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MEMORANDUM FOR: Walter R. Butler, Chief
Containment Systems Branch, DSI

FROM: Peter S. Kapo, DSI

SUBJECT: NUREG-0737, ANALYTICAL SOLUTIONS TO TWO PROBLEMS PERTINENT TO ITEMS II.F.1.4,5,6
(A) Statistical Treatment of Hysteresis and Deadband Errors
(B) Determination of the Time Constant of a First Order Transfer Component from Variation with Frequency of Sinusoidal Output Level.

To evaluate some of the licensees submittals on NUREG-0737 items, we require analytical solutions to the two subject problems indicated above. These problems are discussed in some detail in attachment A and B. Here we will briefly state the problems and their solution.

## (A) STATISTICAL TREATMENT OF HYSTERESIS AND DEADBAND ERRORS

For the Safety Evaluations of certain NUREG-0737 items which involve measurement systems displaying hysteresis and deadband effects, we require an algorithm that provides a conservative measure of system total error. A simple algorithm which serves this end is developed in Attachment A. We will here quote the algorithm and give a description of its interpretation.

The error analysis in Attachment A specifically treats three error sources - random biases, the hysteresis effect, and the deadband effect. Random errors are lumped together with random biases to form what are referred to as random biases. Hence there is no specific treatment of random errors.

Throughout Attachment A we use indices $I, J, K$, which have the following meaning:

$I$ is the system design type.

$J$ is the module specification, which for design $I$ runs from $1$ to $M$. $J$ indicates the module type, range, vendor, model number, etc.

$K$ is an index which sequentially labels individual items of any generic type runs from $1$ to $N$.

We indicate an estimate of a parent population parameter obtained by examining a sample from the parent population by a double bar, i.e., $\bar{\bar{S}}$. Also we indicate a conservative estimate of a parent population parameter with a double bar.

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For the measurement systems we are discussing here, it is necessary to have the system design completed and approved before the modules which make up the system can be procured. This being the case, the system measurement uncertainty must be computed from vendor specifications for the modules, some of which are necessarily probabilistic descriptions of the parameters of the generic module type. Thus the algorithm for system measurement uncertainty must give, at least in part, a probabilistic, rather than an absolute, indication of the system measurement error.

The random bias, $B(I,J,K)$, associated with a generic module type varies from module to module, and for the systems we are discussing we have assumed that this is the only part of the module parameter specification that must be treated statistically. It is desirable that the random biases be small, and it is here assumed that the manufacturing process has been adjusted so that the average value of the random bias of each generic module type is zero. For computing system uncertainty, for each module type in the system we need the standard deviation of random bias, which is given by

$$\bar{\bar{S}}^2(I,J,B) = \sum_{K=1}^{N} B^2(I,J,K)/N$$

In practice the vendor may quote either the standard deviation of biases or the upper limit of biases, $\bar{\bar{B}}(I, J, Max)$. In the latter case it is generally appropriate to use

$$\bar{\bar{S}}(I,J,B) = \bar{\bar{B}}(I,J,Max)/3$$

For system design $I$ the standard deviation of the system random bias is given by

$$\bar{\bar{S}}^2(I,B) = \sum_{J=1}^{M} \bar{\bar{S}}^2(I,J,B)$$

$\bar{\bar{S}}^2(I,B)$ has the following meaning: In principle, an infinite number of measurements systems of system design $I$ could be constructed. Each of these measurement systems would have a unique random bias due to the random biases of the constituent modules. However, for any one system the module random biases are not known in advance, and hence the system random bias cannot be computed in advance. The value of $\bar{\bar{S}}^2(I,B)$ combined with the assertion that population of system random biases has an approximately normal density function provides a probabilitic description of the random bias that can be expected for a system that is yet to be constructed.

For each generic module type which exhibits the hysteresis or deadband effect the vendor specifies hysteresis loop and deadband half widths, $\bar{\bar{H}}(I,J)$ and $\bar{\bar{D}}(I,J)$, which are fixed numbers. (i.e., there is no dependence on $K$ as there is for module biases). The system half widths, $\bar{\bar{H}}(I)$ and $\bar{\bar{D}}(I)$ are the sums of the module half widths.

In systems that exhibit hysteresis and deadband effects the system total error is not fixed by the system design, but is also a function of the time dependence of the monitored variable. In Attachment A three possible patterns of time dependence of the monitored variable are considered, the standard deviation of system total error is computed for each of the three cases, and the following simple algorithm is devised which forms an upper bound for the three cases:

$$\overline{\overline{S}}^2(I) = \overline{\overline{S}}^2(I,B) + \overline{\overline{E}}^2(I) + \overline{\overline{H}}(I) * \overline{\overline{D}}(I) + \overline{\overline{D}}^2(I)/2$$

While it is not exact, this equation provides a convenient conservative estimator for the standard deviation of the system design $I$ total error. The value of $\overline{\overline{S}}^2(I)$ combined with the assertion that the population of system total errors has and approximately normal density function provides a conservative probabilistic description of the total error that can be expected for a system that is yet to be constructed.

The basis for the assertion that the population of system design $I$ biases and the population of system design $I$ total errors have approximately normal density functions is discussed in Attachment A. The argument for the normality of system design $I$ total errors requires that $\overline{\overline{H}}(I)$ and $\overline{\overline{D}}(I)$ are much smaller than $\overline{\overline{S}}(I,B)$.

(B) DETERMINATION OF THE TIME CONSTANT OF A FIRST ORDER LINEAR SYSTEM FROM THE VARIATION WITH FREQUENCY OF SINUSOIDAL OUTPUT LEVEL

The most common measure of the time response of a linear system is the time constant, $\tau$, which is defined as the time required for the system to attain 63.2% of its final response after having a step change impressed on the system input. Another useful measure of time response is the Ramp Asymptotic Delay Time (RADT), which is the time lag between a ramp input function and the ramp output function at times large enough that the initial transient has died out.

For First Order Transfer Functions (FOTFs), $\tau$ and RADT are equal, and for Higher Order Transfer Function (HOTFs) $\tau$ is slightly larger than RADT, the maximum difference being about 2%.

One way the time response of a linear system can be evaluated is to impress upon the system a sinusoidal input signal of two different angular frequencies, $\omega_1$, and $\omega_2$ [to simplify verbiage we assume that both input signals have amplitude unity], and to measure the amplitudes, $A(\omega_1)$ and $A(\omega_2)$, of the output signal.

When time response is specified by this type of measurement, we require a method for computing a conservative estimate of $\tau$ or RADT from the values of $\omega_1, \omega_2$, $A(\omega_1)$, $A(\omega_2)$. This method is as follows:

In Attachment B we have shown that for FOTFs we obtain the exact result that $\tau$ = RADT is given by

(eq. B1)

$$A^2(\omega_1) * [1 + \omega_1^2\tau^2] = A^2(\omega_2) * [1 + \omega_2^2\tau^2]$$

For HOTFs $\omega_1, \omega_2$, $A(\omega_1)$, $A(\omega_2)$ is too small a data set to determine the transfer function, so no exact result can be obtained. However in Attachment B it is shown that if $\omega_1$ and $\omega_2$ are in a proper range then the solution, $\tau$, to equation B1 is a conservative estimate of the HOTF RADT. Thus for systems of all orders the solution to equation B1 provides a conservative estimate of RADT.

While both these solutions were developed for use in evaluating NUREG-0737 items, they both have a much wider range of applicability.

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Attachments:
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# ATTACHMENT A

# STATISTICAL TREATMENT OF HYSTERESIS AND DEADBAND

## TABLE OF CONTENTS

## A.1.0 INTRODUCTION AND BACKGROUND

We will begin by discussing a few statistical concepts pertinent to the analysis of measurement systems. The measurement systems we are discussing here are composed of modules, each module being designed to have a specific transfer function or Input/Output (I/O) relationship. In practice the modules exhibit a behavior slightly different from that specified by their design transfer functions which introduces errors into the measurement results. The statistics of any total measurement error is a function of the statistics of the deviations of the actual transfer functions of the modules from their design transfer functions.

In this discussion we will treat four sources of measurement error -- module random biases, module random errors, module hysterysis errors, and module deadband errors. We will say a few words about random biases and random errors here and discuss hysteresis and deadband errors in later sections.

The difference between a module random bias and a module random error should be clearly understood. A module random bias is a bias in the transfer function of the module which is (in the present context) unknown, but for each individual module has a unique fixed value. This bias is present in the transfer function every time the module is used. A module random error, on the other hand, is an error in the transfer function of the module which occurs in a random fashion whenever the module is used.

In the measurement systems we are discussing here, the random errors are generally small compared to the random biases, and the statistical treatment can be simplified by combining these two error sources together and labeling the combination "random biases." In practice the vendor experimentally evaluates the error in a module's transfer function by a single measurement, and labels this error the module bias. When the vendor performs his measurement both the module random bias and the module random error contribute to the error he observes, and hence the error he observes which he is calling the module random bias is actually a combination of module random bias and module random error.

Generally, when a module is purchased to be used as part of a measurement system, the vendor does not specify the individual module bias, but rather specifies the distribution of biases over the population of modules he has for sale. The design philisophy which leads to treating biases as random is described in section A.5.0.

Usually the bias exhibited by a module is a function of I/O level. To simplify specification statements and analyses, generally the bias quoted is the maximum which occurs in the module's I/O range. This leads to conservative results.

## A.2.0 NOTATION

A measurement system consists of a string of modules, the first being a sensor, the last being a readout device, and those in between being whatever is required to get the signal from the sensor to the readout device. We need to deal with systems to measure a number of different quantities, and for each quantity to be measured we can specify a number of different system designs which can satisfactorily perform the desired measurement. By a single system design we mean a pictorial configuration of components in which each component is specified by generic type (i.e., type of component, range, vendor, model number, etc.). We will label each single system design by an index $I$.

We will label the module generic types in a measurement system by an index $J$, where $J$ runs from $1$ to $M$. For example a pressure measurement system might consist of a Pressure Transducer ($J = 1$), an Amplifier ($J = 2$), a Current to Voltage Converter ($J = 3$), and a Strip Chart Recorder ($J = 4$).

We will label individual items of any generic type by an index $K$, where $K$ runs from $1$ to $N$.

An individual item of any generic type, $X(I,J,K)$, is drawn from some Parent Population. We will call this Parent Population $X(I,J,PP)$.

In statistics we need to consider true parameters of a Parent Population and approximations to these parameters obtained by examining samples taken from the Parent Population. In this discussion the approximate parameters well be designated by a double bar, i.e., $\bar{\bar{S}}$. Also conservative estimates of Parent Population parameters will be designated by a double bar.

## A.3.0 STANDARD DEVIATION

As noted in section A.1.0, generally the vendor does not measure and specify a bias, $B(J,K)$, for each module, $K$, but rather specifies some statistical property of $B(J,K)$ as the measure of the bias a module selected at random is expected to exhibit. The statistical property most commonly specified is the Standard deviation $S(J,B)$. The index $B$ is included here to indicate bias.

Ideally the vendor would like to know the density function, $f(J,B)$, of $B(J,PP)$. $f(J,B)$ is a function such that the fraction of the $PP$ that lies between $B$ and $B + \Delta B$ is given by $f(J,B) * \Delta B$. If the density function is known, then $S^2(J,B)$ is given by

$$S^2(J,B) = \int_{B=-\infty}^{+\infty} (B - \bar{B})^2 f(J,B)\, dB \qquad [\bar{B} = \text{mean value of } B].$$

Generally the manufacturing process is adjusted so that the mean of the random bias is zero. Thus the $\bar{B}$ can usually be dropped from the above definition. In practice, the vendor never has the density function, and instead approximates $S^2(J,B)$ by the following equation, where the $N$ biases are measurements of a random sample of his $PP$ of modules.

$$\bar{\bar{S}}^2(J,B) = \sum_{K=1}^{N} B^2(J,K)/N$$

When discussing system design $I$, we will relabel $B(J,K)$ and $S(J,B)$ as $B(I,J,K)$ and $S(I,J,B)$, the index $I$, being added to indicate that we are dealing with system design $I$.

Having measurement system design $I$, we may assemble $N$ systems labeling each system with an index, $K$, and also labeling each component in the $K'th$ system with the same index, $K$. The overall bias of the $K'th$ system is given by

$$B(I,K) = \sum_{J=1}^{M} B(I,J,K) \; .$$

If we assume the $[B(I,J,K),\ J=1,M]$ are not correlated, than from statistics $S^2(I,B)$ can be approximated by

$$\overline{\overline{S}}^2(I,B) = \sum_{J=1}^{M} \overline{\overline{S}}^2(I,J,B) \; .$$

## A.4.0 NORMAL DISTRIBUTIONS

Almost all discussions of the propagation of random errors involve normal, or Gaussian, density functions. A normal density function is defined by (suppress the indices $I,J,K$)

$$f(B) = \frac{1}{S * \sqrt{2\pi}} * exp\left[\frac{-1}{2} \left[\frac{B - \overline{B}}{S}\right]^2\right] \; .$$

Normal distributions occur so frequently in error analysis that usually the expected error in a measurement system or module is specified simply by quoting the $S$ and saying nothing about the error distribution, it tacitly being assumed that the error distribution is normal. The many properties of the normal distribution are discussed in any good textbook on statistics, and we will here mention only two of these properties, which we require for our discussion.

## A.4.1 CENTRAL LIMIT THEOREM

Usually each $B(I,J,PP)$ is normal, and from statistics it follows that $B(I,PP)$ is normal. However $B(I,PP)$ will be nearly normal under much weaker assumptions. The Central Limit Theorem of statistics states that if $B(I,J,PP)$ have arbitrary density functions, and no highly non-normal $B(I,J,PP)$ has an $S(J)$ so large that it overwelms the other $S(J)s$, then $B(I,PP)$ will be approximately normal, and its standard deviation will be given by the last equation in section A.3.0.

## A.4.2 COMPARISON OF $S$ AND SPREAD OF THE NORMAL DENSITY FUNCTION

To give a feel for the relationship between $S$ and the spread of a normal density function, we note the following: If the distribution of biases of a vendor's modules is truly normal, and $1000$ of these modules are selected at random [i.e., $K=1,1000$], then about $683$ of them will exhibit a $|B(I,J,K)|$ less than $1*S(I,J,B)$ and about $997$ of them will exhibit a $|B(I,J,K)|$ less than $3*S(I,J,B)$. For practical purposes $3*S(I,J,B)$ is considered the upper limit of $|B(I,J)|$.

## A.5.0 SUMMARY OF MEASUREMENT SYSTEM DESIGN PHILOSOPHY

In constructing the measurement systems we are discussing here it is necessary to take steps in the following order:

(1) Establish measurement system reliability and accuracy criteria.

Because of variations in module parameters it is necessary that the system accuracy criteria be expressed in a probabilistic, rather than an absolute, fashion. A statement of the measurement system error density function would provide a complete probabilistic description of the system accuracy. However, in practice, a much simpler recourse is taken, and the system accuracy specification is expressed by giving: (1) a statement of the measurement system standard deviation, and (2) an expressed or implied statement that the measurement system error density function is reasonably approximated by a normal density function.

As noted in section A.4.2, three standard deviations is a practical upper limit for system error, so the probabilistic specification that the measurement system standard deviation be no larger than $X/3$ is tantamount to the absolute specification that the system measurement error will not exceed $X$.

(2) Design the measurement system using vendor specifications for module parameters.

(3) Compute the measurement system error standard deviation using the design from step 2 and the vendor specifications for module accuracy parameters.

The vendor should quote the module error standard deviation as his module specification. As in step 1, if the vendor instead quotes an upper limit of module error, $X$, then it is generally reasonable to use $X/3$ as the module standard deviation.

(4) Purchase modules from the vendors.

(5) Assemble measurement system.

When only a small variety of modules of any one type are available it may be necessary to modify the procedure and perform steps 2 and 3 first, and then, instead of step 1, simply state the system error standard deviation computed in step 3 and ask if this is acceptable.

The main points to note here are that: (1) for the measurement systems we are discussing it is necessary to have the system designed and approved before any parts can be procured, and (2) this being the case, the system design can, at best, only be assured to be good enough on a probabilistic basis.

## A.6.0 HYSTERESIS AND DEADBAND

Suppose the hysteresis and deadband we wish to consider is in a strip chart recorder. Then there are three sources of error within the recorder -- the bias, the hysteresis, and the deadband. To conceptually simplify the reasoning we can imagine that an extra module is added to the system just before the recorder, and the bias is removed from the recorder and placed in this module. Now the only error sources in the recorder we need consider are the hysteresis and the deadband.

We should here clearly define the terms hysteresis and deadband. Let $V(true)$ be the voltage that would be delivered to the recorder if all biases could be removed from the system. Then the voltage actually delivered to the recorder is $V(input) = V(true) + B(I,K)$. To simplify verbiage, in the remainder of the discussion of hysteresis and deadband, we will assume that the scale on the recorder is calibrated in volts, and we will call the recorder pen position $V(readout)$, where ideally $V(readout)$ would the the same as $V(input)$.

The hysteresis effect could have either of two causes: (1) hysteresis in some magnetic part of the system, or (2) backlash in some mechanical part of the system. The hysteresis effect causes the readout of the recorder, $V(readout)$, to lag behind $V(input)$ by a constant amount, $H$, where $H$ is the half width of the hysteresis loop. That is, when $V(input)$ is ramping up, or is stationary after ramping up $V(readout) = V(input) - H$. When $V(input)$ is ramping down, or is stationary after ramping down $V(readout) = V(input) + H$.

The deadband effect is caused by the sticking of mechanical parts in modules such as pressure transducers, indicators, and strip chart recorders. In this paragraph we will describe an idealized deadband effect, and in this discussion we will ignore the hysteresis effect. When $V(input)$ becomes stationary at $V(input,stat)$ the recorder pen becomes stationary at $V(input,stat)$. At this point the deadband effect causes the recorder pen to remain stationary at $V(input,stat)$ if $V(input)$ fluctuates within the band $[V(input,stat) \pm D]$, where $D$ is the half width of the deadband. At the moment $V(input)$ ramps through the boundary of this band $V(readout)$ jumps to $V(input)$ and continues to follow $V(input)$ exactly as long as it continues ramping. When $V(input)$ levels out, $V(readout)$ becomes stationary and again exhibits the deadband behavior described above.

Actually the deadband effect is rather erratic compared to this idealized description. The recorder does not jump exactly the distance $D$ every time it jumps out of its deadband, and sometimes after a jump the recorder pen sticks again and then, when $V(input)$ has ramped a sufficient distance, jumps again. Repeated jumps are often evident in a recorder trace with a ramp $V(input)$, in which case part of the $V(readout)$ trace forms a staircase rather than a ramp.

## A.7.0 STATISTICAL TREATMENT FOR LARGE OSCILLATIONS IN THE MEASURED PARAMETER

For large oscillations in the measured parameter the statistical treatment is quite straightforward. The measurement system sweeps through its deadband quickly whenever the measured parameter reverses its direction of ramping so that the deadband effect contributes practically nothing to the total system error and can be ignored in the statistical treatment.

It is obvious that if several modules exhibit hysteresis, their effect is additive, so for system $(I,K)$ we need consider only the total system hysteresis.

If we assume that the measured parameter is ramping up half the time, and ramping down the other half, then half the time the error in $V(readout)$ is $B(I,K) + H(I,K)$, and half the time the error is $B(I,K) - H(I,K)$. Thus

$$\begin{aligned}
&\textit{Time Average [System } (I,K) \textit{ Total Error]}^2 \;=\; S^2(I,K) \\
&= 0.5 * \left[ [B(I,K) + H(I,K)]^2 + [B(I,K) - H(I,K)]^2 \right] \\
&= B^2(I,K) + H^2(I,K) \;.
\end{aligned}$$

Averaging over all systems, $K$, this gives

$$\overline{S^2}(I) \;=\; \sum_{K=1}^{N} [B^2(I,K) + H^2(I,K)]/N \;=\; \overline{S^2}(I,B) + \overline{H^2}(I)$$

where $\overline{H}^2(I) = \sum_{K=1}^{N} H^2(I,K)/N$ .

If $H^2(I) \ll S^2(I,B)$, and, as we previously supposed, the density function of $B(I,K)$ is nearly normal, then from the central limit theorem the system design $I$ total error has a nearly normal density function.

Here note that the density function of errors due to hysteresis is double humped, which is highly non-normal. We have gone out of our way to make a point of the central limit theorem in order to assure the reader that the highly non-normal density functions we encounter will not distroy the validity of our results.

## A.8.2 STATISTICAL TREATMENT FOR SLOW DRIFT IN THE MEASURED PARAMETER

As in the large oscillation case, if more than one module exhibits hysteresis, we can perform our analysis simply using the total system hysteresis, $H(I,K)$. However, here and in section A.9.1 we will assume that only the last module, the recorder, has a deadband, and we will call its half width $D(I,K)$. If we assumed other modules had a deadband the analysis would become entirely too complicated for our purposes, While our model requires that only the last module has a deadband, the error for the case in which other modules have a deadband is bounded by the error predicted by our model. This can be seen by working through a few examples, however we will not go into these examples here.

Suppose that $V(input)$ is a ramp with slope $dV/dt = \pm G$ ($G$ positive) that is slow enough that $V(readout)$ forms a staircase as shown in figure A.8-1. Then

$$Time\ Average\ [System\ (I,K)\ Total\ Error]^2 = S^2(I,K)$$

$$= \frac{0.5 * G}{D(I,K)} * \int_{t=0}^{D(I,K)/G} \left[ \left[ B(I,K) - [H(I,K) + G*t] \right]^2 + \left[ B(I,K) + [H(I,K) + G*t] \right]^2 \right] * dt$$

$$= B^2(I,K) + [H(I,K) + D(I,K)/2]^2 + D^2(I,K)/12 \cong B^2(I,K) + [H(I,K) + D(I,K)/2]^2$$

Up to this point we have, for the sake of generality, treated $H(I,J,K)$ and $D(I,J,K)$ as being functions of $K$. We must now drop this dependence on $K$, for otherwise the formula for $\bar{\bar{S}}(I)$ will involve the covariance of $H$ and $D$, a quantity for which we have no estimate. With this simplification we have

$$S^2(I,K) = B^2(I,K) + [\bar{\bar{H}}(I) + \bar{\bar{D}}(I)/2]^2$$

In vendor specifications $H(I,J)$ and $D(I,J)$ are always treated as fixed numbers, so this simplification is consistent with general practice.

Averaging $S(I,K)$ over all systems, $K$, we obtain the standard deviation of system total error for system design $I$:

$$\bar{\bar{S}}^2(I) = \bar{\bar{S}}^2(I,B) + [\bar{\bar{H}}(I) + \bar{\bar{D}}(I)/2]^2$$

[ In section A.9.2 we will again make the same simplification without repeating the argument above. ]

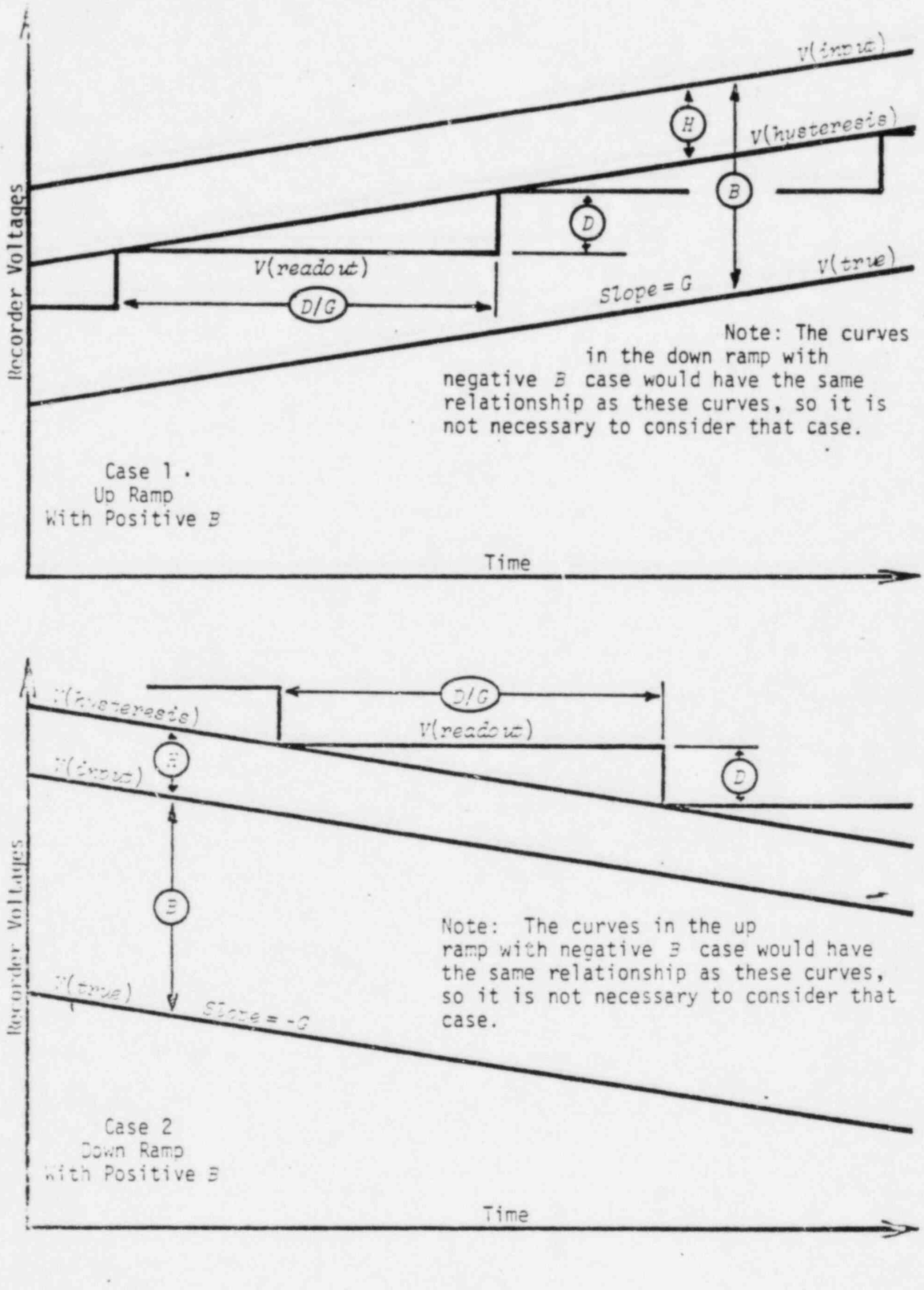


Figure A.8-1 Strip Chart Recorder Voltages for the Slow Drift Model

## A.9.0 STATISTICAL TREATMENT FOR V(READOUT) STATIONARY

### A.9.1 MODEL FOR COMPUTING HYSTERESIS PLUS DEADBAND ERROR

If both the hysteresis abd deadband effects are present the total measurement error is a complicated function of the two effects combined, as well as a function of the detailed shape of $V(input)$. In any practical situation with $V(readout)$ stationary the shape of $V(input)$ is not known because the fluctuations in $V(input)$ are washed out of the recorder trace by the hysteresis and deadband effects. Thus to evaluate the hysteresis plus deadband error we must construct a model for $V(input)$ fluctuations which we feel is conservative compared with the actual $V(input)$ fluctuations, and this model will provide a conservative upper bound for the hysteresis plus deadband error.

In choosing a model for the $V(input)$ fluctuations we note that in tracking various phenomena we sometimes observe a sine wave fluctuation on our strip chart. The sine wave may represent an expected or unexpected property of the phenomena or may be due to slight oscillations of some component in the measuring system, but in any case the occurrence of a sine wave is not uncommon. Thus a fluctuation in $V(input)$ of $A * sin(\omega t)$ in which $A$ is chosen to maximize the hysteresis plus deadband error must be considered a realistic possibility. Most realistic random fluctuations in $V(input)$ would be less conservative than the sine wave, and we therefore consider the sine wave model to be adequately conservative.

In section A.7.0 we considered the case of large oscillations (large $A$), so to avoid repeating previously derived results, we will here consider only the case where $A$, $H(I,K)$ and $D(I,K)$ are of comparable magnitude.

## A.9.2 COMPUTATION OF TOTAL MEASUREMENT ERROR

On figure A.9-1 is shown a plot of our model $V(input)$ and a plot of $V(hysteresis)$, the trace which would appear on the strip chart recorder which suffered from hysteresis, but not from deadband. The relationship between the two curves should be obvious to the reader from inspection.

Suppose we choose $A = H(I,K) + D(I,K)$. According to the deadband effect described in section A.6.0, each time $V(hysteresis)$ rises or falls from a plateau, $V(readout)$ jumps to catch up with $V(hysteresis)$ somewhere between that plateau and the next, and then follows $V(hysteresis)$ to the next plateau. However, as noted in section A.6.0, the deadband jumping is somewhat erratic, and after a number of cycles $V(readout)$ is bound to stick exactly halfway between two $V(hysteresis)$ plateaus, and after that $V(readout)$ will remain flat. An example of how an actual strip chart recorder trace might look is given in figure A.9-1. In the stable $V(readout) = constant$ configuration the time average square of the error in $V(readout)$ is

$$Time\ Average\ [System\ (I,K)\ Total\ Error]^2 = S^2(I,K)$$

$$= \frac{\omega}{2\pi} * \int_{t=0}^{2\pi/\omega} \left[ B(I,K) + [H(I,K) + D(I,K)] * sin(\omega t) \right]^2 dt$$

$$= B^2(I,K) + [H(I,K) + D(I,K)]^2/2$$

Averaging over all systems, $K$, this gives $\quad \overline{\overline{S}}^2(I) = \overline{\overline{S}}^2(I,B) + [\overline{\overline{H}}(I) + \overline{\overline{D}}(I)]^2/2$

As in section A.7.0 if $[\overline{\overline{H}}(I) + \overline{\overline{D}}(I)]^2/2 << \overline{\overline{S}}^2(I,B)$ and the density function of $B(I,K)$ is nearly normal, then the system design $I$ total error density function is nearly normal

Note that either increasing or decreasing $A$ would decrease the hysteresis plus deadband error. If $A = H(I,K) + D(I,K)$ were smaller than assumed, then the integrand in the error integral would be smaller and the error would decrease. If $A$ were larger than assumed, then the amplitude of $V(hysteresis)$ would be larger than $D(I,K)$, and the stable configuration described above with $V(readout)$ constant could not exist, and the error would decrease. Thus we conclude that we have chosen $A$ to maximize the hysteresis plus deadband error.

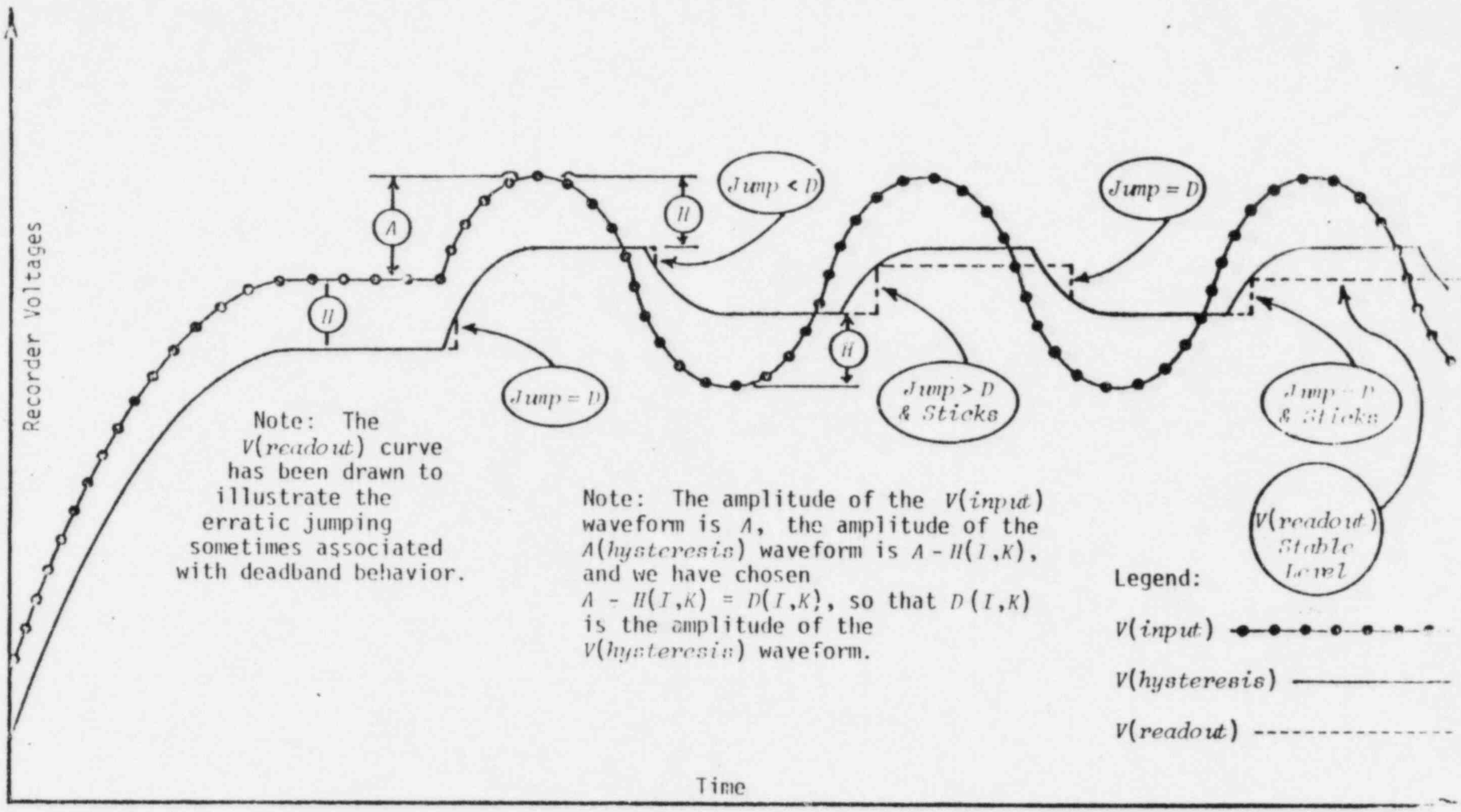


Figure A.9-1 Strip Chart Recorder Voltages for the $V(input) = A * sin(\omega t)$ Model.

## A.10.0 SUMMARY

In any measurement system there exist errors of various types which contribute uncertainty to the measurement. The error analysis done here specifically treats three error sources -- random biases, hysteresis effect, and deadband effect. Random errors are lumped together with random biases to form what are referred to as random biases. Hence in this development there is no specific treatment of random errors.

The indices $I$, $J$, $K$, here have the following meaning:

$I$ is the system design type.

$J$ is the module specification, which for design $I$ runs from $1$ to $M$. $J$ indicates the module type, range, vendor, model number, etc.

$K$ is an index which sequentially labels individual items of any generic type. $K$ runs from $1$ to $N$.

We indicate an estimate of a parent population parameter obtained by examining a sample from the parent population by a double bar, i.e., $\bar{\bar{S}}$. Also we indicate a conservative estimate of a parent population parameter with a double bar.

Each module in a measurement system contains a certain bias, $B(I,J,K)$ from the ideal transfer function. The vendor specifies the standard deviation of biases for the population of modules he has for sale:

$$\bar{\bar{S}}^2(I,J,B) = \sum_{K=1}^{N} B^2(I,J,K)/N$$

In practice the vendor may quote either the standard deviation of biases or an upper limit of biases, $\bar{\bar{B}}(I,J,Max)$. In the latter case it is generally appropriate to use $\bar{\bar{S}}(I,J,B) = \bar{\bar{B}}(I,J,Max)/3$.

The standard deviation of the total bias for system design $I$ is

$$\bar{\bar{S}}^2(I,B) = \sum_{J=1}^{M} \bar{\bar{S}}^2(I,J,B) .$$

$\overline{\overline{H}}(I,J)$ and $\overline{\overline{D}}(I,J)$ are the half widths of the hysteresis loop and deadband of the $(I,J)$ module. $\overline{\overline{H}}(I,J)$ and $\overline{\overline{D}}(I,J)$ are treated as fixed numbers for each generic module type. (i.e., there is no dependence on $K$ as there is for module biases.) For most module types $\overline{\overline{H}}(I,J)$ and $\overline{\overline{D}}(I,J)$ are zero, and when nonzero are assumed to be small compared with the total system bias. The system design $I$ total half widths are given by

$$\overline{\overline{H}}(I) = \sum_{J=1}^{M} \overline{\overline{H}}(I,J) \quad \text{and} \quad \overline{\overline{D}}(I) = \sum_{J=1}^{M} \overline{\overline{D}}(I,J) .$$

In computing the total measurement error we found that the time dependence of $V(readout)$ significantly affected the total error. We considered three cases: (1) large oscillations in $V(readout)$, (2) slow drift in $V(readout)$, and (3) $V(readout)$ stationary. The first two cases were straightforward. For case (3) it was necessary to assume a model input voltage, and we chose to use $V(true) = A * sin(\omega t)$. The standard deviations of total measurement error we obtained in the three cases were as follows:

(1) [Large Oscillations in $V(readout)$] $\equiv$ $\overline{\overline{S}}^2(I) = \overline{\overline{S}}^2(I,B) + \overline{\overline{H}}^2(I)$

(2) [Slow Drift in $V(readout)$] $\equiv$ $\overline{\overline{S}}^2(I) = \overline{\overline{S}}^2(I,B) + [\overline{\overline{H}}(I) + \overline{\overline{D}}(I)/2]^2$

(3) [$V(readout)$ Stationary] $\equiv$ $\overline{\overline{S}}^2(I) = \overline{\overline{S}}^2(I,B) + [\overline{\overline{H}}(I) + \overline{\overline{D}}(I)]^2/2$

$\overline{\overline{S}}^2(I) = \overline{\overline{S}}^2(I,B) + \overline{\overline{H}}^2(I) + \overline{\overline{H}}(I) * \overline{\overline{D}}(I) + \overline{\overline{D}}^2(I)/2$ is a simple algorithm which forms an upper bound for all three cases, and therefore is a convenient conservative estimator for $S^2(I)$.

If $\overline{\overline{H}}(I)$ and $\overline{\overline{D}}(I)$ are small compared to $\overline{\overline{S}}(I,B)$, as we have assumed, then from the central limit theorem in all three cases the density function of the total measurement error is nearly normal. Thus the conservative estimator for $\overline{\overline{S}}(I)$ given above combined with the assertion of normality provides a conservative estimate of the total measurement error density function.

The following points should be clearly understood:

(1) $\overline{\overline{S}}(I)$ is the standard deviation of errors exhibited by a collection of system design $I$ measurement systems. For an individual system we cannot predict the system error, but the value of $\overline{\overline{S}}(I)$, along with the assertion of normality, gives a probabilistic description of the system error.

(2) Ideally we would like to know $S(I,K)$. However, due to the practical considerations explained in section A.5.0 we must design our system to the $\overline{\overline{S}}(I)$ value.

# ATTACHMENT B

# DETERMINATION OF THE TIME CONSTANT OF A FIRST ORDER TRANSFER COMPONENT FROM THE VARIATION WITH FREQUENCY OF SINUSOIDAL OUTPUT LEVEL

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# ATTACHMENT B

# DETERMINATION OF THE TIME CONSTANT OF A FIRST ORDER TRANSFER COMPONENT FROM THE VARIATION WITH FREQUENCY OF SINUSOIDAL OUTPUT LEVEL

## B.1 INTRODUCTION

For transfer components the most common measure of time response is the time constant, $\tau$. For components with transfer functions of any order, $\tau$ is defined as the time required for the output to reach 63.2% of its final response after a step change is made in the input. However, this definition is irrelevent to the present discussion. The property of $\tau$ that is of current interest is the fact that for First Order Transfer Functions (FOTFs), $\tau$ is equal to the Ramp Asymptotic Delay Time (RADT), i.e., the time by which an input ramp function leads the output ramp function after the initial transient has died out. In most real situations the input signal being measured is a ramp function, which makes the RADT an important parameter for evaluating time response.

In principal $\tau$ and RADT are different quantities. They have different definitions and different numerical values. However, in practice it is found that $\tau$ is either equal to or slightly greater than RADT, the maximum difference being about 2%. This difference is less than the error which inevidably occurs when measuring $\tau$ or RADT, and hence for measurement purposes $\tau$ and RADT can be considered to be identical.

One specification for the time response of a component which is sometimes quoted is the Variation with Frequency of Sinusoidal Signal Output Level. The purpose of this attachment is to develop the mathematical algorithm to determine RADT from this specification.

## B.2 ANALYSIS STRATEGY

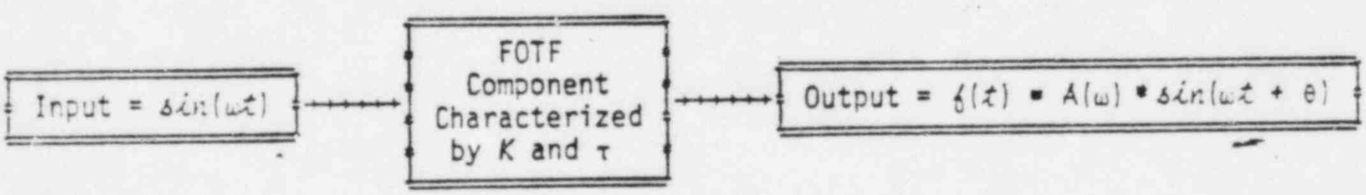
We will first consider the case of a component with a FOTF and develop a formula for finding $\tau$. This $\tau$ is equal to the component RADT exactly.

We will then consider the case of a component with a Second Order Transfer Function (SOTF), which we will analyze with the FOTF formula. Since the FOTF formula is not correct, the FOTF $\tau$ is not correct, and varies with the choice of test frequencies. We will show that if the test frequencies used are in a proper range, then the FOTF $\tau$ is a conservative measure of the true RADT.

From the SOTF case we consider, it is obvious how higher order cases could be analyzed, but we will end our analysis with the SOTF case.

## B.3 ANALYSIS OF A FOFT COMPONENT

The system we consider is the following:




The Laplace Transform of the output of the above system, $F(s)$, is

$$F(s) = \left[\frac{\omega}{s^2 + \omega^2}\right] \cdot \left[\frac{K}{\tau s + 1}\right]$$

Resolving the right member into partial fractions, we obtain

$$F(s) = \left[\frac{\omega \tau K}{1 + \omega^2\tau^2}\right]\left[\frac{1}{s + 1/\tau}\right] + \left[\frac{K}{1 + \omega^2\tau^2}\right]\left[\frac{\omega}{s^2 + \omega^2}\right] - \left[\frac{\omega \tau K}{1 + \omega^2\tau^2}\right]\left[\frac{s}{s^2 + \omega^2}\right]$$

Taking the inverse Laplace Transform of both members, we obtain

$$f(t) = \left[\frac{\omega \tau K}{1 + \omega^2\tau^2}\right] * exp(-t/\tau) + \left[\frac{K}{1 + \omega^2\tau^2}\right] * sin(\omega t) - \left[\frac{\omega \tau K}{1 + \omega^2\tau^2}\right] * cos(\omega t)$$

The first term vanishes for large $t$. Rearranging the coefficients of the remaining part of the equation, we have

$$f(t) = \frac{K}{\sqrt{1 + \omega^2\tau^2}} * \left[\left[\frac{1}{\sqrt{1 + \omega^2\tau^2}}\right] * sin(\omega t) - \left[\frac{\omega \tau}{\sqrt{1 + \omega^2\tau^2}}\right] * cos(\omega t)\right]$$

The sum of the squares of the coefficients of $sin(\omega t)$ and $cos(\omega t)$ is one. Therefore we can find a θ such that

$$\frac{1}{\sqrt{1 + \omega^2\tau^2}} = cos(\theta) \qquad \frac{\omega\tau}{\sqrt{1 + \omega^2\tau^2}} = sin(\theta)$$

$$\delta(t) = \frac{K}{\sqrt{1 + \omega^2\tau^2}} * \left[ cos(\theta)\, sin(\omega t) + sin(\theta)\, cos(\omega t) \right]$$

$$\delta(t) = A(\omega) * sin(\omega t + \theta) = \frac{K}{\sqrt{1 + \omega^2\tau^2}} * sin(\omega t + \theta)$$

$$A(\omega) = \frac{K}{\sqrt{1 + \omega^2\tau^2}}$$

$\tau$ = RADT is measured by measuring $A(\omega)$ at two different angular frequencies, $\omega_1$ and $\omega_2$ and then computing $\tau$ from

$$\boxed{A^2(\omega_1) * [1 + \omega_1^2\tau^2] = A^2(\omega_2) * [1 + \omega_2^2\tau^2]}$$

## B.4 ANALYSIS OF A SOTF COMPONENT

We now have a formula which gives $\tau = \text{RADT}$ for a FOTF component exactly. The following question arises: "Suppose the component being analyzed is actually a SOTF component. Not knowing that it is a SOTF component the experimentor applies the FOTF formula and obtains a value of $\tau$. What is the relationship between the value of $\tau$ measured in this way and the actual SOTF parameters?" The answer is that if $\omega_1$ and $\omega_2$ are chosen properly, $\tau$ will be a conservative measure of the SOTF RADT. We will now demonstrate this.

In the remainder of this discussion we will consistently use $\tau$ to indicate the solution to the FOTF formula, and RADT to indicate the true RADT of the component being analyzed. It turns out that this development gives us information about the SOTF RADT, but provides no information about the SOTF time constant. This is why we started emphacizing the RADT and playing down the time constant in the introduction.

The SOTF Laplace Transform is $K/[(T_1 s + 1)(T_2 s + 1)]$, i.e., simply the product of two FOTF Laplace Transforms. Thinking of a SOTF component as two FOTF components in tandem, if the input to the first FOTF is a ramp, the output from the first FOTF is a ramp delayed from the input ramp by $T_1$. Then the output of the second FOTF is a ramp delayed by $T_2$ from the first FOTF output ramp. Thus $T_1 + T_2$ is the SOTF RADT.

Henceforth we will write $A(\omega)$ as $A[K|\omega|T_1|T_2]$, where the list of $T$s contains as many $T$s as the order of the transfer function. In section B.3 we showed that the attenuation a sinusoidal signal suffers upon being processed by a FOTF component is $A[K|\omega|T_i] = K/\sqrt{1 + \omega^2 T_i^2}$. Again thinking of the SOTF component as two FOTF components in tandem, the attenuation a sinusoidal signal suffers upon being processed by a SOTF component is

$$A^2[K|\omega|T_1|T_2] = \frac{K^2}{(1+T_1^2\omega^2)(1+T_2^2\omega^2)}$$

$$= \left[\frac{K^2}{1+(T_1+T_2)^2\omega^2}\right] * \left[\frac{1+(T_1+T_2)^2\omega^2}{(1+T_1^2\omega^2)(1+T_2^2\omega^2)}\right]$$

$$= A^2[K|\omega|T_1+T_2] * B[\omega|T_1|T_2]$$

For the moment we will be focusing our attention on $B[\omega|T_1|T_2]$. To aid us in visualizing $B[\omega|T_1|T_2]$ we will want to examine its first derivative.

$$B' = \frac{dB[\omega|T_1|T_2]}{d\omega} = \frac{dB[\omega|T_1|T_2]}{d(\omega^2)} * \frac{d(\omega^2)}{d\omega}$$

$$= \frac{(1+T_1^2\omega^2)(1+T_2^2\omega^2)(T_1+T_2)^2 - [1+(T_1+T_2)^2\omega^2][T_1^2(1+T_2^2\omega^2) + T_2^2(1+T_1^2\omega^2)]}{(1+T_1^2\omega^2)^2 * (1+T_2^2\omega^2)^2} * 2\omega$$

$$= \frac{2T_1T_2 - 2T_1^2T_2^2\omega^2 - T_1^2T_2^2(T_1+T_2)^2\omega^4}{(1+T_1^2\omega^2)^2 * (1+T_2^2\omega^2)^2} * 2\omega$$

We note the following values (Here we suppress the arguments $T_1$ and $T_2$).

$$B(\omega=0) = 1 \qquad B(\omega=\infty) = \frac{(T_1+T_2)^2}{T_1^2T_2^2} * \frac{1}{\omega^2}$$

$$B'(\omega=0) = 0, \qquad B'\left[\omega^2 = \frac{\sqrt{1+2(T_1+T_2)^2/T_1T_2} - 1}{(T_1+T_2)^2}\right] = 0$$

Neither $B$ nor $B'$ has any real poles. With this information we can sketch $B$ vs $\omega$, which is shown in figure B-1.

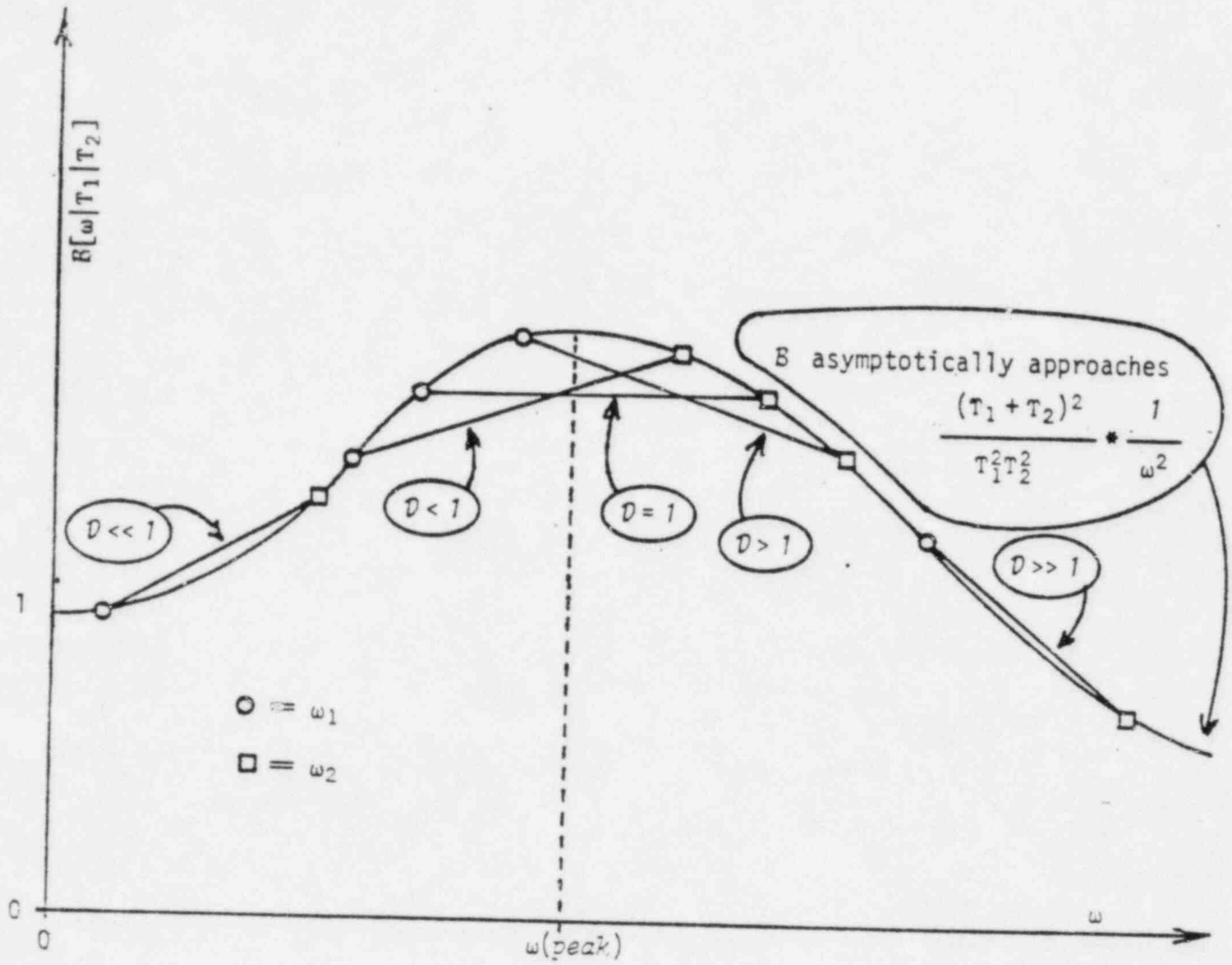



Figure B-1. Plot of $B[\omega|T_1|T_2]$ vs. $\omega$.

The relative values of $\mathcal{D}$ are indicated for various choices of $\omega_1$ and $\omega_2$.

$$\omega^2(peak) = \frac{\sqrt{1 + 2(T_1 + T_2)^2/T_1T_2} - 1}{(T_1 + T_2)^2}$$

We now write the FOTF equation for $\tau$ using the attenuations that would be obtained in a SOTF component. We will adopt the convention that $\omega_2$ is greater than $\omega_1$.

$$A^2[K|\omega_1|T_1|T_2] \star [1+\omega_1^2\tau^2] \;=\; A^2[K|\omega_2|T_1|T_2] \star [1+\omega_2^2\tau^2]$$

$$A^2[K|\omega_1|T_1+T_2] \star B[\omega_1|T_1|T_2] \star [1+\omega_1^2\tau^2] \;=\; A^2[K|\omega_2|T_1+T_2] \star B[\omega_2|T_1|T_2] \star [1+\omega_2^2\tau^2]$$

Write $$C[K|\omega_1|\omega_2|T_1+T_2] \;=\; A^2[K|\omega_1|T_1+T_2]/A^2[K|\omega_2|T_1+T_2]$$

$$D[\omega_1|\omega_2|T_1|T_2] \;=\; B[\omega_1|T_1|T_2]/B[\omega_2|T_1|T_2]$$

Note that $A$ decreases with increasing $\omega$, so $C$ is always greater than $1$.

In the following we suppress the argument list in $C$ and $D$.

$$C \star D \star [1+\omega_1^2\tau^2] \;=\; [1+\omega_2^2\tau^2]$$

$$\tau^2 \;=\; \frac{C \star D - 1}{\omega_2^2 - C \star D \star \omega_1^2}$$

We can make the following observations:

For $\mathcal{D} = 1$ $\quad \tau = T_1 + T_2$ is the correct RADT for the SOTF component.

For $\mathcal{D} > 1$ The numerator is too big and the denominator is too small. $\tau$ is an overestimate of the SOTF RADT (conservative).

For $\mathcal{D} > C*\omega_2/\omega_1$ The numerator is negative and the equation cannot be solved for real $\tau$.

For $\mathcal{D} < 1$ The numerator is too small and the denominator is too big. $\tau$ is an underestimate of the SOTF RADT (nonconservative)

For $\mathcal{D} < 1/C$ The numerator is negative and the equation cannot be solved for real $\tau$.

The choices of $\omega_1$ and $\omega_2$ which give the various relative values of $\mathcal{D}$ discussed above are indicated in figure B-1.

From figure B-1 we can see that to obtain an accurate value for the SOTF RADT we should choose $\omega_1$ and $\omega_2$ to lie roughly symmetrically about $\omega(peak)$. To obtain a conservative value for the SOTF RADT we should choose $\omega_1$ and $\omega_2$ to be somewhat higher than this. However, choosing $\omega_1$ and $\omega_2$ too high will lead to an excessively conservative value for the SOTF RADT.

In the experimental situation the values of $T_1$ and $T_2$ are not known. Good values to use for $\omega_1$ and $\omega_2$ will have to be determined from previous experiments on the same type of SOTF component.

## B.5 HIGHER ORDER SYSTEMS

From the analysis of a SOTF component, it is obvious how the analysis of higher order components could be performed. However, the mathematical manipulations for the higher order components would be unwieldly, and the results would be of little value to the experimentor.

## B.6 EMPERICAL STUDIES

For pressure transducers, for which the above development is intended, typically $T_1 = 0.1$ sec and $T_2 = 0.01$ sec. $\omega(peak)$ would be $18.23$ for this case. We tested this case for several different pairs of input frequencies, and obtained the following results.

| $T_1$ | $T_2$ | SOTF RADT | $\omega_1$ | $\omega_2$ | $\tau$ | % Nonconservatism |
|---|---|---|---|---|---|---|
| 0.1 | 0.01 | 0.11 | 4 | 12 | 0.1014 | 7.81 |
| 0.1 | 0.01 | 0.11 | 10 | 20 | 0.1051 | -4.47 |
| 0.1 | 0.01 | 0.11 | 14 | 23.13 | 0.1100 | 0 |
| 0.1 | 0.01 | 0.11 | 16 | 28 | 0.1181 | -7.32 |
| 0.1 | 0.01 | 0.11 | 22 | 40 | 0.2324 | -111.27 |

For this case the worst choice of $\omega_1$ and $\omega_2$ gave a result which is nonconservative by 7.81%. Thus we see that if the Ts are well separated, as they are here, then the worst achievable nonconservatism is not too bad.

By comparison, with $T_1 = 2$ and $T_2 = 1$ the worst achievable nonconservatism is about 25%, which would be unacceptable in some applications

Another interesting emperical study is performed in the following table.

| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | RADT | $\omega_1$ | $\omega_2$ | $\tau$ | % Nonconservatism |
|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 1.0 | 0 | 0 | 0 | 0 | 3.0 | 0.39 | 0.60 | 3.0042 | -0.14 |
| 2.0 | 1.0 | 0.5 | 0 | 0 | 0 | 3.5 | 0.39 | 0.60 | 3.4498 | 1.44 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0 | 0 | 3.75 | 0.39 | 0.60 | 3.5872 | 4.34 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0.125 | 0 | 3.875 | 0.39 | 0.60 | 3.6237 | 6.48 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0.125 | .0625 | 3.9375 | 0.39 | 0.60 | 3.6330 | 7.73 |

Here we see that an $(\omega_1, \omega_2)$ pair that works well for a SOTF component still works pretty well on higher order components if the two lowest $T$ values are the same.

By comparison in the table below we see than an $(\omega_1, \omega_2)$ pair that works poorly for a SOTF component works even less well on higher order components with the same two lowest $T$ values.

| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | RADT | $\omega_1$ | $\omega_2$ | $\tau$ | % Nonconservatism |
|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 1.0 | 0 | 0 | 0 | 0 | 3.0000 | 0.2 | 0.5 | 2.5331 | 15.56 |
| 2.0 | 1.0 | 0.5 | 0 | 0 | 0 | 3.5000 | 0.2 | 0.5 | 2.6922 | 23.08 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0 | 0 | 3.7500 | 0.2 | 0.5 | 2.7342 | 27.09 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0.125 | 0 | 3.8750 | 0.2 | 0.5 | 2.7448 | 29.17 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0.125 | .0625 | 3.9375 | 0.2 | 0.5 | 2.7475 | 30.22 |
| 2.0 | 1.0 | 0 | 0 | 0 | 0 | 3.0000 | 0.4 | 0.9 | 4.2940 | -43.13 |
| 2.0 | 1.0 | 0.5 | 0 | 0 | 0 | 3.5000 | 0.4 | 0.9 | 7.5847 | -116.70 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0 | 0 | 3.7500 | 0.4 | 0.9 | 10.694 | -185.18 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0.125 | 0 | 3.8750 | 0.4 | 0.9 | 12.293 | -217.24 |
| 2.0 | 1.0 | 0.5 | 0.25 | 0.125 | .0625 | 3.9375 | 0.4 | 0.9 | 12.816 | -225.49 |

## B.7 SUMMARY

We can measure the RADT of a linear transfer component by impressing sinusoidal signals of frequencies $\omega_1$ and $\omega_2$ on the input of the component and observing the attenuation in the sinusoidal output amplitudes, $A(\omega_1)$ and $A(\omega_2)$.

For a FOTF component, RADT is given exactly by the solution, $\tau$, to the equation

$$\boxed{A^2(\omega_1) * [1 + \omega_1^2\tau^2] = A^2(\omega_2) * [1 + \omega_2^2\tau^2]}$$

We will use this equation for estimating the RADT for components of all orders.

For a SOTF component, if $\omega_1$ and $\omega_2$ are chosen optimally, the solution, $\tau$, will be very close to the RADT. If $\omega_1$ and $\omega_2$ are chosen higher than the optimum value, $\tau$ will be a conservative estimate of the RADT.

For higher order components a good choice of $\omega_1$ and $\omega_2$ is the same as the optimal values of $\omega_1$ and $\omega_2$ for a SOTF component with the same $T_1$ and $T_2$.

If the values $\omega_1$ and $\omega_2$ are chosen too low when measuring a component of higher order than first, then $\tau$ will be a nonconservative estimate of the RADT.

The larger the separation between the $T$s, the more the component behaves like a FOTF component. Thus, for components with the $T$s well separated, the solution, $\tau$, is relatively insensitive to the choice of $\omega_1$ and $\omega_2$, and the worst achievable nonconservatism is not too bad. For components with the $T$s closely spaced, choosing $\omega_1$ and $\omega_2$ too low can lead to unacceptably nonconservative results.