^{3i}\langle P_k \rangle_{LF} + \tilde{P}_{kLF} \right) \left(\underline{U}_k' - \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \right\rangle$$

$$+ v^{-1} \left\langle \int_{A_k'} P_k' \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\tilde{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \right\rangle$$

$$+ v^{-1} \left\langle \int_{A_k'} P_k' \left(\underline{U}_k' - \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \right\rangle . \tag{6.11.10}$$

Accordingly,

$$v^{-1}\left\langle \int_{A_k} P_k(\underline{U}_k - \underline{W}_k)\cdot \underline{n}_k \, dA \right\rangle$$

$$= -\,{}^{3i}\langle ?_k\rangle_{LF}\left(\frac{\partial \alpha_{kLF}}{\partial t} + {}^{3i}\langle \underline{U}_k\rangle_{LF}\cdot \nabla \alpha_{kLF}\right) - (PWI)_k^{(u)} + (PWI)_k^{(h)}\,, \quad (6.11.11)$$

where $(PWI)_k^{(u)}$ is defined in Eq. 6.9.11 and

$$(PWI)_k^{(h)} = v^{-1}\int_{A_{kLF}} \breve{P}_{kLF}\left({}^{3i}\langle \underline{U}_k\rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF}\right)\cdot \underline{n}_{kLF}\, dA$$

$$+ v^{-1}\int_{A_{kLF}} \left\langle P_k'(\underline{U}_k' - \underline{W}_k')\right\rangle \cdot \underline{n}_{kLF}\, dA$$

$$+ v^{-1}\left\langle \int_{A_k'} \breve{P}_{kLF}(\underline{U}_k' - \underline{W}_k')\cdot \underline{n}_k'\, dA\right\rangle$$

$$+ v^{-1}\left\langle \int_{A_k'} P_k'\left({}^{3i}\langle \underline{U}_k\rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF}\right)\cdot \underline{n}_k'\, dA\right\rangle$$

$$+ v^{-1}\left\langle \int_{A_k'} P_k'(\underline{U}_k' - \underline{W}_k')\cdot \underline{n}_k'\, dA\right\rangle . \quad (6.11.12)$$

In deriving Eq. 6.11.11, the relationship given by Eq. 6.2.5c has been used.

It is easy to show that

$$-(PWI)_k^{(u)} + (PWI)_k^{(h)} = -(PWI)_k - v^{-1}\int_{A_{kLF}}\left(\breve{P}_{kLF}\underline{W}_{kLF} + {}^{t}\langle P_k'\underline{W}_k'\rangle\right)\cdot \underline{n}_k\, dA$$

$$- v^{-1}\left\langle \int_{A_k'} \left(\breve{P}_{kLF}\underline{W}_k' + P_k'\underline{W}_{kLF} + P_k'\underline{W}_k'\right)\cdot \underline{n}_k'\, dA\right\rangle , \quad (6.11.13)$$

where $(PWI)_k$ is defined by Eq 6.7.25.

$$\bullet \quad -v^{-1}\left\langle \int_{A_k} \rho_k h_k(\underline{U}_k - \underline{W}_k)\cdot \underline{n}_k\, dA\right\rangle$$

$$= {}^{t}\langle \Gamma_k\rangle\, {}^{3i}\langle h_k\rangle_{LF} + (EPYTI)_k\,, \quad (6.11.14)$$

where the interfacial enthalpy transfer integral $(EPYTI)_k$ is defined by

$$
\begin{aligned}
(EPYTI)_k = &- v^{-1} \int_{A_{kLF}} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} \right) \breve{h}_{kLF} + {}^{t}\langle \rho_k' h_k' \rangle \right] \\
&\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA \\
&- v^{-1} \int_{A_{kLF}} \left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} \right) {}^{t}\left\langle h_k' (\underline{U}_k' - \underline{W}_k') \right\rangle \cdot \underline{n}_{kLF} \, dA \\
&- v^{-1} \int_{A_{kLF}} \breve{h}_{kLF} \, {}^{t}\left\langle \rho_k' (\underline{U}_k' - \underline{W}_k') \right\rangle \cdot \underline{n}_{kLF} \, dA \\
&- v^{-1} \, {}^{t}\Big\langle \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} + \rho_k' \right) h_k' + \rho_k' \breve{h}_{kLF} \right] \\
&\left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \breve{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \Big\rangle \\
&- v^{-1} \, {}^{t}\Big\langle \int_{A_k'} \left[\left({}^{3i}\langle \rho_k \rangle_{LF} + \breve{\rho}_{kLF} \right) \left(\breve{h}_{kLF} + h_k' \right) + \rho_k' \breve{h}_{kLF} \right] \left(\underline{U}_k' - \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \Big\rangle \\
&- v^{-1} \, {}^{t}\Big\langle \int_{A_k'} \rho_k' h_k' (\underline{U}_k' - \underline{W}_k') \cdot \underline{n}_k' \, dA \Big\rangle \, . \qquad (6.11.15)
\end{aligned}
$$

Hence, the time-averaged interfacial enthalpy source per unit volume is

$$
\begin{aligned}
{}^{t}\langle \mathcal{Q}_k \rangle = &- {}^{3i}\langle P_k \rangle_{LF} \frac{d \alpha_{kLF}}{d t_k} + {}^{t}\langle \dot{Q}_k \rangle + {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle h_k \rangle_{LF} \\
&+ (PWI)_k^{(h)} + (VDI)_k + (EFYTI)_k \, , \qquad 6.11.16)
\end{aligned}
$$

in which the substantive time derivative $\frac{d}{d t_k}$ is defined in Eq. 6.3.12. It may be noted that the term $-(PWI)_k^{(u)}$ in Eq. 6.11.11 cancels with $+(PWI)_k^{(u)}$ arising from the time-average of $-\alpha_k \, {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle$ and that $(VDI)_k$ arises from the time-average of $\alpha_k \, {}^{3i}\langle \phi_k \rangle$.

Performing the time averaging of Eq. 5.8, followed by introducing the results given by Eqs. 6.11.1, 6.11.3, 6.11.7, 6.7.8, 6.9.9b, 6.7.12, 6.7.15, 6.9.12b, 6.11.11, and 6.11.14, one obtains, after combinination of terms and rearrangement, the following time-volume averaged enthalpy equation:

$$\frac{\partial}{\partial t}\left(\alpha_{kLF}\,{}^{3i}\langle\rho_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'\rho_k'\rangle\right){}^{3i}\langle h_k\rangle_{LF}$$

$$+\nabla\cdot\left(\alpha_{kLF}\,{}^{3i}\langle\rho_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'\rho_k'\rangle\right){}^{3i}\langle\underline{U}_k\rangle_{LF}\,{}^{3i}\langle h_k\rangle_{LF}$$

$$+\frac{\partial\Phi_{hk}}{\partial t} + \nabla\cdot\Phi_{hk}\,{}^{3i}\langle\underline{U}_k\rangle_{LF} + \nabla\cdot\underline{\Psi}_{mk}\,{}^{3i}\langle h_k\rangle_{LF}$$

$$=\frac{\partial}{\partial t}\left(\alpha_{kLF}\,{}^{3i}\langle P_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'P_k'\rangle\right)$$

$$+\nabla\cdot\left(\alpha_{kLF}\,{}^{3i}\langle P_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'P_k'\rangle\right){}^{3i}\langle\underline{U}_k\rangle_{LF} + \nabla\cdot\underline{\Psi}_{Pk}$$

$$-\left(\alpha_{kLF}\,{}^{3i}\langle P_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'P_k'\rangle\right)\nabla\cdot\,{}^{3i}\langle\underline{U}_k\rangle_{LF} - \Phi_{Pk}$$

$$+\left(\alpha_{kLF}\,{}^{3i}\langle\underline{\underline{\tau}}_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'\underline{\underline{\tau}}_k'\rangle\right) : \nabla,\,{}^{3i}\langle\underline{U}_k\rangle_{LF} + \Phi_{\tau k}$$

$$-\nabla\cdot\alpha_{kLF}\left({}^{3i}\langle\underline{h}_k^{T}\rangle + {}^{3i}\langle\underline{\tilde{h}}_k\rangle + {}^{3i}\langle\underline{\tilde{h}}_k^{T}\rangle\right)$$

$$-\nabla\cdot\left(\alpha_{kLF}\,{}^{3i}\langle\underline{J}_{qk}\rangle_{LF} + {}^{t3i}\langle\alpha_k'\underline{J}_{qk}'\rangle\right)$$

$$+\alpha_{kLF}\,{}^{3i}\langle J_{Ek}\rangle_{LF} + {}^{t3i}\langle\alpha_k'J_{Ek}'\rangle + {}^{t}\langle\mathcal{H}_k\rangle\,, \qquad (6.11.17a)$$

in which Φ_{hk} is defined by Eq. 6.11.2, $\underline{\Psi}_{mk}$ by Eq. 6.3.5, $\underline{\Psi}_{Pk}$ by Eq. 6.7.9, Φ_{Pk} by Eq. 6.9.10, $\Phi_{\tau k}$ by Eq. 6.9.13, and ${}^{t}\langle\mathcal{H}_k\rangle$ by Eq. 6.11.16.

Alternatively, Eq. 6.11.17a can be rewritten as

$$\frac{\partial}{\partial t}\left(\alpha_{kLF}\,{}^{3i}\langle\rho_k\rangle_{LF} + {}^{t3i}\langle\alpha_k'\rho_k'\rangle\right){}^{3i}\langle h_k\rangle_{LF}$$

$$+ \nabla \cdot \left(\alpha_{kLF}\ {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) {}^{3i}\langle \underline{U}_k \rangle_{LF}\ {}^{3i}\langle h_k \rangle_{LF}$$

$$+ \frac{\partial\ \Phi_{hk}}{\partial t} + \nabla \cdot \Phi_{hk}\ {}^{3i}\langle \underline{U}_k \rangle_{LF} + \nabla \cdot \underline{\Psi}_{mk}\ {}^{3i}\langle h_k \rangle_{LF}$$

$$= \alpha_k \frac{d\ {}^{3i}\langle P_k \rangle_{LF}}{d\ t_k} + \frac{d}{d\ t_k}\ {}^{t3i}\langle \alpha_k' P_k' \rangle + \nabla \cdot \underline{\Psi}_{Pk} - \Phi_{Pk}$$

$$+ \left(\alpha_{kLF}\ {}^{3i}\langle \underline{\underline{\tau}}_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle\right) : \nabla,\ {}^{3i}\langle \underline{U}_k \rangle_{LF} + \Phi_{\tau k}$$

$$- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle \underline{h}_k^T \rangle + {}^{3i}\langle \tilde{\underline{h}}_k \rangle + {}^{3i}\langle \tilde{h}_k^T \rangle\right)$$

$$- \nabla \cdot \left(\alpha_{kLF}\ {}^{3i}\langle \underline{J}_{qk} \rangle_{LF} + {}^{t3i}\langle \alpha_k' \underline{J}_{qk}' \rangle\right) + \alpha_{kLF}\ {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha_k' J_{Ek}' \rangle$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + {}^{t}\langle \Gamma_k \rangle\ {}^{3i}\langle h_k \rangle$$

$$+ \left(\mathrm{PWI}\right)_k^{(h)} + \left(\mathrm{VDI}\right)_k + \left(\mathrm{EPYTI}\right)_k . \qquad (6.11.17b)$$

Another form of Eq. 6.11.17a can be obtained by using Eq. 6.3.13 to eliminate ${}^{t}\langle \Gamma_k \rangle$ in Eq. 6.11.17b. The result is

$$\left(\alpha_{kLF}\ {}^{3i}\langle \rho_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' \rho_k' \rangle\right) \left(\frac{\partial\ {}^{3i}\langle h_k \rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla\ {}^{3i}\langle h_k \rangle_{LF}\right)$$

$$\frac{\partial \Phi_{hk}}{\partial t} + \nabla \cdot \Phi_{hk}\ {}^{3i}\langle \underline{U}_k \rangle_{LF} + \underline{\Psi}_{mk} \cdot \nabla\ {}^{3i}\langle h_k \rangle_{LF}$$

$$= \alpha_{kLF}\ \frac{d}{dt_k}\ {}^{3i}\langle P_k \rangle_{LF} + \frac{d}{dt_k}\ {}^{t3i}\langle \alpha_k' P_k' \rangle + \nabla \cdot \underline{\Psi}_{Pk} - \Phi_{Pk}$$

$$+\left(\alpha_{kLF}\,{}^{3i}\langle\underline{\underline{\tau}}_k\rangle_{LF}+{}^{t3i}\langle\alpha'_k\underline{\underline{\tau}}'_k\rangle\right):\nabla,\ {}^{3i}\langle\underline{U}_k\rangle_{LF}+\Phi_{\tau k}$$

$$-\nabla\cdot\alpha_{kLF}\left({}^{3i}\langle\underline{h}^{T}_{k}\rangle+{}^{3i}\langle\underline{\tilde{h}}_k\rangle+{}^{3i}\langle\underline{\tilde{h}}^{T}_{k}\rangle\right)$$

$$-\nabla\cdot\left(\alpha_{kLF}\,{}^{3i}\langle\underline{J}_{qk}\rangle+{}^{t3i}\langle\alpha'_k\underline{J}'_{qk}\rangle\right)$$

$$+\alpha_{kLF}\,{}^{3i}\langle J_{Ek}\rangle_{LF}+{}^{t3i}\langle\alpha'_k J'_{Ek}\rangle$$

$$+{}^{t}\langle\dot{Q}_k\rangle+\left(PWI\right)^{(h)}_{k}+\left(VDI\right)_k+\left(EPYTI\right)_k\ . \qquad (6.11.17c)$$

When ρ_k = constant, Eq. 6.11.17b reduces to

$$\rho_k\left(\frac{\partial}{\partial t}\alpha_{kLF}\,{}^{3i}\langle h_k\rangle_{LF}+\nabla\cdot\alpha_{kLF}\,{}^{3i}\langle\underline{U}_k\rangle_{LF}\,{}^{3i}\langle h_k\rangle_{LF}\right)$$

$$+\frac{\partial\,{}^{o}\Psi_{hk}}{\partial t}+\nabla\cdot{}^{o}\Phi_{hk}\,{}^{3i}\langle\underline{U}_k\rangle_{LF}+\nabla\cdot{}^{o}\underline{\Psi}_{mk}\,{}^{3i}\langle h_k\rangle_{LF}$$

$$=\alpha_{kLF}\frac{d}{dt_k}\,{}^{3i}\langle P_k\rangle_{LF}+\frac{d}{dt_k}\,{}^{t3i}\langle\alpha'_k P'_k\rangle+\nabla\cdot\underline{\Psi}_{Pk}$$

$$+\left(\alpha_{kLF}\,{}^{3i}\langle{}^{o}\underline{\underline{\tau}}_k\rangle_{LF}+{}^{t3i}\langle\alpha'_k\,{}^{o}\underline{\underline{\tau}}'_k\rangle\right):\nabla,\ {}^{3i}\langle\underline{U}_k\rangle_{LF}+{}^{o}\Phi_{\tau k}$$

$$-\nabla\cdot\alpha_{kLF}\left({}^{3i}\langle{}^{o}\underline{h}^{T}_{k}\rangle+{}^{3i}\langle{}^{o}\underline{\tilde{h}}_k\rangle+{}^{3i}\langle{}^{o}\underline{\tilde{h}}^{T}_{k}\rangle\right)$$

$$-\nabla\cdot\left(\alpha_{kLF}\,{}^{3i}\langle\underline{J}_{qk}\rangle_{LF}+{}^{t3i}\langle\alpha'_k\underline{J}_{qk}\rangle\right)$$

$$+\alpha_{kLF}\,{}^{3i}\langle J_{Ek}\rangle_{LF}+{}^{t3i}\langle\alpha'_k J'_{Ek}\rangle$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + {}^{ot}\langle T_k \rangle \, {}^{3i}\langle h_k \rangle_{LF}$$

$$+ \left(PWI\right)_k^{(h)} - \left(PWI\right)_k^{(u)} + {}^{o}\left(VDI\right)_k + {}^{o}\left(EPYTI\right)_k \, , \qquad (6.11.18a)$$

in which

$${}^{o}\phi_{hk} = \rho_k \, {}^{t3i}\langle \alpha_k' h_k' \rangle \qquad (6.11.19)$$

$${}^{3i}\langle {}^{o}\underline{h}_k^{T} \rangle = \rho_k \, {}^{t3i}\langle \underline{U}_k' h_k' \rangle \qquad (6.11.20)$$

$${}^{3i}\langle {}^{o}\underset{\sim}{\underline{h}}_k \rangle = \rho_k \, {}^{3i}\langle \check{\underline{U}}_{kLF} \check{h}_{kLF} \rangle \qquad (6.11.21)$$

$$\alpha_{kLF} \, {}^{3i}\langle {}^{o}\underset{\sim}{\underline{h}}_k^{T} \rangle = \rho_k \left({}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \check{\underline{U}}_{kLF} h_k' \rangle \right\rangle + {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \underline{U}_k' \check{h}_{kLF} \rangle \right\rangle \right) \qquad (6.11.22)$$

and

$$\begin{aligned} {}^{o}\left(EPYTI\right)_k = & - v^{-1} \rho_k \int_{A_{kLF}} \check{h}_{kLF} \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_{kLF} \, dA \\ & - v^{-1} \rho_k \int_{A_{kLF}} {}^{t}\left\langle h_k' \left(\underline{U}_k' - \underline{W}_k' \right) \right\rangle \cdot \underline{n}_{kLF} \, dA \\ & - v^{-1} \rho_k \, {}^{t}\left\langle \int_{A_k'} h_k' \left({}^{3i}\langle \underline{U}_k \rangle_{LF} + \check{\underline{U}}_{kLF} - \underline{W}_{kLF} \right) \cdot \underline{n}_k' \, dA \right\rangle \\ & - v^{-1} \rho_k \, {}^{t}\left\langle \int_{A_k'} \left(\check{h}_{kLF} + h_k' \right) \left(\underline{U}_k' - \underline{W}_k' \right) \cdot \underline{n}_k' \, dA \right\rangle . \end{aligned} \qquad (6.11.23)$$

Likewise, Eq. 6.11.17c reduces to

$$\alpha_{kLF} \rho_k \left(\frac{\partial \, {}^{3i}\langle h_k \rangle_{LF}}{\partial t} + {}^{3i}\langle \underline{U}_k \rangle_{LF} \cdot \nabla \, {}^{3i}\langle h_k \rangle_{LF} \right)$$

$$+ \frac{\partial \, {}^{o}\phi_{hk}}{\partial t} + \nabla \cdot {}^{o}\phi_{hk} \, {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\underline{\Psi}_{mk} \cdot \nabla \, {}^{3i}\langle h_k \rangle_{LF}$$

$$= \alpha_{kLF} \frac{d}{dt_k} {}^{3i}\langle P_k \rangle_{LF} + \frac{d}{dt_k} {}^{t3i}\langle \alpha_k' P_k' \rangle + \nabla \cdot \underline{\Psi}_{Pk} - \Phi_{Pk}$$

$$+ \left(\alpha_{kLF} {}^{3i}\langle {}^{o}\underline{\tau}_k \rangle_{LF} + {}^{t3i}\langle \alpha_k' {}^{o}\underline{\tau}_k' \rangle \right) : \nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + {}^{o}\Phi_{\tau k}$$

$$- \nabla \cdot \alpha_{kLF} \left({}^{3i}\langle {}^{o}\underline{h}_k^{T} \rangle + {}^{3i}\langle {}^{o}\underline{\tilde{h}}_k \rangle + {}^{3i}\langle {}^{o}\underline{\tilde{h}}_k^{T} \rangle \right)$$

$$- \nabla \cdot \left(\alpha_{kLF} {}^{3i}\langle \underline{J}_{qk} \rangle + {}^{t3i}\langle \alpha_k' \underline{J}_{qk}' \rangle \right)$$

$$+ \alpha_{kLF} {}^{3i}\langle J_{Ek} \rangle_{LF} + {}^{t3i}\langle \alpha_k' J_{Ek}' \rangle$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + \left(PWI \right)_k^{(h)} - \left(PWI \right)_k^{(u)} + {}^{o}\left(VDI \right)_k + {}^{o}\left(EPYTI \right)_k . \qquad (6.11.18b)$$

In Eqs. 6.11.18a and 6.11.18b, ${}^{o}\underline{\Psi}_{mk}$, ${}^{3i}\langle {}^{o}\underline{\tau}_k \rangle_{LF}$, ${}^{t3i}\langle \alpha_k' {}^{o}\underline{\tau}_k' \rangle$, ${}^{o}\Phi_{\tau k}$, ${}^{ot}\langle \Gamma_k \rangle$, and ${}^{o}\left(VDI \right)_k$ are given by Eqs. 6.3.15, 6.5.26a, 6.5.26b, 6.9.28, 6.5.31, and 6.9.29 respectively. The difference, $\left(PWI \right)_k^{(h)} - \left(PWI \right)_k^{(u)}$, is given by Eq. 6.11.13.

6.12 Time-Volume-averaged Interfacial Enthalpy Balance Equation $(\gamma_V = \gamma_A = 1)$

When the capillary energy is ignored, the local volume-averaged enthalpy balance relation for the interface A_{kf} is given by Eq. 5.2.6. Upon performing time averaging and making use of Eqs. 6.11.11, 6.11.14, and 6.11.16, one obtains

$$ {}^{t}\langle \mathcal{H}_k \rangle + {}^{t}\langle \mathcal{H}_f \rangle = 0 \qquad (6.12.1a)$$

or, equivalently,

$$- {}^{3i}\langle P_k \rangle_{LF} \frac{d\,\alpha_{kLF}}{d\,t_k} + {}^{t}\langle \dot{Q}_k \rangle + {}^{t}\langle \Gamma_k \rangle \, {}^{3i}\langle h_k \rangle_{LF}$$

$$+ \left(\mathrm{PWI}\right)_k^{(h)} + \left(\mathrm{VDI}\right)_k + \left(\mathrm{EPYTI}\right)_k$$

$$- {}^{3i}\langle P_f \rangle_{LF} \frac{d\,\alpha_{fLF}}{d\,t_f} + {}^{t}\langle \dot{Q}_f \rangle + {}^{t}\langle \Gamma_f \rangle\, {}^{3i}\langle h_f \rangle_{LF}$$

$$+ \left(\mathrm{PWI}\right)_f^{(h)} + \left(\mathrm{VDI}\right)_f + \left(\mathrm{EPYTI}\right)_f = 0 \ . \qquad (6.12.1b)$$

When ρ_k = constant, ${}^{t}\langle \Gamma_k \rangle$, $\left(\mathrm{VDI}\right)_k$, and $\left(\mathrm{EPYTI}\right)_k$ should be replaced by their corresponding quantities with superscript ${}^{o}( )$. Similar changes should be made for the corresponding terms with subscript f. Finally, we note that

$$\frac{d}{d\,t_f} = \frac{\partial}{\partial t} + {}^{3i}\langle \underline{U}_f \rangle_{LF} \cdot \nabla \ . \qquad (6.12.2)$$

7. TIME-VOLUME VERSUS VOLUME-TIME AVERAGING

The significance of first performing volume-averaging of the phasic conservation equations and their associated interfacial balance equations, followed by time-averaging, has been pointed out earlier. This order of averaging preserves the distinction of the dynamic phases in a multiphase system, such as droplets or bubbles of different sizes, or particles of the same size and material but of different electric charges. Eulerian time averaging from the very beginning will remove such distinction unless suitable conditional averaging is used. Simple time averaging leads to fractional residence time* of a phase rather than volume fraction of a phase. This fractional residence time of a phase becomes equal to the physical volume fraction only in the case of one-dimensional uniform motion of incompressible phases. Volume fraction relates naturally to cumulative thermodynamic relations, while fractional residence time does not.

The foregoing discussion clearly contradicts the conclusion reached by Delhaye and Achard [14], who stated that the order of time-volume averaging is interchangeable and gave a mathematical proof to support their claim. Unfortunately, their proof was in error due to improper application of the Leibnitz rule for the differentiation of an integral. A brief recapitulation of their derivation follows.

Consider the variation of any property f_k associated with phase k, such as density, temperature, or velocity, as seen by an observer at a fixed point in multiphase flows. Since phase k passes through the point intermittently, f_k would have the appearance shown in Fig. 3 [13,14].

*Fractional residence time was referred to as local time fraction or time averaged phase density function in Ref. 13.

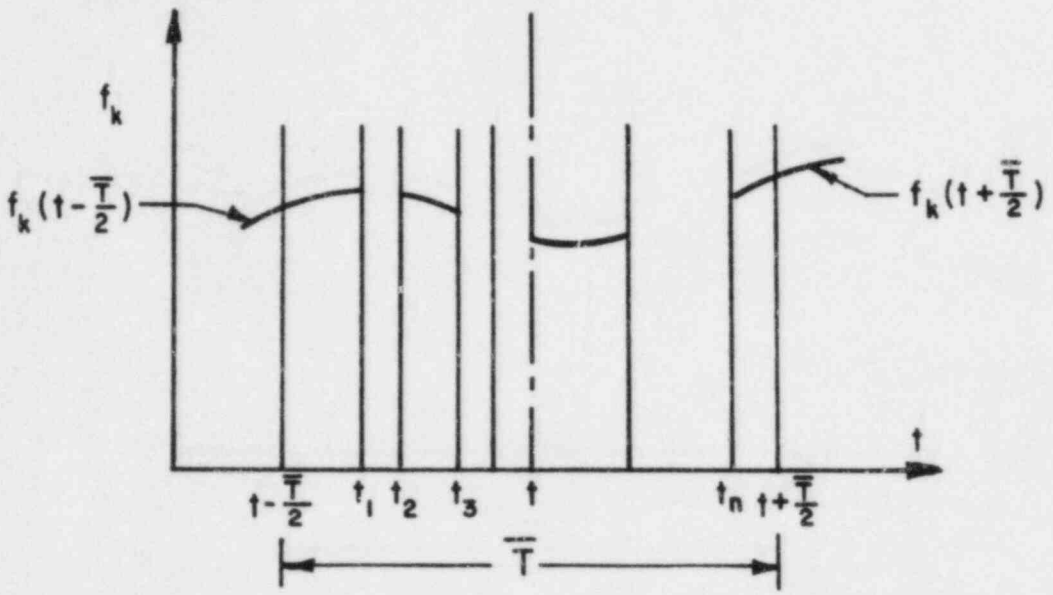


Fig. 3. Variation of f_k with Time at a Fixed Point

Delhaye and Achard[14] considered the time interval $\left(t - \frac{\bar{T}}{2}, t + \frac{\bar{T}}{2}\right)$ centered at the instant t and denoted the cumulated residence time of phase k in the interval by $[T_k]$. The averaging time interval $\bar{T}$ was taken to be a constant. Referring to Fig. 3, we may write

$$\int_{[T_k]} f_k \, dt = \int_{\eta=t-\bar{T}/2}^{\eta=t_1} f_k(\eta) \, d\eta + \int_{\eta=t_2}^{\eta=t_3} f_k(\eta) \, d\eta + \cdots + \int_{\eta=t_n}^{\eta=t+\bar{T}/2} f_k(\eta) \, d\eta \,, \tag{7.1}$$

where η is the dummy variable of integration. Delhaye and Achard[14] then applied the Leibnitz rule for differentiation of an integral and, at the same time, set

$$\frac{dt_i}{dt} = 0, \text{ for } i = 1, 2, \ldots n \,. \tag{7.2}$$

The result was

$$\frac{\partial}{\partial t} \int_{[T_k]} f_k \, dt = f_k\left(t + \frac{\bar{T}}{2}\right) - f_k\left(t - \frac{\bar{T}}{2}\right), \tag{7.3}$$

i.e., the time derivative of the integral defined in Eq. 7.1 depends <u>only</u> on the values of the integrand f_k evaluated at the two end points.

Let us pause and inquire into the physical meaning of

$$\frac{\partial}{\partial t}\int_{[T_k]} f_k \, dt$$

under the condition defined in Eq. 7.2. One one hand, we have from elementary calculus

$$\frac{\partial}{\partial t}\int_{[T_k]} f_k \, dt = \lim_{\Delta t \to 0} \frac{\left(\int_{[T_k]} f_k \, dt\right)_{t+\Delta t} - \left(\int_{[T_k]} f_k \, dt\right)_t}{\Delta t} \, . \tag{7.4}$$

On the other hand, we note that for time averaging to be physically meaningful in multiphase flows, the averaging duration $\bar{T}$ must encompass a sufficiently large number of interfaces, i.e., it must be large relative to the inverse of the passage frequency of the phase interfaces ν_s. At the same time, it must be small compared with that required for the mixture flowing at a representative velocity U through the characteristic dimension L of the system. Hence,

$$(L/U) \gg \bar{T} \gg (1/\nu_s) \, . \tag{7.5}$$

In other words, Δt in Eq. 7.4 should never be allowed to approach zero. It must be finite. Now for time t, phase k will first leave the observation point at time $t_1 - (t - \bar{T}/2)$ after the initial arrival of phase k. For time $t + \Delta t$, phase k will, in general, not leave the observation point at the same time subsequent to its initial arrival. The same can be said for the "arrival" time t_2 and "departure" time t_3, etc. Hence, t_i is not independent of t, and the use of Eq. 7.2 is not physically realizable in multiphase flows. Consequently, Eqs. 7.2 and 7.3 are invalid; so is the conclusion reached by Delhaye and Achard.

There is an additional difficulty associated with the application of Eulerian time averaging to the phasic conservation equations from the beginning. This difficulty stems from the fact that the time interval $\bar{T}$ chosen for averaging is not intrinsic to the structure of the multiphase medium under consideration, but depends strongly on the convection velocity. In most engineering systems, wide ranges of velocities often exist, and hence, they may not be characterized by a single time scale. This is in contrast to the length scale associated with local volume averaging, which is independent of the flow velocity.

8. SIMPLIFICATIONS AND COMPARISONS WITH CURRENTLY "ACCEPTED" TWO-PHASE FLOW GOVERNING EQUATIONS

A number of two-phase flow equations are available in the literature. It should be instructive to compare them with the set of equations given in Sec. 6 of this report. This is particularly appropriate since the procedure and the results presented herein are new and we are not aware of a comparable analysis published in the open literature. Ishii's monograph[13] considered time averaging only; hence, the results may not be compared directly. For instance, the local volume fraction α_k of phase k that appears in the present set of equations is, in general, not the same as the local time fraction in

Ishii's equations.* Despite such difficulties, we proceed to examine in Sec. 8.1 their similarities and differences.

8.1 Comparison with Ishii's Two-Fluid Equations

Since Ishii's analysis was based on time averaging only, comparison may be made when the local averaging volume v in the present analysis is sufficiently small such that all spatial deviations $\tilde{\psi}_k$ can be neglected. We are mindful of the fact that such simplification is usually not physically realizable. Assuming that the said simplification can be made, the decomposition of an instantaneous point variable ψ_k, which may be a scalar, a vector, or a second-order tensor, becomes:

$$\psi_k = \bar{\psi}_{kLF} + \psi_k' \,, \tag{8.1.1}$$

in which $\bar{\psi}_k$ denotes the time average, and ψ_k' denotes the high-frequency fluctuation. The duration T over which the averaging is to be made satisfies the inequality defined by Eq. 6.1.4. Hence, $\bar{\psi}_k$ is the low-frequency component of ψ_k, including the time-independent case in the limit. Upon comparing Eq. 8.1.1 with Eq. 6.1.8, one sees that ${}^{3i}\langle\psi_k\rangle_{LF} = \bar{\psi}_k$, since $\tilde{\psi}_{kLF}$ is negligible. Accordingly, all quantities identified by ${}^{3i}\langle\psi_k\rangle_{LF}$ in the time-volume averaged conservation equations and the interfacial balance equations presented in Sec. 6 will be replaced by $\bar{\psi}_k$. The low-frequency component of the local volume fraction α_{kLF} becomes $\bar{\alpha}_k$. Furthermore, quantities identified by ${}^{t3i}\langle f_k'\psi_k'\rangle$ become ${}^{t}\langle f_k'\psi_k'\rangle$. With the foregoing simplifications, the time-volume averaged conservation equations for mass, momentum, and energy and their interfacial balance relations reduce to:

- Mass Conservation Equation (from Eq. 6.3.13)

$$\frac{\partial}{\partial t}\left(\bar{\alpha}_k\bar{\rho}_k + {}^{t}\langle\alpha_k'\rho_k'\rangle\right) + \nabla\cdot\left(\bar{\alpha}_k\bar{\rho}_k + {}^{t}\langle\alpha_k'\rho_k'\rangle\right)\bar{\underline{U}}_k$$

$$- \nabla\cdot D_{mk}^{T}\,\nabla\,\bar{\alpha}_k\bar{\rho}_k = {}^{t}\langle\Gamma_k\rangle^{zd} \,, \tag{8.1.2}$$

where

$$D_{mk}^{T}\,\nabla\,\bar{\alpha}_k\bar{\rho}_k = -\bar{\alpha}_k\,{}^{t}\langle\rho_k'\underline{U}_k'\rangle - \bar{\rho}_k\,{}^{t}\langle\alpha_k'\underline{U}_k'\rangle \,, \tag{8.1.3}$$

and

$${}^{t}\langle\Gamma_k\rangle^{zd} = \bar{\rho}_k\left(\frac{\partial\,\bar{\alpha}_k}{\partial t} + \bar{\underline{U}}_k\cdot\nabla\,\bar{\alpha}_k\right) + \left(\mathrm{MTI}\right)_k^{zd} \,, \tag{8.1.4}$$

in which

---

*The equality of local time fraction and local volume fraction is implied in Ishii's work. See p. 67 of Ref. 13.

$$(\mathrm{MTI})_k^{zd} = - v^{-1} \int_{A_{kLF}} {}^t\langle \rho_k'(\underline{U}_k' - \underline{W}_k')\rangle \cdot \underline{n}_{kLF}\, dA$$

$$- v^{-1} \left\langle \int_{A_k'} \left[(\bar{\rho}_k + \rho_k')(\underline{U}_k' - \underline{W}_k') + \rho_k'(\bar{\underline{U}}_k - \bar{\underline{W}}_k)\right] \cdot \underline{n}_k'\, dA \right\rangle . \quad (8.1.5)$$

The superscript zd stands for zero spatial deviation.

If one further stipulates that

(a) $\alpha_k' \approx 0$, and hence, $A_k' \approx 0$ and

(b) all correlations involving ρ_k' are insignificant,* } (8.1.6)

then Eq. 8.1.2 reduces to

$$\frac{\partial}{\partial t} \bar{\alpha}_k \bar{\rho}_k + \nabla \cdot \bar{\alpha}_k \bar{\rho}_k \bar{\underline{U}}_k = {}^t\langle \Gamma_k \rangle^s , \quad (8.1.7)$$

where

$${}^t\langle \Gamma_k \rangle^s = \bar{\rho}_k \left(\frac{\partial\, \bar{\alpha}_k}{\partial t} + \bar{\underline{U}}_k \cdot \nabla\, \bar{\alpha}_k \right), \quad (8.1.8)$$

and superscript s denotes further simplification resulting from assumptions stated in Eq. 8.1.6 with the consequence that $(\mathrm{MTI})_k^{zd} = 0$. For a two-fluid system, k = 1 or 2, Ishii gave the following equation for void propagation (Eq. VII 3.15 in Ref. 13):

$$\Gamma_k^I = \frac{\bar{\bar{\rho}}_1 \bar{\bar{\rho}}_2}{\sum_{k=1}^{2} \alpha_k^I \bar{\bar{\rho}}_k} \left(\frac{\partial\, \alpha_k^I}{\partial t} + \underline{C}_k \cdot \nabla\, \alpha_k^I \right), \quad (8.1.9)$$

in which superscript (I) refers to Ishii, and superscript (=) denotes Ishii's phase average. α_k^I is the local time fraction and it relates the phase average $\bar{\bar{\Psi}}_k$ to the Eulerian time average $\bar{\Psi}_k$ according to

$$\alpha_k^I \bar{\bar{\psi}}_k = \bar{\psi}_k . \quad (8.1.10)$$

*This is a weaker restriction than the condition that ρ_k = constant since the variation of ρ_k with temperature and pressure can still be considered even though $\rho_k' = 0$.

$\underline{C}_k$ is Ishii's kinematic wave velocity, and Γ_k^I is the rate of production of phase k at the interface for which Ishii wrote (Eq. V 2.1 in Ref. 13)

$$\Gamma_k^I = -\frac{1}{\Delta t}\sum_j \frac{1}{W_{kn}}\rho_k(\underline{U}_k - \underline{W}_k)\cdot\underline{n}_k \ , \tag{8.1.11}$$

where Δt is a fixed time interval for averaging, W_{kn} is the magnitude of the normal component of the interfacial velocity, and index j refers to the number of times the interface passes through a fixed observation point during Δt. Ishii's time-averaged mass conservation equation (Eq. IX 1.1 in Ref. 13) is

$$\frac{\partial}{\partial t}\alpha_k^I \bar{\rho}_k + \nabla\cdot\alpha_k^I \bar{\rho}_k \underline{\bar{U}}_k^I = \Gamma_k^I \ , \tag{8.1.12}$$

where $\underline{\bar{U}}_k^I$ is the mass-weighted mean velocity defined by

$$\underline{\bar{U}}_k^I = \overline{\rho_k \underline{U}_k}/\bar{\rho}_k \ . \tag{8.1.13}$$

In Eqs. 8.1.9 and 8.1.12, α_k^I is the local time fraction which is generally not the same as local volume fraction. Despite these differences, the similarities between Eqs. 8.1.7 and 8.1.12, and between Eqs. 8.1.8 and 8.1.9 are apparent.

- Interfacial Mass Balance Equation (from Eq. 6.4.1a)

For a two-fluid system, the interfacial mass balance equation associated with Eq. 8.1.7 is

$$\sum_{k=1}^{2} {}^t\langle\Gamma_k\rangle^s = 0 \ . \tag{8.1.14}$$

In conjunction with Eq. 8.1.12, Ishii gave

$$\sum_{k=1}^{2} \Gamma_k^I = 0 \ . \tag{8.1.15}$$

Both Eqs. 8.1.14 and 8.1.15 express the conservation of mass at the interfaces.

- Linear Momentum Equation (from Eqs. 6.5.22 and 6.5.23a)

$$\frac{\partial}{\partial t}\left(\bar{\alpha}_k\bar{\rho}_k + {}^t\langle\alpha_k'\rho_k'\rangle\right)\underline{\bar{U}}_k + \nabla\cdot\left(\bar{\alpha}_k\bar{\rho}_k + {}^t\langle\alpha_k'\rho_k'\rangle\right)\underline{\bar{U}}_k\underline{\bar{U}}_k$$

$$-\frac{\partial}{\partial t} D_{mk}^T \nabla\bar{\alpha}_k\bar{\rho}_k - 2\nabla\cdot\underline{\bar{U}}_k D_{mk}^T \nabla\bar{\alpha}_k\bar{\rho}_k$$

$$= - \nabla \bar{\alpha}_k \bar{P}_k - \nabla \, {}^t\langle \alpha_k' P_k' \rangle$$

$$+ \nabla \cdot \bar{\alpha}_k \left(\bar{\underline{\underline{\tau}}}_k + \underline{\underline{\tau}}_k^T \right) + \nabla \cdot {}^t\langle \alpha_k' \underline{\underline{\tau}}_k' \rangle$$

$$+ \left(\bar{\alpha}_k \bar{\rho}_k + {}^{t3i}\langle \alpha_k' \rho_k' \rangle \right) \underline{f}$$

$$+ {}^t\langle \Gamma_k \rangle^{zd} \underline{\bar{U}}_k + \bar{P}_k \nabla \bar{\alpha}_k - \bar{\underline{\underline{\tau}}}_k \cdot \nabla \bar{\alpha}_k$$

$$+ \left(PTI \right)_k^{zd} - \left(VSTI \right)_k^{zd} + \left(MMTI \right)_k^{zd} , \qquad (8.1.16)$$

where

$$\underline{\underline{\tau}}_k^T = - \bar{\rho}_k \, {}^t\langle \underline{U}_k' \underline{U}_k' \rangle \qquad (8.1.17)$$

$$\left(PTI \right)_k^{zd} = - v^{-1} \left\langle {}^t \int_{A_k'} P_k' \, \underline{n}_k' \, dA \right\rangle \qquad (8.1.18)$$

$$\left(VSTI \right)_k^{zd} = - v^{-1} \left\langle {}^t \int_{A_k'} \underline{\underline{\tau}}_k' \cdot \underline{n}_k' \, dA \right\rangle \qquad (8.1.19)$$

$$\left(MMTI \right)_k^{zd} = - v^{-1} \int_{A_{kLF}} \left[\bar{\rho}_k \, {}^t\langle \underline{U}_k' (\underline{U}_k' - \underline{W}_k') \rangle + {}^t\langle \rho_k' \underline{U}_k' \rangle \left(\bar{\underline{U}}_k - \bar{\underline{W}}_k \right) \right] \cdot \underline{n}_{kLF} \, dA$$

$$- v^{-1} \left\langle {}^t \int_{A_k'} \left(\bar{\rho}_k + \rho_k' \right) \underline{U}_k' \left[\bar{\underline{U}}_k + \underline{U}_k' - \left(\bar{\underline{W}}_k + \underline{W}_k' \right) \right] \cdot \underline{n}_k' \, dA \right\rangle \qquad (8.1.20)$$

and ${}^t\langle \Gamma_k \rangle^{zd}$ is defined by Eq. 8.1.4.

If additional simplifications specified in Eq. 8.1.6 are introduced, Eq. 8.1.16 becomes, for $\underline{f} = \underline{g}$,

$$\frac{\partial}{\partial t} \bar{\alpha}_k \bar{\rho}_k \underline{\bar{U}}_k + \nabla \cdot \bar{\alpha}_k \bar{\rho}_k \underline{\bar{U}}_k \underline{\bar{U}}_k$$

$$= - \nabla \bar{\alpha}_k \bar{P}_k + \nabla \cdot \bar{\alpha}_k \left(\bar{\underline{\underline{\tau}}}_k + \underline{\underline{\tau}}_k^T \right) + \bar{\alpha}_k \bar{\rho}_k \underline{g} + {}^t\langle \underline{M}_k \rangle^s , \qquad (8.1.21)$$

where ${}^t\langle \underline{M}_k \rangle^s$ denotes the interfacial momentum sources under the stated simplifying conditions:

$${}^t\langle \underline{M}_k \rangle^s = {}^t\langle \Gamma_k \rangle^s \underline{\bar{U}}_k + \bar{P}_k \nabla \bar{\alpha}_k - \bar{\underline{\underline{\tau}}}_k \cdot \nabla \bar{\alpha}_k + \left(MMTI \right)_k^s , \qquad (8.1.22)$$

where ${}^{t}\langle\Gamma_k\rangle^{s}$ is defined by Eq. 8.1.8 and

$$\left(\text{MMTI}\right)_k^s = -v^{-1}\int_{A_k} {}^{t}\langle \underline{U}_k'\left(\underline{U}_k' - \underline{W}_k'\right)\rangle \cdot \underline{n}_k \, dA \, . \qquad (8.1.23)$$

It is seen that $\left(\text{MMTI}\right)_k^s$ represents a "modified" Reynolds stress due to interfacial turbulence.

- Interfacial Momentum Balance Equation

For a two-fluid system, Eq. 6.6.2, when simplified for conditions consistent with those used in deriving Eq. 8.1.20, becomes

$$\sum_{k=1}^{2} {}^{t}\langle \underline{M}_k\rangle^{s} = v^{-1}\int_{A_k} \left(\nabla_i \bar{\sigma} - 2\bar{\sigma}\,\bar{H}_k\,\underline{n}_k\right) dA \, , \qquad (8.1.24a)$$

where ∇_i is the interfacial surface gradient operator, $\bar{\sigma}$ is the temporal mean interfacial tension, $\bar{H}_k$ is the temporal mean of the average principal curvature of the interface, and subscript k may be either 1 or 2, referring to fluids on either side of the interface. It is understood that A_k and $\underline{n}_k$ are also temporal means. In writing Eq. 8.1.24a, the effect of deviation of local curvature from the average curvature change is ignored. It can be added if desired.

When the averaging volume is sufficiently small, the integral on the right-hand side of Eq. 8.1.24a may be approximated by

$$v^{-1}\int_{A_k} \left(\nabla_i \bar{\sigma} - 2\bar{\sigma}\,\bar{H}_k\,\underline{n}_k\right) dA \simeq \frac{A_k}{v}\nabla_i \bar{\sigma} + 2\bar{\sigma}\,\bar{H}_k\,\nabla\bar{\alpha}_k \, . \qquad (8.1.25)$$

The first term on the right-hand side accounts for the variation of surface tension along the interface, which is probably small in dispersed systems due to the random nature of its distribution over the particles. The second term accounts for the curvature effect. In many practical systems, both contributions relative to $\underline{M}_k^s$ often are small and can be neglected. If this is indeed the case, Eq. 8.1.24a reduces to

$$\sum_{k=1}^{2} {}^{t}\langle \underline{M}_k\rangle^{s} \simeq 0 \, . \qquad (8.1.24b)$$

The time-averaged linear momentum equation given by Ishii (Eq. IX 1.14 in Ref 13) is

$$\frac{\partial}{\partial t}\alpha_k^I \bar{\bar{\rho}}_k \underline{\bar{U}}_k^I + \nabla\cdot\alpha_k^I \bar{\bar{\rho}}_k \underline{\bar{U}}_k^I\underline{\bar{U}}_k^I$$

$$= -\nabla\alpha_k^I \bar{\bar{P}}_k + \nabla\cdot\alpha_k^I\left(\bar{\bar{\tau}}_k + \bar{\bar{\tau}}_k^T\right) + \alpha_k^I \bar{\bar{\rho}}_k \underline{g} + \underline{M}_k^I \, , \qquad (8.1.26)$$

in which $\underline{\underline{\tau}}_k$ and $\underline{\underline{\tau}}_k^T$ are the viscous and turbulent stress tensors and $\underline{M}_k^I$ is the interfacial momentum source given by

$$\underline{M}_k^I = \Gamma_k^I \underline{\bar{U}}_{ki}^I + \bar{\bar{P}}_{ki} \nabla \alpha_k^I + \underline{M}_k^d - \underline{\underline{\tau}}_i \cdot \nabla \alpha_k^I \tag{8.1.27}$$

where $\underline{M}_k^d$ is the total drag force. In Eq. 8.1.27, the last term was added in accordance with Refs. 15 and 16. $\underline{M}_k^d$ consists of a tangential component giving rise to skin friction drag and a normal component, which gives rise to the form drag.

The associated interfacial transfer condition given by Ishii (Eqs. IX 1.12 and VIII 2.7 in Ref. 13) is

$$\sum_{k=1}^{2} \underline{M}_k^I = \sum_j \frac{1}{L_j} 2\left(H_{21} - \bar{\bar{H}}_{21}\right)\bar{\bar{\sigma}}\underline{n}_1 + 2\bar{\bar{\sigma}}H_{21}\nabla\alpha_2^I + \text{force due to } \nabla_i\bar{\bar{\sigma}} \ , \tag{8.1.28a}$$

where L_j^{-1} denotes the area concentration per unit volume. The first term on the right-hand side of Eq. 8.1.28a accounts for the effect of the change in mean curvature, which is not included in Eqs. 8.1.24a and 8.1.25. When the terms on the right-hand side of Eq. 8.1.28a, taken collectively, are small compared with terms in $\underline{M}_k^I$, one can write

$$\sum_{k=1}^{2} \underline{M}_k^I \simeq 0 \ . \tag{8.1.28b}$$

In two recent papers, Ishii and Mishima[15], and Ishii and Kocamustafaogullari[16] gave the following "simplified" form of the time-averaged momentum equation (Eq. 2 in Refs. 15 and 16):

$$\frac{\partial}{\partial t}\alpha_k^I \bar{\bar{\rho}}_k \underline{\bar{U}}_k + \nabla \cdot \alpha_k^I \bar{\bar{\rho}}_k \underline{\bar{U}}_k \underline{\bar{U}}_k$$

$$= -\alpha_k^I \nabla \bar{\bar{P}}_k + \nabla \cdot \alpha_k^I \left(\underline{\underline{\tau}}_k + \underline{\underline{\tau}}_k^T\right)$$

$$+ \alpha_k^I \bar{\bar{\rho}}_k \underline{g} + \Gamma_k^I \underline{\bar{U}}_{ki} - \underline{\underline{\tau}}_i \cdot \nabla \alpha_k^I + \underline{M}_{ik} + \left(\bar{\bar{P}}_{ki} - \bar{\bar{P}}_k\right) \nabla \alpha_k^I \tag{8.1.29}^*$$

in which $\underline{\underline{\tau}}_i$ is the interfacial shear stress, and $\underline{M}_{ik}$ is the "generalized" interfacial drag. By comparing Eq. 8.1.26 with Eq. 8.1.29, one sees that $\underline{M}_k^d = \underline{M}_{ik}$. The authors of Refs. 15 and 16 suggested that an equation of the following constituents would be suitable for the dispersed phase:

---

*The last term in Eq. 8.1.29 was added per personal communication between Dr. Ishii and W. T. Sha, December 14, 1984.

$\underline{M}_{ik}$ = sum of standard drag force, virtual mass force, and Basset force, all computed on the basis of a unit volume. (8.1.30)**

Equation 8.1.30, which is Eq. 6 in Ref. 15 or Eq. 9 in Ref. 16, is not a derived result. While it appears to be physically meaningful, there is no assurance that $\underline{M}_{ik}$ in Eq. 8.1.29 can be expressed as such.

The associated interfacial momentum balance equation stated by Ishii and Kocamustafaogullari for a two-fluid system is

$$\sum_{k=1}^{2} \underline{M}_{ik} = 0 \ . \tag{8.1.31}$$

Despite the difference between our $\bar{\alpha}_k$ and Ishii's α_k^I, it is instructive to compare the simplified momentum equation 8.1.21 with Ishii's result, Eq. 8.1.26. If we assume that ${}^t\langle\Gamma_k\rangle^s = \Gamma_k^I$, $\bar{\rho}_k = \bar{\bar{\rho}}_k$, $\bar{P}_k = \bar{\bar{P}}_k$, etc., and in addition $\bar{\alpha}_k = \alpha_k^I$, then Eq. 8.1.21 becomes formally identical to Eq. 8.1.26, provided that

$${}^t\langle \underline{M}_k \rangle^s = \underline{M}_k^I \ , \tag{8.1.32}$$

or, equivalently,

$$\underline{M}_k^d + \left(\bar{P}_{ki} - \bar{P}_k\right) \nabla \bar{\alpha}_k + {}^t\langle\Gamma_k\rangle^s \left(\bar{\underline{U}}_{ki} - \bar{\underline{U}}_k\right)$$

$$- \left(\underline{\underline{\tau}}_i - \bar{\underline{\underline{\tau}}}_k\right) \cdot \nabla \bar{\alpha}_k = \left(\mathrm{MMTI}\right)_k^s \ . \tag{8.1.33}$$

We reiterate that $\left(\mathrm{MMTI}\right)_k^s$ represents interfacial momentum transfer due to turbulence. The two interfacial momentum balance relationships, Eqs. 8.1.24b and 8.1.28b, are formally identical.

By using the assmptions just cited, namely, ${}^t\langle\Gamma_k\rangle^s = \Gamma_k^I$, $\bar{\rho}_k = \bar{\bar{\rho}}_k$, $\bar{\alpha}_k = \alpha_k^I$, etc., one may readily demonstrate that Eqs. 8.1.21 and 8.1.29 of Ref. 15 become formally identical if

$$- {}^t\langle\Gamma_k\rangle^s \left(\bar{\underline{U}}_{ki} - \bar{\underline{U}}_k\right) - \left(\bar{P}_{ki} - \bar{P}_k\right) \nabla \bar{\alpha}_k$$

** The "generalized" drag force was represented by $\underline{M}_{id}$ in Eq. 6 of Ref. 15 and in Eq. 9 of Ref 16 instead of $\underline{M}_{ik}$. Presumably the subscript d refers to dispersed phase.

$$+ \left(\underline{\tau}_i - \underline{\bar{\tau}}_k\right) \cdot \nabla \bar{\alpha}_k + \left(\mathrm{MMTI}\right)_k^s = \underline{M}_{ik} \,. \tag{8.1.34}$$

Clearly, Eq. 8.1.34 is equivalent to Eq. 8.1.33 since $\underline{M}_{ik} = \underline{M}_k^d$.

An alternative form of the momentum equation is given by Eq. 6.5.23c, which, upon simplification under the stated conditions, becomes (for $\underline{f} = \underline{g}$):

$$\bar{\alpha}_k \bar{\rho}_k \frac{d\,\underline{\bar{U}}_k}{dt_k^{zd}} = -\bar{\alpha}_k \nabla \bar{P}_k + \nabla \cdot \bar{\alpha}_k \left(\underline{\bar{\tau}}_k + \underline{\tau}_k^T\right) + \bar{\alpha}_k \bar{\rho}_k \underline{g}$$

$$- \underline{\bar{\tau}}_k \cdot \nabla \bar{\alpha}_k + \left(\mathrm{MMTI}\right)_k^s \,, \tag{8.1.35}$$

where $\dfrac{d}{d\,t_k^{zd}} = \dfrac{\partial}{\partial t} + \underline{\bar{U}}_k \cdot \nabla$.

Equation IX 1.15 in Ref. 13 reads:

$$\alpha_k^I \bar{\rho}_k^I \frac{D_k \underline{\bar{U}}_k^I}{D\,t} = -\alpha_k^I \nabla \bar{\bar{P}}_k + \nabla \cdot \alpha_k^I \left(\underline{\bar{\tau}}_k + \underline{\tau}_k^T\right) + \alpha_k^I \bar{\rho}_k^I \underline{g}$$

$$+ \left(\bar{\bar{P}}_{ki} - \bar{\bar{P}}_k\right) \nabla \alpha_k^I + \Gamma_k^I \left(\underline{\bar{U}}_{ki}^I - \underline{\bar{U}}_k^I\right) + \underline{M}_k^d - \underline{\tau}_i \cdot \nabla \alpha_k^I \tag{8.1.36}$$

where $\dfrac{D_k}{Dt} = \dfrac{\partial}{\partial t} + \underline{\bar{U}}_k^I \cdot \nabla$ and $\underline{M}_k^d$ is the total drag force.

If one assumes that $\bar{\alpha}_k = \alpha_k^I$, $\bar{\rho}_k = \bar{\bar{\rho}}_k$, $\underline{\bar{U}}_k = \underline{\bar{U}}_k^I$, $\bar{P}_k = \bar{\bar{P}}_k$, etc., then Eq. 8.1.35 becomes formally identical to Eq. 8.1.36 when

$$\underline{M}_k^d + \left(\bar{P}_{ki} - \bar{P}_k\right) \nabla \bar{\alpha}_k + {}^t\langle \Gamma_k \rangle^s \left(\underline{\bar{U}}_{ki} - \underline{\bar{U}}_k\right)$$

$$- \left(\underline{\tau}_i - \underline{\bar{\tau}}_k\right) \cdot \nabla \bar{\alpha}_k = \left(\mathrm{MMTI}\right)_k^s \tag{8.1.37}$$

which is identical to Eq. 8.1.33, as one would expect.

- Enthalpy Equation (from Eq. 6.11.17b)

$$\frac{\partial}{\partial t}\left(\bar{\alpha}_k \bar{\rho}_k + {}^t\langle \alpha_k' \rho_k' \rangle\right) \bar{h}_k + \nabla \cdot \left(\bar{\alpha}_k \bar{\rho}_k + {}^t\langle \alpha_k' \rho_k' \rangle\right) \underline{\bar{U}}_k \bar{h}_k$$

$$+ \frac{\partial \Phi_{hk}^{zd}}{\partial t} + \nabla \cdot \Phi_{hk}^{zd} \underline{\bar{U}}_k - \nabla \cdot \left(D_{mk}^{T} \nabla \bar{\alpha}_k \bar{\rho}_k \right) \bar{h}_k$$

$$= \bar{\alpha}_k \frac{d \bar{P}_k}{d t_k^{zd}} + \frac{d}{d t_k^{zd}} {}^{t}\langle \alpha_k' P_k' \rangle + \nabla \cdot \underline{\Psi}_{Pk}^{zd} - \Phi_{Pk}^{zd}$$

$$+ \left(\bar{\alpha}_k \bar{\underline{\tau}}_k + {}^{t}\langle \alpha_k' \underline{\tau}_k' \rangle \right) : \nabla \underline{\bar{U}}_k + \Phi_{\tau k}^{zd}$$

$$- \nabla \cdot \left(\bar{\alpha}_k \underline{\bar{J}}_{qk} + {}^{t}\langle \alpha_k' \underline{J}_{qk}' \rangle \right) + \nabla \cdot \bar{\alpha}_k \underline{h}_k^{T}$$

$$+ \bar{\alpha}_k \bar{J}_{Ek} + {}^{t}\langle \alpha_k' J_{Ek}' \rangle$$

$$+ {}^{t}\langle \dot{Q}_k \rangle + {}^{t}\langle \Gamma_k \rangle^{zd} \bar{h}_k$$

$$+ \left(PWI \right)_k^{(h),zd} + \left(VDI \right)_k^{zd} + \left(EPYTI \right)_k^{zd} , \qquad (8.1.38)$$

in which

$$\frac{d}{d t_k^{zd}} = \frac{\partial}{\partial t} + \underline{\bar{U}}_k \cdot \nabla \qquad (8.1.39)$$

$$\Phi_{hk}^{zd} = \bar{\rho}_k {}^{t}\langle \alpha_k' h_k' \rangle + \bar{\alpha}_k {}^{t}\langle \rho_k' h_k' \rangle \qquad (8.1.40)$$

$$\underline{\Psi}_{Pk}^{zd} = \bar{P}_k {}^{t}\langle \alpha_k' \underline{U}_k' \rangle + \bar{\alpha}_k {}^{t}\langle P_k' \underline{U}_k' \rangle \qquad (8.1.41)$$

$$\Phi_{Pk}^{zd} = \bar{P}_k {}^{t}\langle \nabla \cdot \alpha_k' \underline{U}_k' \rangle + \bar{\alpha}_k {}^{t}\langle P_k' \nabla \cdot \underline{U}_k' \rangle \qquad (8.1.42)$$

$$\Phi_{\tau k}^{zd} = \bar{\underline{\tau}}_k : {}^{t}\langle \nabla \alpha_k' \underline{U}_k' \rangle + \bar{\alpha}_k {}^{t}\langle \underline{\tau}_k' : \nabla \underline{U}_k' \rangle \qquad (8.1.43)$$

$$\underline{h}_k^{T} = \bar{\rho}_k {}^{t}\langle \underline{U}_k' h_k' \rangle \qquad (8.1.44)$$

$$(PWI)_k^{(h),zd} = v^{-1} \int_{A_{kLF}} {}^t\langle P_k'(\underline{U}_k' - \underline{W}_k')\rangle \cdot \underline{n}_{kLF}\, dA$$

$$+ v^{-1} \left\langle {}^t\int_{A_k'} P_k'[(\bar{\underline{U}}_k + \underline{U}_k') - (\bar{\underline{W}}_k + \underline{W}_k')] \cdot \underline{n}_k'\, dA \right\rangle \qquad (8.1.45)$$

$$(VDI)_k^{zd} = v^{-1} \bar{\underline{\underline{\tau}}}_k : \left\langle {}^t\int_{A_k'} \underline{U}_k' \underline{n}_k'\, dA \right\rangle \qquad (8.1.46)$$

$$(EPYTI)_k^{zd} = - v^{-1} \int_{A_{kLF}} \left[\bar{\rho}_k\, {}^t\langle h_k'(\underline{U}_k' - \underline{W}_k')\rangle + {}^t\langle \rho_k' h_k'\rangle (\bar{\underline{U}}_k - \bar{\underline{W}}_k)\right] \cdot \underline{n}_{kLF}\, dA$$

$$- v^{-1} \left\langle {}^t\int_{A_k'} \rho_k' h_k'[(\bar{\underline{U}}_k + \underline{U}_k') - (\bar{\underline{W}}_k + \underline{W}_k')] \cdot \underline{n}_k'\, dA \right\rangle \qquad (8.1.47)$$

and ${}^t\langle \Gamma_k\rangle^{zd}$ is defined in Eq. 8.1.4.

If further simplifications specified in Eq. 8.1.6 are introduced, and, in addition, correlations between pressure and velocity fluctuations are assumed negligible as well as dissipation due to $\bar{\underline{\underline{\tau}}}\langle \underline{\underline{\tau}}_k' : \nabla, \underline{U}_k'\rangle$, then Eq. 8.1.38 becomes for $J_{Ek} = 0$

$$\frac{\partial}{\partial t} \bar{\alpha}_k \bar{\rho}_k \bar{h}_k + \nabla \cdot \bar{\alpha}_k \bar{\rho}_k \bar{\underline{U}}_k \bar{h}_k = \bar{\alpha}_k \frac{d \bar{P}_k}{d\, t_k^{zd}} - \nabla \cdot \bar{\alpha}_k \left(\underline{J}_{qk} + \underline{h}_k^T\right)$$

$$+ \phi_k^s + {}^t\langle \dot{Q}_k\rangle^s + {}^t\langle \Gamma_k\rangle^s \bar{h}_k + (EPYTI)_k^s, \qquad (8.1.48)$$

in which ϕ_k^s is the dissipation function in the bulk fluid. It is

$$\phi_k^s = \bar{\alpha}_k \bar{\underline{\underline{\tau}}}_k : \nabla, \bar{\underline{U}}_k \,. \qquad (8.1.49)$$

We recall that ${}^t\langle \dot{Q}_k\rangle$ is the time-averaged interfacial heat transfer rate per unit volume of the mixture and is defined by Eq. 6.7.16. Under the simplifying conditions used in this section, it becomes

$${}^t\langle \dot{Q}_k\rangle^s = - v^{-1} \int_{A_k} {}^t\langle \underline{J}_{qk}\rangle \cdot \underline{n}_k\, dA$$

$$= - \frac{A_k}{v} \frac{1}{A_k} \int_{A_k} {}^t\langle \underline{J}_{qk}\rangle \cdot \underline{n}_k\, dA$$

$$= \frac{A_k}{v} q_{ki} \tag{8.1.50}$$

where the scalar q_{ki} is the area-averaged temporal mean interfacial heat flux into phase k.

The interfacial mass generation rate ${}^t\langle\Gamma_k\rangle^s$ is given by Eq. 8.1.8 and

$$\left(\text{EPYTI}\right)_k^s = - v^{-1} \int_{A_k} \bar{\rho}_k \, {}^t\langle h_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_k \, dA \, , \tag{8.1.51}$$

which represents the interfacial enthalpy transfer due to turbulence.

- Interfacial Enthalpy Balance Equation

For a two-fluid system, Eq. 6.12.1b, when simplified for conditions consistent with those used in deriving Eq. 8.1.48, becomes

$$\sum_{k=1}^{2} \left[{}^t\langle \dot{Q}_k \rangle^s + {}^t\langle \Gamma_k \rangle^s \bar{h}_k - \bar{P}_k \frac{d \bar{\alpha}_k}{d t_k^{zd}} + \left(\text{EPYTI}\right)_k^s \right] = 0 \, . \tag{8.1.52}$$

The enthalpy equation recommended in Refs. 15 and 16 is

$$\frac{\partial}{\partial t} \alpha_k^I \bar{\bar{\rho}}_k \bar{h}_k + \nabla \cdot \alpha_k^I \bar{\bar{\rho}}_k \underline{\bar{U}}_k \bar{h}_k$$

$$= \alpha_k^I \frac{D_k}{Dt} \bar{\bar{P}}_k - \nabla \cdot \alpha_k \left(\underline{J}_{qk} + \underline{J}_{qk}^T \right) + \frac{q_{ki}''}{L_s} + \Gamma_k^I \bar{h}_{ki} + \phi_k \, , \tag{8.1.53}$$

in which $\underline{J}_{qk}^T$ is the turbulent heat flux, q_{ki}'' is the interfacial heat flux, and L_s^{-1} denotes interfacial area per unit volume.

The substantive derivative in Eq. 8.1.53 is defined by

$$\frac{D_k}{Dt} = \frac{\partial}{\partial t} + \underline{\bar{U}}_k \cdot \nabla \tag{8.1.54}$$

and hence is identical to $\dfrac{d}{d t_k^{zd}}$ defined by Eq. 8.1.39.

It is thus seen that Eq. 8.1.48 becomes identical to Eq. 8.1.53 if

a. $\bar{\rho}_k = \bar{\bar{\rho}}_k$, $\bar{P}_k = \bar{\bar{P}}_k$, ${}^t\langle \Gamma_k \rangle^s = \Gamma_k^I$, and $\phi_k^s = \phi_k$ (all of these are most likely true),

b. $\underline{h}_k^T = \underline{J}_{qk}^T$, i.e., the turbulent enthalpy flux is the same as the turbulent heat flux,

c. $\bar{\alpha}_k = \alpha_k^I$,

and

d. ${}^t\langle \Gamma_k \rangle^s (\bar{h}_{ki} - \bar{h}_k) - (\mathrm{EPYTI})_k^s = 0$.

The interfacial enthalpy balance equation given in Refs. 15 and 16 is

$$\sum_{k=1}^{2} \left(\frac{q''_{ki}}{L_s} + \Gamma_k^I \bar{h}_{ki} \right) = 0 , \tag{8.1.55}$$

which becomes identical to Eq. 8.1.52 when $\bar{P}_k \dfrac{d\,\bar{\alpha}_k}{d\,t_k^{zd}}$ is negligible.

8.2 Comparison with Conservation Equation used in the TRAC Computer Code

The field equations describing the two-phase, two-fluid flow used in the TRAC code were based on: (1) mixture mass equation, (2) vapor mass equation, (3) vapor equation of motion, (4) liquid equation of motion, (5) mixture energy equation, and (6) vapor energy equation. Since the energy equation is written in terms of internal energy, it is selected for comparison.

Using the simplifications introduced in the beginning of Sec. 8.1, including those specified by Eq. 8.1.6, Eq. 6.9.19a reduces to:

$$\frac{\partial}{\partial t} \bar{\alpha}_k \bar{\rho}_k \bar{u}_k + \nabla \cdot \bar{\alpha}_k \bar{\rho}_k \underline{\bar{U}}_k \bar{u}_k$$

$$= - \bar{\alpha}_k \bar{P}_k \nabla \cdot \underline{\bar{U}}_k - \nabla \cdot \alpha_k \left(\underline{\bar{J}}_{qk} + \underline{u}_k^T \right) + \bar{\alpha}_k \bar{J}_{Ek}$$

$$+ {}^t\langle \dot{Q}_k \rangle^s + \phi_k^s + {}^t\langle \Gamma_k \rangle^s \bar{u}_k + (\mathrm{IETI})_k^s , \tag{8.2.1}$$

in which all contributions due to pressure-velocity correlations and the interfacial viscous dissipation are neglected. The turbulent internal energy flux $\underline{u}_k^T$ is given by

$$\underline{u}_k^T = \bar{\rho}_k \, {}^t\langle \underline{U}_k' u_k' \rangle \tag{8.2.2}$$

and

$$(\mathrm{IETI})_k^s = - v^{-1} \int_{A_k} \bar{\rho}_k \, {}^t\langle u_k' (\underline{U}_k' - \underline{W}_k') \rangle \cdot \underline{n}_k \, dA \, . \tag{8.2.3}$$

The similarity between Eqs. 8.2.2 and 8.1.44 and that between Eqs. 8.2.3 and 8.1.51 are readily seen. The dissipation function ϕ_k^s is defined by Eq. 8.1.49 and the interfacial mass generation rate per unit volume ${}^t\langle \Gamma_k \rangle^s$ is defined by Eq. 8.1.8. The corresponding interfacial internal energy balance equation is

$$\sum_{k=1}^{2} \left({}^t\langle \dot{Q}_k \rangle^s + {}^t\langle \Gamma_k \rangle^s \, \bar{u}_k + (\mathrm{IETI})_k^s \right) = 0 \, . \tag{8.2.4}$$

Since

$$ {}^t\langle \Gamma_k \rangle^s = \bar{\rho}_k \left(\frac{\partial \bar{\alpha}_k}{\partial t} + \underline{\bar{U}}_k \cdot \nabla \bar{\alpha}_k \right) ,$$

$$ {}^t\langle \Gamma_k \rangle^s \, \bar{u}_k = {}^t\langle \Gamma_k \rangle^s \left(\bar{h}_k - \frac{\bar{P}_k}{\bar{\rho}_k} \right)$$

$$= {}^t\langle \Gamma_k \rangle^s \, \bar{h}_k - \bar{P}_k \left(\frac{\partial \bar{\alpha}_k}{\partial t} + \underline{\bar{U}}_k \cdot \nabla \bar{\alpha}_k \right). \tag{8.2.5}$$

Substituting Eq. 8.2.5 into Eq. 8.2.1, followed by combining and rearranging terms, yields

$$\frac{\partial}{\partial t} \bar{\alpha}_k \bar{\rho}_k \bar{u}_k + \nabla \cdot \bar{\alpha}_k \bar{\rho}_k \underline{\bar{U}}_k \bar{u}_k$$

$$= - \bar{P}_k \frac{\partial \bar{\alpha}_k}{\partial t} - \bar{P}_k \nabla \cdot \bar{\alpha}_k \underline{\bar{U}}_k - \nabla \cdot \bar{\alpha}_k \left(\underline{\bar{J}}_{qk} + u_k^T \right) + \phi_k^s$$

$$+ {}^t\langle \dot{Q}_k \rangle^s + {}^t\langle \Gamma_k \rangle^s \, \bar{h}_k + \bar{\alpha}_k \, \bar{J}_{Ek} + (\mathrm{IETY})_k^s \, . \tag{8.2.6}$$

The molecular and turbulent conduction fluxes can be expressed in terms of molecular and eddy diffusivities D_{uk} and D_{uk}^T. Thus,

$$- \nabla \cdot \bar{\alpha}_k \left(\underline{J}_{qk} + \underline{u}_k^T \right) = \nabla \cdot \bar{\alpha}_k \, \bar{\rho}_k \left(D_{uk} + D_{uk}^T \right) \nabla \, \bar{u}_k \, . \qquad (8.2.7)$$

The vapor internal energy equation used in the TRAC code (Eq. 65 in Ref. 17), written in the present notation, is (with subscript g replaced by k):

$$\frac{\partial}{\partial t} \bar{\alpha}_k \, \bar{\rho}_k \, \bar{u}_k + \nabla \cdot \bar{\alpha}_k \, \bar{\rho}_k \, \underline{\bar{U}}_k \, \bar{u}_k$$

$$= - \bar{P}_k \frac{\partial \, \bar{\alpha}_k}{\partial t} - \bar{P}_k \nabla \cdot \bar{\alpha}_k \, \underline{\bar{U}}_k + {}^t\langle \dot{Q}_k \rangle^s + {}^t\langle \Gamma_k \rangle^s \, \bar{h}_{ki} + \dot{Q}_{wk} \, , \qquad (8.2.8)$$

where $\dot{Q}_{wk}$ denotes the wall heat transfer rate per unit volume. Equation 8.2.6 becomes identical to Eq. 8.2.8 when

a. Molecular and turbulent conduction are absent,

b. Viscous dissipation is negligible,

c. $\bar{\alpha}_k \, \bar{J}_{Ek} = \dot{Q}_{wk}$, and

d. ${}^t\langle \Gamma_k \rangle^s \left(\bar{h}_{ki} - \bar{h}_k \right) - \left(\mathrm{IETI} \right)_k^s = 0$.

In boiling reactor applications, the first two conditions are probably quite reasonable. The third is merely a statement that the wall heat transfer (such as that from fuel rods) is treated as a distributed heat source. The last is analogous to that of Condition d listed following Eq. 8.1.54 of Sec. 8.1.

We have demonstrated in this section that when the local averaging volume is made small and when several additional simplifications are introduced, the set of rigorously derived conservation equations prescribed in Sec. 6 reduces to various forms that compare reasonably well with those given in Refs. 13, 15, 16, and 17. Perhaps a more important finding from the comparison stems not so much from their agreement, but rather revealing the missing terms in the currently "accepted" two-phase flow formulations. It would be a useful and logical step forward to assess the importance of these missing terms under a range of conditions encountered in practical applications.

9.0 DISCUSSION AND CONCLUDING REMARKS

This report presents the basic time-volume-averaged conservation equations for multiphase flow in systems without internal solid structures. The starting point of the analysis is the well established phasic conservation equations of mass, momentum, and energy, and their interfacial balance relations. Within the framework of generalized multiphase mechanics first suggested by Soo[8], particles of different ranges of sizes, densities, and shapes are treated as different dynamic phases.

The local volume averaging is performed first, followed by time averaging. In this way, the identity of the dynamic phases is preserved.

Delhaye and Achard[14] suggested that interchanging of the order of volume and time averaging would lead to identical results. The fallacy of this conclusion was the result of improper application of the Leibnitz rule for the differentiation of an integral as has shown in Sec. 7 of this report.

The local volume-averaged conservation equations were deduced from the phasic equations by using the theorems of local volume averaging developed by Slattery, Whitaker, and others [3,5,6,7]. Time-averaging was subsequently applied to these equations. The analysis is rigorous, subject only to the restrictions on (1) characteristic length scales of the system as prescribed by the inequalities defined in Eq. 3.4.3, and (2) characteristic time scale inequalities prescribed by Eq. 6.1.4. Because of the length scale restrictions, the resulting equations are strictly valid for highly dispersed systems. When these equations are applied to systems that are not highly dispersed, the extent and nature of errors involved remain to be a subject of further research.

The analysis of multiphase flow calls for the solution of the time-volume averaged differential-integral equations of conservation with appropriate initial and boundary conditions. An examination of these equations reveals immediately that they are incomplete in that constitutive relations for the diffusive, dispersive, turbulent, and interfacial transport need to be developed. Collectively, this constitutes the closure problem. It should be noted that the integrand of the interfacial integrals consists of the local values of the dependent variables. Equivalently, it contains the deviation of the local value of the variable from its intrinsic local volume average and, in the presence of high-frequency fluctuations, its turbulent component. The closure problem is not unlike that in the analysis of turbulent flow, but with additional complications. In the absence of turbulence, a closure scheme for the determination of the spatial deviation of the dependent variable for systems involving diffusion and first-order chemical reaction has been given by Crapiste, Rotstein, and Whitaker [18]. A rigorous approach to treat the general closure problem including convective transport and turbulence will, no doubt, remains a challenge.

If the flow and thermodynamic conditions are such that the spatial deviations of the dependent variables, denoted by $(\tilde{\ })$, are small and may be deleted, and if, in addition, that $\alpha_k' \sim 0$ and all time correlations involving ρ_k' and P_k' are negligible, then the resulting set of simplified equations reduces to a form closely resembling Ishii's time-averaged equations although some differences remain. These differences are not unexpected since Ishii's local time fraction is not identical to the local volume fraction. Strictly speaking, his mass-weighted mean densities, velocities, etc. are not the same as the low-frequency component of their intrinsic volume averages, even when the averaging volume is small. The internal energy equation used in the development of the TRAC code has also been demonstrated to be in reasonably close agreement with the more complete internal energy equation obtained from the present study and simplified for applications to nuclear reactor systems.

At the present time, the evaluation of the interface transfer integrals in the time-volume-averaged conservation equations is not generally known. An order-of-magnitude analysis to assess the relative importance of these interface transfer integrals would be helpful. One of the fundamental problems in understanding multiphase flow is the lack of knowledge of mass, momentum, and

energy transfer at the interface. In the past, empirical correlations were developed from experimental data to quantify interfacial mass, momentum, and energy transfer rates, often without sound theoretical basis. These correlations therefore are valid only in the range of operating conditions for which the experimental data are obtained. Other urgently needed information is the quantification of transport properties such as eddy and dispersive diffusivities of mass, Reynolds and dispersive stresses, and eddy and dispersive conductivities of heat, etc., by performing planned experiments in conjunction with analysis.

In summary, a set of rigorously derived conservation equations of mass, momentum, and energy for multiphase systems without internal solid structures via time-volume averaging has been presented. Similar derivation will be presented in Part Two for multiphase systems with stationary internal solid structures. These equations are in differential-integral form and are not a set of partial differential equations as currently "appear" in most literature on multiphase flow. This set of conservation equations serves as a reference point for modeling multiphase flow with simplified approximations and provides theoretical guidance and physical insight that may be useful to develop correlations for quantifying interfacial mass, momentum, and energy transfer between phases.

Finally, it is important to note that, for the conservation equations presented in this report, the local averaging volume is unrelated to the volume of a computational cell used in the numerical computations.

ACKNOWLEDGEMENTS

We are indebted to Drs. R. T. Curtis, C. N. Kelber, and Mr. P. M. Wood of the U.S. Nuclear Regulatory Commission, for without their support, this work would not have been possible. The painstaking care which Mrs. S. Moll exercised in the typing of this manuscript is also sincerely appreciated.

The work is performed under the auspices of the Division of Accident Evaluation, Office of Nuclear Regulatory Research, U. S. Nuclear Regulatory Commission under Contract A2045.

REFERENCES

1. W. T. Sha, B. T. Chao, and S. L. Soo, "Local Volume-Averaged Transport Equations for Multiphase Flow in Regions Containing Distributed Solid Structures," NUREG/CR-2354, ANL-81-69 (December 1981).

2. W. T. Sha, B. T. Chao, and S. L. Soo, "Time Averaging of Local Volume-Averaged Conservation Equations of Multiphase Flow," NUREG/CR-3434, ANL-83-49 (July 1983).

3. S. Whitaker, "Advances in Theory of Fluid Motion in Porous Media," Ind. Eng. Chem. 61:14-28 (1969).

4. S. Whitaker, "Diffusion and Dispersion in Porous Media," AIChE Journal 13:420-427 (1967).

5. J. C. Slattery, "Flow of Viscoelastic Fluids through Porous Media," AIChE Journal 13:1066-1071 (1967).

6. T. B. Anderson and R. Jackson, "A Fluid Mechanical Description of Fluidized Beds," I&EC Fundamentals 6:527-539 (1967).

7. W. G. Gray and P. C. Y. Lee, "On the Theorems for Local Volume Averaging of Multiphase Systems," Int. J. Multiphase Flow 3:333-340 (1977).

8. S. L. Soo, "Dynamics of Multiphase Flow," I&EC Fundamentals 4:425-433 (1965).

9. W. G. Gray, "A Derivation of the Equations for Multiphase Transport," Chem. Eng. Sci., 30:229-233 (1975).

10. J. P. Hinz, "Turbulence," McGraw Hill Book Co. (1978).

11. S. Winnikow and B. T. Chao, "Droplet Motion in Purified Systems," Physics of Fluids, 9:50-61 (1966).

12. L. W. Florschuetz and B. T. Chao, "On the Mechanics of Vapor Bubble Collapse," Journal of Heat Transfer, Trans. ASME, 87C:209-220 (1965).

13. M. Ishii, "Thermo-Fluid Dynamic Theory of Two-Phase Flow," Eyrolles, Paris (1975).

14. J. M. Delhaye and J. L. Achard, "On the Averaging Opeators Introduced in Two-Phase Flow Modeling," Proc. OF OECD Specialists' Meeting on Transient Two-Phase Flow, Toronto, Pergamon Press (1976).

15. M. Ishii and K. Mishima, "Study of Two-Fluid Model and Interfacial Area," NUREG/CR-1873, ANL-80-111 (December 1980).

16. M. Ishii and G. Kocamustafaogullari, "Two-Phase Flow Models and Their Limitations," in Advances in Two-Phase Flow and Heat Transfer, Vol. 1, pp. 1-14, Martinus Nijhoff Publishers, Series E: Applied Sciences, No. 63 (1983).

17. D. R. Liles, et al., "TRAC-P1A, An Advanced Best-Estimate Computer Program for PWR LOCA Analysis," NUREG/CR-0665, LA-7777-MS (1979).

18. Personal communication from S. Whitaker to B. T. Chao (April 24, 1984).

APPENDIX A. PHYSICAL INTERPRETATION OF $\nabla \alpha_k = - v^{-1} \int_{A_k} n_k \, dA$. (A.1)

To provide a physical interpretation of Eq. A.1, which is Eq. 3.4.9 with $\gamma_v = 1$, we consider a dispersed system and an averaging volume in the shape of a rectangular parallelopiped $\Delta x \Delta y \Delta z$ with its centroid located at (x,y,z), as illustrated in Fig. A-1a. Its top view is shown Fig. A-1b.

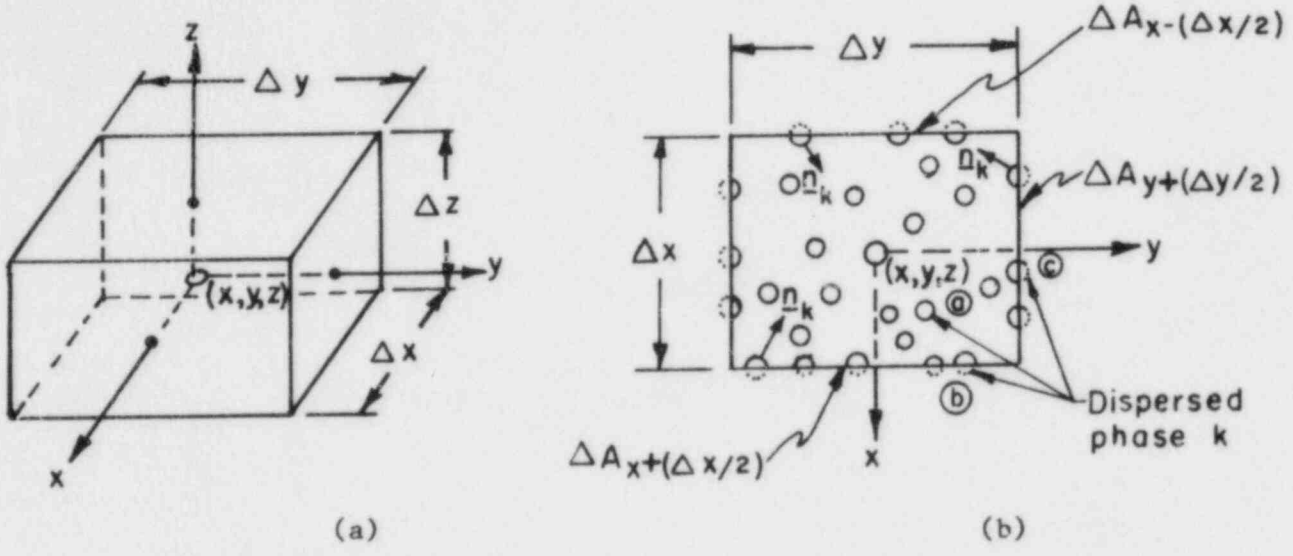


Fig. A-1. Physical Interpretation of Eq. 3.4.9 for $\gamma_v = 1$

Clearly, for those elements of the dispersed phase k that are completely inside the averaging volume,

$$\int_{\delta A_k} \underline{n}_k \, dA_k = 0 \, , \tag{A.2}$$

where δA_k is the closed surface of the element. Such an element, labeled ⓐ in Fig. A-1b, may be a bubble or a droplet, spherical or nonspherical. Next, we consider those elements of the dispersed phase that are intersected by the boundary surface $\Delta A_{x+(\Delta x/2)}$. One such element is labeled ⓑ in Fig. A-1b. The unit outdrawn normal vector $\underline{n}_k$ can be represented by

$$\underline{n}_k = \underline{i} \, e_1 + \underline{j} \, e_2 + \underline{k} \, e_3 \, , \tag{A.3}$$

where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are unit vectors pointing in the positive directions of x-, y-, and z-axis, respectively, and e_1, e_2, and e_3 are the direction cosines of $\underline{n}_k$. If we denote the portion of the interfacial area of element ⓑ that is inside the averaging volume v by $\delta A_{k,[x+(\Delta x/2)]}$, and its area of intersection with the surface $\Delta A_{x+(\Delta x/2)}$ by $\delta A_{k,x+(\Delta x/2)}$, then

$$\int_{\delta A_{k,[x+(\Delta x/2)]}} e_1 \, dA = \delta A_{k,x+(\Delta x/2)} \tag{A.3a}$$

and

$$\int_{\delta A_{k,[x+(\Delta x/2)]}} e_2 \, dA = \int_{\delta A_{k,[x+(\Delta x/2)]}} e_3 \, dA = 0 \, . \qquad \text{(A.3b,c)}$$

Likewise, for an element of phase k that is interesected by the surface $\Delta A_{x-(\Delta x/2)}$, we have

$$\int_{\delta A_{k,[x-(\Delta x/2)]}} e_1 \, dA = \delta A_{k,x-(\Delta x/2)} \qquad \text{(A.4a)}$$

and

$$\int_{\delta A_{k,[x-(\Delta x/2)]}} e_2 \, dA = \int_{\delta A_{k,[x-(\Delta x/2)]}} e_3 \, dA = 0 \, . \qquad \text{(A.4b,c)}$$

Following the same procedure, we have for an element of phase k that is intersected by the surface $\Delta A_{y+(\Delta y/2)}$ (labeled ⓒ in Fig. A-1b)

$$\int_{\delta A_{k,[y+(\Delta y/2)]}} e_2 \, dA = \delta A_{k,y+(\Delta y/2)} \qquad \text{(A.5a)}$$

and

$$\int_{\delta A_{k,[y+(\Delta y/2)]}} e_1 \, dA = \int_{\delta A_{k,[y+(\Delta y/2)]}} e_3 \, dA = 0 \, , \qquad \text{(A.5b,c)}$$

where $\delta A_{k,[y+(\Delta y/2)]}$ denotes the portion of the interfacial area of the element ⓒ that is inside v, and its intersection by the surface $\Delta A_{y+(\Delta y/2)}$ is $\delta A_{k,y+(\Delta y/2)}$. Similar expressions can be written for elements of phase k that are intersected by the bounding surface $\Delta A_{y-(\Delta y/2)}$, $\Delta A_{z+(\Delta z/2)}$, and $\Delta A_{z-(\Delta z/2)}$.

The x-component of the integral on the right-hand side of Eq. A.1 is

$$\left(- v^{-1} \int_{A_k} \underline{n}_k \, dA \right)_x = - \frac{1}{\Delta x \Delta y \Delta z} \left(- \sum \delta A_{k,x+(\Delta x/2)} + \sum \delta A_{k,x-(\Delta x/2)} \right) , \qquad \text{(A.6)}$$

where the summation is taken for all elements of phase k cut through by the bounding surfaces $\Delta A_{x+(\Delta x/2)}$ and $\Delta A_{x-(\Delta x/2)}$. Using the relationship given by Eq. 3.3.11a, one has

$$\frac{\sum \delta A_{k,x+(\Delta x/2)}}{\Delta y \Delta z} = \alpha_{k,x+(\Delta x/2)}$$

and

$$\frac{\sum \delta A_{k,x-(\Delta x/2)}}{\Delta y \Delta z} = \alpha_{k,x-(\Delta x/2)} \cdot$$

Thus,

$$\left(- v^{-1} \int_{A_k} \underline{n}_k \, dA \right) = \frac{\Delta \alpha_{k,x}}{\Delta x} , \tag{A.7}$$

where $\Delta \alpha_{k,x} = \alpha_{k,x+(\Delta x/2)} - \alpha_{k,x-(\Delta x/2)}$. Similar expressions can be written for the y- and z-component of the indicated integral. As has been pointed out in Sec. 3, for highly dispersed systems $\Delta \alpha_{k,x} = \Delta \alpha_{k,y} = \Delta \alpha_{k,z} = \Delta \alpha_k$. It follows, then, that

$$- v^{-1} \int_{A_k} \underline{n}_k \, dA = \underline{i} \frac{\Delta \alpha_k}{\Delta x} + \underline{j} \frac{\Delta \alpha_k}{\Delta y} + \underline{k} \frac{\Delta \alpha_k}{\Delta z} , \tag{A.8a}$$

for which we can write

$$\nabla \alpha_k = - v^{-1} \int_{A_k} \underline{n}_k \, dA \tag{A.8b}$$

in view of the length scale restrictions of Eq. 3.4.3.

APPENDIX B. EVALUATION OF $^{t}\left\langle \alpha_k \, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle \right\rangle$ FOR NON-NEWTONIAN FLUIDS

For a Newtonian fluid, the stress and strain rate of a fluid phase k are linearly related and are expressible as

$$\underline{\underline{\tau}}_k = \left(\lambda_k - \frac{2}{3}\mu_k\right) \nabla \cdot \underline{U}_k \, \underline{\underline{I}} + \mu_k \left[\nabla, \underline{U}_k + (\nabla, \underline{U}_k)_c\right], \tag{B.1}$$

in which all quantities have been defined previously. When the viscosity coefficients λ_k and μ_k are dependent on the strain rate, they are decomposed in accordance with Eq. 6.1.8, i.e.,

$$\lambda_k = {}^{3i}\langle \lambda_k \rangle_{LF} + \tilde{\lambda}_{kLF} + \lambda'_k \tag{B.2}$$

$$\mu_k = {}^{3i}\langle \mu_k \rangle_{LF} + \tilde{\mu}_{kLF} + \mu'_k \, . \tag{B.3}$$

It is straightforward to demonstrate that

$$\begin{aligned}
{}^{3i}\langle \underline{\underline{\tau}}_k \rangle = {} & {}^{3i}\left\langle \lambda_k - \frac{2}{3}\mu_k \right\rangle_{LF} \left(\nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF}\right) \underline{\underline{I}} + {}^{3i}\left\langle \lambda_k - \frac{2}{3}\mu_k \right\rangle_{LF} {}^{3i}\langle \nabla \cdot \tilde{\underline{U}}_{kLF} \rangle \, \underline{\underline{I}} \\
& + {}^{3i}\left\langle \lambda_k - \frac{2}{3}\mu_k \right\rangle_{LF} {}^{3i}\langle \nabla \cdot \underline{U}'_k \rangle \, \underline{\underline{I}} \\
& + {}^{3i}\left\langle \left(\tilde{\lambda}_{kLF} - \frac{2}{3}\tilde{\mu}_k\right) \nabla \cdot \tilde{\underline{U}}_{kLF} \right\rangle \underline{\underline{I}} + {}^{3i}\left\langle \left(\tilde{\lambda}_k - \frac{2}{3}\tilde{\mu}_k\right) \nabla \cdot \underline{U}'_k \right\rangle \underline{\underline{I}} \\
& + {}^{3i}\left\langle \lambda'_k - \frac{2}{3}\mu'_k \right\rangle \left(\nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF}\right) \underline{\underline{I}} + {}^{3i}\left\langle \left(\lambda'_k - \frac{2}{3}\mu'_k\right) \nabla \cdot \tilde{\underline{U}}_{kLF} \right\rangle \underline{\underline{I}} \\
& + {}^{3i}\left\langle \left(\lambda'_k - \frac{2}{3}\mu'_k\right) \nabla \cdot \underline{U}'_k \right\rangle \underline{\underline{I}} \\
& + {}^{3i}\langle \mu_k \rangle_{LF} \left[\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \left(\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF}\right)_c\right] \\
& + {}^{3i}\langle \mu_k \rangle_{LF} \, {}^{3i}\langle \nabla, \tilde{\underline{U}}_{kLF} + (\nabla, \tilde{\underline{U}}_{kLF})_c \rangle \\
& + {}^{3i}\langle \mu_k \rangle_{LF} \, {}^{3i}\langle \nabla, \underline{U}'_k + (\nabla, \underline{U}'_k)_c \rangle
\end{aligned}$$

$$+ {}^{3i}\langle\tilde{\mu}_{kLF}\,[\nabla,\underline{\tilde{U}}_{kLF} + (\nabla,\underline{\tilde{U}}_{kLF})_c]\rangle + {}^{3i}\langle\tilde{\mu}_{kLF}\,[\nabla,\underline{U}'_k + (\nabla,\underline{U}'_k)_c]\rangle$$

$$+ {}^{3i}\langle\mu'_k\rangle\left[\nabla,{}^{3i}\langle\underline{U}_k\rangle_{LF} + \left(\nabla,{}^{3i}\langle\underline{U}_k\rangle_{LF}\right)_c\right]$$

$$+ {}^{3i}\langle\mu'_k[\nabla,\underline{\tilde{U}}_{kLF} + (\nabla,\underline{\tilde{U}}_{kLF})_c]\rangle$$

$$+ {}^{3i}\langle\mu'_k[\nabla,U'_k + (\nabla,\underline{U}'_{kL})_c]\rangle\,, \qquad \text{(B.4)}$$

for which the relations given by Eq. 6.2.21b and c have been used. Subsequent time averaging gives

$$\begin{aligned}
{}^{t}\left\langle\alpha_k\,{}^{3i}\langle\underline{\tau}_k\rangle\right\rangle &= \alpha_{kLF}\,{}^{3i}\left\langle\lambda_k - \tfrac{2}{3}\mu_k\right\rangle_{LF}\left(\nabla\cdot{}^{3i}\langle\underline{U}_k\rangle_{LF}\right)\underline{\underline{I}} \\
&+ \alpha_{kLF}\,{}^{3i}\langle\mu_k\rangle_{LF}\left[\nabla,{}^{3i}\langle\underline{U}_k\rangle_{LF} + \left(\nabla,{}^{3i}\langle\underline{U}_k\rangle_{LF}\right)_c\right] \\
&+ \alpha_{kLF}\,{}^{3i}\left\langle\lambda_k - \tfrac{2}{3}\mu_k\right\rangle_{LF}\,{}^{3i}\langle\nabla\cdot\underline{\tilde{U}}_k\rangle_{LF}\,\underline{\underline{I}} \\
&+ \alpha_{kLF}\,{}^{3i}\langle\mu_k\rangle_{LF}\,{}^{3i}\langle\nabla,\underline{\tilde{U}}_{kLF} + (\nabla,\underline{\tilde{U}}_{kLF})_c\rangle \\
&+ \alpha_{kLF}\,{}^{3i}\left\langle\left(\tilde{\lambda}_{kLF} - \tfrac{2}{3}\tilde{\mu}_{kLF}\right)\nabla\cdot\underline{\tilde{U}}_{kLF}\right\rangle\underline{\underline{I}} \\
&+ \alpha_{kLF}\,{}^{3i}\langle\tilde{\mu}_{kLF}\,[\nabla,\underline{\tilde{U}}_{kLF} + (\nabla,\underline{\tilde{U}}_{kLF})_c]\rangle \\
&+ \alpha_{kLF}\,{}^{t3i}\left\langle\left(\lambda'_k - \tfrac{2}{3}\mu'_k\right)\nabla\cdot\underline{U}'_k\right\rangle\underline{\underline{I}} \\
&+ \alpha_{kLF}\,{}^{t3i}\langle\mu'_k\,[\nabla,\underline{U}'_k + (\nabla,\underline{U}'_k)_c]\rangle \\
&+ {}^{3i}\left\langle\lambda_k - \tfrac{2}{3}\mu_k\right\rangle_{LF}\,{}^{t3i}\langle\alpha'_k\,\nabla\cdot\underline{U}'_k\rangle\,\underline{\underline{I}} \\
&+ {}^{3i}\langle\mu_k\rangle_{LF}\,{}^{t3i}\langle\alpha'_k\,[\nabla,\underline{U}'_k + (\nabla,\underline{U}'_k)_c]\rangle \\
&+ {}^{t3i}\left\langle\alpha'_k\left(\lambda'_k - \tfrac{2}{3}\mu'_k\right)\right\rangle\left(\nabla\cdot{}^{3i}\langle\underline{U}_k\rangle_{LF}\right)\underline{\underline{I}} \\
&+ {}^{t3i}\langle\alpha'_k\,\mu'_k\rangle\left[\nabla,{}^{3i}\langle\underline{U}_k\rangle_{LF} + \left(\nabla,{}^{3i}\langle\underline{U}_k\rangle_{LF}\right)_c\right]
\end{aligned}$$

$$+ \left\langle \alpha_k' \, {}^{3i}\left\langle \left(\lambda_k' - \frac{2}{3}\mu_k'\right) \nabla \cdot \underline{\tilde{U}}_{kLF} \right\rangle \right\rangle \underline{\underline{I}}$$

$$+ \left\langle \alpha_k' \, {}^{3i}\langle \mu_k' \left[\nabla, \underline{\tilde{U}}_{kLF} + (\nabla, \underline{\tilde{U}}_{kLF})_c\right] \rangle \right\rangle$$

$$+ \left\langle \alpha_k' \, {}^{3i}\left\langle \left(\tilde{\lambda}_{kLF} - \frac{2}{3}\tilde{\mu}_{kLF}\right) \nabla \cdot \underline{U}_k' \right\rangle \right\rangle \underline{\underline{I}}$$

$$+ \left\langle \alpha_k' \, {}^{3i}\langle \tilde{\mu}_{kLF} \left[\nabla, \underline{U}_k' + (\nabla, \underline{U}_k')_c\right] \rangle \right\rangle$$

$$+ \text{terms involving triple time correlations.} \qquad \text{(B.5)}$$

In deriving Eq. B.5, the relation ${}^{t3i}\langle \nabla \cdot \underline{U}_k' \rangle = 0$ has been used.

When λ_k and μ_k are independent of velocity gradients, ${}^{3i}\langle \lambda_k \rangle_{LF} = \lambda_k$, ${}^{3i}\langle \mu_k \rangle_{LF} = \mu_k$, $\tilde{\lambda}_{kLF} = \lambda_k' = 0$, and $\tilde{\mu}_{kLF} = \mu_k' = 0$. In addition, ${}^{3i}\langle \nabla \cdot \underline{\tilde{U}}_{kLF} \rangle = 0$, ${}^{3i}\langle \nabla, \underline{\tilde{U}}_{kLF} \rangle = 0$, and ${}^{3i}\langle (\nabla, \underline{\tilde{U}}_{kLF})_c \rangle = 0$. Consequently, for Newtonian fluids, Eq. B.5 simplifies to

$$\left\langle \alpha_k \, {}^{3i}\langle \underline{\underline{\tau}}_k \rangle \right\rangle = \alpha_{kLF} \left\{ \left(\lambda_k - \frac{2}{3}\mu_k\right) \left(\nabla \cdot {}^{3i}\langle \underline{U}_k \rangle_{LF}\right) \underline{\underline{I}} \right.$$

$$\left. + \mu_k \left[\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF} + \left(\nabla, {}^{3i}\langle \underline{U}_k \rangle_{LF}\right)_c\right] \right\}$$

$$+ \left(\lambda_k - \frac{2}{3}\mu_k\right) {}^{t3i}\langle \alpha_k' \nabla \cdot \underline{U}_k' \rangle \underline{\underline{I}}$$

$$+ \mu_k \, {}^{t3i}\langle \alpha_k' \left[\nabla, \underline{U}_k' + (\nabla, \underline{U}_k')_c\right] \rangle \, , \qquad \text{(B.6)}$$

which is precisely the result given in Eqs. 6.5.7, 6.5.8d, and 6.5.8g.

APPENDIX C. EVALUATION OF $^{t}\left\langle \alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle \right\rangle$ FOR ISOTROPIC CONDUCTION WITH VARIABLE CONDUCTIVITY

The Fourier law of isotropic conduction for fluid phase k is

$$\underline{J}_{qk} = - \kappa_k \nabla T_k \, , \tag{C.1}$$

which is valid for variable conductivity, since $du_k = c_{vk} dT_k$, Eq. C.1 can be written in a form relating the heat flux vector and the gradient of internal energy. Thus,

$$\underline{J}_{qk} = - \beta_k \nabla u_k \, , \tag{C.2}$$

where $\beta_k = \kappa_k / c_{vk}$. When κ_k or c_{vk}, or both, vary with temperature, we write

$$\beta_k = {}^{3i}\langle \beta_k \rangle + \breve{\beta}_{kLF} + \beta_k' \, . \tag{C.3}$$

Accordingly,

$$\begin{aligned} {}^{3i}\langle \underline{J}_{qk} \rangle = & - {}^{3i}\langle \beta_k \rangle_{LF} \left(\nabla \, {}^{3i}\langle u_k \rangle_{LF} + {}^{3i}\langle \nabla \breve{u}_{kLF} \rangle + {}^{3i}\langle \nabla u_k' \rangle \right) \\ & - {}^{3i}\left\langle \breve{\beta}_{kLF} \left(\nabla \, {}^{3i}\langle u_k \rangle_{LF} + \nabla \breve{u}_{kLF} + \nabla u_k' \right) \right\rangle \\ & - {}^{3i}\left\langle \beta_k' \left(\nabla \, {}^{3i}\langle u_k \rangle_{LF} + \nabla \breve{u}_{kLF} + \nabla u_k' \right) \right\rangle . \end{aligned} \tag{C.4}$$

In deriving Eq. C.4, the relation $^{3i}\left\langle \nabla \, {}^{3i}\langle u_k \rangle_{LF} \right\rangle = \nabla \, {}^{3i}\langle u_k \rangle_{LF}$ has been used. Multiplying Eq. C.4 by $(\alpha_{kLF} + \alpha_k')$, followed by time averaging, leads to

$$\begin{aligned} {}^{t}\left\langle \alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle \right\rangle = & - \alpha_{kLF} \, {}^{3i}\langle \beta_k \rangle_{LF} \left(\nabla \, {}^{3i}\langle u_k \rangle_{LF} + {}^{3i}\langle \nabla \breve{u}_{kLF} \rangle \right) \\ & - \alpha_{kLF} \, {}^{3i}\langle \breve{\beta}_{kLF} \nabla \breve{u}_{kLF} \rangle - \alpha_{kLF} \, {}^{t3i}\langle \beta_k' \nabla u_k' \rangle \\ & - {}^{3i}\langle \beta_k \rangle_{LF} \, {}^{t3i}\langle \alpha_k' \nabla u_k' \rangle - {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \breve{\beta}_{kLF} \nabla u_k' \rangle \right\rangle \\ & - {}^{t3i}\langle \alpha_k' \beta_k' \rangle \nabla \, {}^{3i}\langle u_k \rangle_{LF} - {}^{t}\left\langle \alpha_k' \, {}^{3i}\langle \beta_k' \nabla \breve{u}_{kLF} \rangle \right\rangle . \end{aligned} \tag{C.5}$$

When β_k is a constant, $^{3i}\langle \beta_k \rangle_{LF} = \beta_k$, and $\breve{\beta}_{kLF} = \beta_k' = 0$. In addition, $^{3i}\langle \nabla \breve{u}_{kLF} \rangle = 0$. Consequently, Eq. C.5 simplifies to

$$^{t}\left\langle \alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle \right\rangle = - \alpha_{kLF} (\kappa_k / c_{vk}) \nabla \, {}^{3i}\langle u_k \rangle_{LF} - (\kappa_k / c_{vk}) \, {}^{t3i}\langle \alpha_k' \nabla u_k' \rangle \, , \tag{C.6}$$

which is precisely the result given by Eqs. 6.7.12, 6.7.13d, and 6.7.13e.

Distribution for NUREG/CR-3989 (ANL-84-66)

Internal:

C. E. Till
R. S. Zeno
Y. S. Cha
B. C. Chen
F. F. Chen
H. N. Chi
T. H. Chien
H. M. Domanus
E. M. Gelbard
M. Ishii
K. E. Kasza
G. K. Leaf
R. W. Lyczkowski
P. C. Messina
N. M. O'Fallon
W. T. Sha (55)
V. L. Shah
R. M. Singer
R. P. Stein
C. I. Yang
S. K. Zussman
ANL Patent Dept.
ANL Contract File
ANL Libraries (2)
TIS Files (6)

External:

NRC, for distribution per R7 (250)
P. M. Wood, Div. of Accident Evaluation, NRR, NRC, Washington
DOE-TIC (2)
Manager, Chicago Operations Office, DOE
Components Technology Division Review Committee:
P. Alexander, Flopetrol Johnston Schlumberger, Houston, Texas 77236-6369
D. J. Anthony, General Electric Company, 175 Curtner Ave., San Jose, Calif. 95125
A. A. Bishop, U. Pittsburgh, 1249 Benedum Hall, Pittsburgh, Pa. 15261
B. A. Boley, Northwestern U., Evanston, Illinois 60201
F. W. Buckman, Delian Corp., 296 Duff Rd., Monroeville, Pa. 15146
R. Cohen, Purdue U., West Lafayette, Ind. 47907
J. Weisman, U. Cincinnati, Cincinnati, Ohio 45221
S. Banerjee, U. of California, Santa Barbara, California 93106
M. Bottoni, KfK-IRE, Postfach 3640, D-7500 Karlsruhe 1, West Germany
B. T. Chao, U. of Illinois, Urbana, Illinois 61801
J. S. Coleman, DOE/BES, Washington
R. T. Curtis, Division of Accident Evaluation, NRR, NRC, Washington
J. M. Delhaye, Centre D'etudes Nucleaires, 38041 Grenoble Cedex, France
D. A. Drew, Rensselaer Polytechnic Institute, Troy, New York 12181
L. T. Fan, Kansas State U., Manhattan, Kansas 66506
D. M. France, U. of Illinois, Chicago, Illinois 60680
D. Gidaspow, IIT, Chicago, Illinois 60616
W. G. Gray, Princeton U., Princeton, New Jersey 08550
P. Griffith, MIT, Cambridge, Massachusetts 02178
J. T. Han, Div. of Accident Evaluation, NRR, NRC, Washington
J. J. Hartnett, U. of Illinois, Chicago, Illinois 60680
G. Hewitt, AERE Harwell, Oxfordshire OX11 ORA, England
C. W. Hirt, Flow Science, Inc., P. O. Box 933, Los Alamos, New Mexico 87544
Y. Y. Hsu, U. of Maryland, College Park, Maryland 20742
L. Kadanoff, U. of Chicago, Chicago, Illinois 60637
M. S. Kazimi, MIT, Cambridge, Massachusetts 02139
C. N. Kelber, Division of Accident Evaluation, NRR, NRC, Washington
J. Kestin, Brown U., Providence Rhode Island 02912
J. H. Kim, Electric Power Research Institute, Post Office Box 10412, Palo Alto, California 94303
S. L. Koh, DOE/BES, Washington

R. T. Lahey, Rensselaer Polytechnic Institute, Troy, New York 12181
J. H. Mahaffy, LASL, Los Alamos, New Mexico 87545
O. Manley, DOE/BES, Washington
W. Minkowycz, U. of Illinois, Chicago, Illinois 60680
M. C. Roco, U. of Kentucky, Lexington, Kentucky 40506
W. M. Rohsenow, MIT, Cambridge, Massachusetts 02139
V. E. Schrock, U. of California, Berkeley, California 94720
L. M. Shotkin, Division of Accident Evaluation, NRR, NRC, Washington
S. L. Soo, U. of Illinois, Urbana, Illinois 61801
D. B. Spalding, Imperial College, Exhibition Rd., London SW7 2BX, England
E. M. Sparrow, U. of Minnesota, Minneapolis, Minnesota 55455
H. B. Stewart, BNL, Upton, New York 11973
N. E. Todreas, MIT, Cambridge, Massachusetts 02139
J. R. Travis, LASL, Los Alamos, New Mexico 87545
G. Wallis, Dartmouth College, Hanover, New Hampshire 03755
S. Whitaker, U. of California, Davis, California 95616
N. Zuber, Division of Accident Evaluation, NRR, NRC, Washington

NRC FORM 335 (2.84) NRCM 1102, 3201, 3202

U.S. NUCLEAR REGULATORY COMMISSION

BIBLIOGRAPHIC DATA SHEET

SEE INSTRUCTIONS ON THE REVERSE

1. REPORT NUMBER (Assigned by TIDC add Vol. No., if any)

NUREG/CR-3989
ANL-84-66

2. TITLE AND SUBTITLE

TIME- AND VOLUME-AVERAGED CONSERVATION EQUATIONS FOR MULTIPHASE FLOW. PART ONE: SYSTEM WITHOUT INTERNAL SOLID STRUCTURES

3. LEAVE BLANK

4. DATE REPORT COMPLETED

MONTH: November YEAR: 1984

5. AUTHOR(S)

W. T. Sha, B. T. Chao, and S. L. Soo

6. DATE REPORT ISSUED

MONTH: December YEAR: 1984

7. PERFORMING ORGANIZATION NAME AND MAILING ADDRESS (Include Zip Code)

Argonne National Laboratory
9700 South Cass Avenue
Argonne, Illinois 60439

8. PROJECT/TASK/WORK UNIT NUMBER

9. FIN OR GRANT NUMBER

A2045

10. SPONSORING ORGANIZATION NAME AND MAILING ADDRESS (Include Zip Code)

Containment Research Systems Branch
Division of Accident Evaluation
Office of Nuclear Regulatory Research
U. S. Nuclear Regulatory Commission
Washington, D.C. 20555

11a. TYPE OF REPORT

Technical

b. PERIOD COVERED (Inclusive dates)

1980-1984

12. SUPPLEMENTARY NOTES

13. ABSTRACT (200 words or less)

A set of rigorously derived conservation equations of mass, momentum, and energy for multiphase systems without internal solid structures via time-volume averaging of point, instantaneous conservation equations is presented. These equations are differential-integral equations in which the area integrals account for the interfacial transport of mass, momentum, and energy. The equations from volume averaging contain averages of the product of the dependent variables which must be expressed in terms of the products of their averages. In nonturbulent flows, this is achieved by expressing the "point" variable as the sum of its intrinsic volume average and a spatial deviation. In turbulent flows for which further time-averaging is required, the "point" variable is then decomposed into a low-frequency component and a high-frequency component. Time averaging following volume averaging preserves the identity of the dynamic phases. Under certain simplifying conditions, the proposed set of rigorously derived conservation equations reduces closely to various forms that are currently "accepted" for two-phase flow analysis. This set of conservation equations serves as a reference point for modeling multiphase flow and provides theoretical guidance and physical insight that may be useful to develop correlations for quantifying interfacial transport of mass, momentum, and energy.

14. DOCUMENT ANALYSIS - a. KEYWORDS/DESCRIPTORS

multiphase flow system
local volume averaging
time averaging
interfacial transfer integrals
differential integral conservation equations

b. IDENTIFIERS/OPEN-ENDED TERMS

15. AVAILABILITY STATEMENT

Unlimited

16. SECURITY CLASSIFICATION

(This page)

(This report)

17. NUMBER OF PAGES

130

18. PRICE