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# POLYNOMIAL REGRESSION FOR THE TEKTRONIX 4052 COMPUTER

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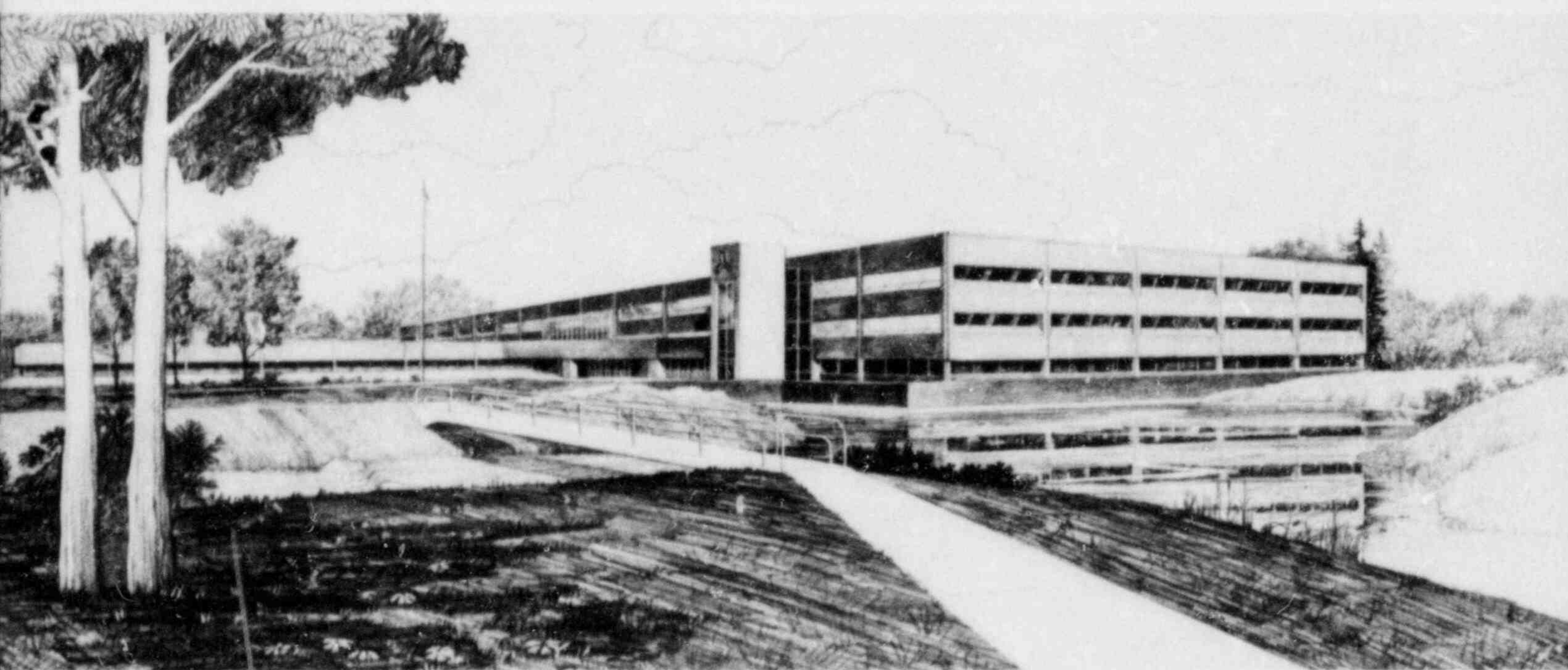

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ABSTRACT

This report is used to perform polynomial regression analyses and is written in BASIC for the Tektronix 4052 computer. Data entry is permitted from the keyboard and the computer program plots descriptive statistics for X and Y values. Analysis of variance table for each degree regression selected is provided. Regression curve plotted against the data. List and plot of residuals. Coefficient for each degree regression with standard error and t-test. Estimated Y for input X. The degree of regression may be changed which does not exceed that selected during initialization, without re-entering the data.

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## SUMMARY

This document is used to perform polynomial regression analyses and is written in BASIC for the 4052 computer. The objective of this report is how the program works and the regression technique used. Also, the program is convenient and fast. The maximum degree of fit, $>1 \leq 12$ for 16K machine $\leq 25$ for 24K 32K machine. Program displays the mean, variance, minimum, maximum, and correlation for X and Y values. Plots the data previously entered. Allows selection of the degree of regression. Any degree that does not exceed that selected during initialization may be designated. Graphs the present selected degree polynomial, displays the coefficients, standard error, and T-statistics. Displays the Y-estimate for any designated X-value. The estimate is based on the current selected degree. Permits storage of data so the machine may be turned off without loss. Additional data may then be added or deleted from the keyboard. Displays the preliminary analysis of variance table to assist in selection of the degree of regression, and for the present selected degree. Produces a plot of X and Y against the standardized residuals for the present selected degree and displays the table of X and Y values and the Y-estimate residuals. A list of various statistics and how they are calculated in the program is given in this report. A numerical example of a linear polynomial is also given in the report.

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CONTENTS

# POLYNOMIAL REGRESSION FOR THE TEKTRONIX 4052 COMPUTER

## 1. INTRODUCTION

Polynomial Regression program for the Tektronix 4052 computer, under highly interactive operator control, will fit a polynomial regression to data pairs of the form X, Y.

If X values are in arithmetic progression, it is possible to greatly reduce the computations by using orthogonal polynomials.

After the maximum degree of regression (25 degree highest) is specified by the user, data may be entered from the keyboard or from previously created magnetic tape files. Any degree of regression less than the selected maximum may be fit, and data may be edited. The reader should be familiar with degree of function that fits the data and make decision accordingly.

When a certain degree of polynomial fits the data, we can estimate the coefficients and thus determine a least square fit, statistics, the residual error (square of the standard error of the estimate) for each degree, R-square value for each degree regression. The appropriate degree of regression may be found quickly by use of preliminary analysis of variance table, where it is not known what function fits the data.

Appendix A has the examples of output. Data entered in the machine, plot of the data points, Preliminary AOV (Preliminary Analysis of Variance Table), statistics, residuals, analysis of variance table, coefficients, etc. Appendix B has the program written in Tektronix 4052 BASIC language. Appendix C contains listing of Tables for F-distribution and Students t-distribution.

## 2. METHODS

A polynomial of the form $y = \beta_0 + \beta_1 x^1 + \beta_2 x^2 + \ldots + \beta_p x^p$ is fit to the data of the form X,Y. Normal equations are formed as data are entered. After all data are entered, normal equations are reduced by the square root (CHOLESKY) method discussed in References 1, 2, 3, 4, and 5. Good results are obtained using the Square Root Procedure.

### 2.1 Specifying Maximum Degree Regression

During the initialization phase of data entry, the maximum degree of regression must be specified (in all cases the maximum should be as minimal as practicable). Required computations depend quadratically upon the maximum degree specified, so four times as much computation is required for Degree 4 as for Degree 2.

Another reason for keeping the maximum degree minimal, is numerical difficulty of high degree polynomial regression. The method used in this program will generally detect an unstable matrix process and will automatically reduce the maximum degree permitted.

### 2.2 Selecting Degree Regression

Several means are available for selecting the appropriate degree of regression. The first method is graphically based, and allows the plot of any regression curve that does not exceed the preselected maximum. The curve is plotted against the data and the degree of regression appearing to fit best is selected.

The second method requires use of the PRELIM AOV (preliminary analysis of variance) key, which initiates a display of a table containing information for each degree of regression from linear to the maximum degree selected. This table contains the following information:

1. The orthogonal polynomial sum of squares for each degree

2. The residual error (square of the standard error of the estimate) for each degree

3. A sequence of independent F-tests for significance of the highest order term (see Table C-1 of Appendix C)

4. The R-square value for each degree regression.

The appropriate degree of regression may be found quickly by use of the preliminary analysis of variance table.

## 3. DEFINITIONS

### 3.1 Matrices

The theory that will be used here will require some of the mathematical techniques of matrix algebra. These techniques are discussed without proof.

1. A matrix is an array of numbers, consisting of m rows and n columns. It is denoted by a bold face capital letter (X, Y).

2. The (i, j) element of a matrix is the element occurring in row i and column j. It is usually denoted by a lower case letter with subscripts ($\alpha_{ij}$, $\sigma_{ij}$).

3. A matrix is called rectangular if m (number of rows) $\neq$ n (number of columns).

4. A matrix is called square if $m = n$.

5. In the transpose of a matrix A, denoted by A'; the element in the j'th row and i'th column of A is equal to the element in the i'th row and j'th column of A'. $A^{-1}$ will denote the inverse of A.

6. A matrix with m rows and 1 column is called a column vector.

7. A matrix with one row and n columns is called a row vector and is usually denoted by a prime.

8. A matrix with one row and one column is called a scaler.

9. Two matrices, A and B, can be added (subtracted) if the number of rows (columns) in A equals the number of rows (columns) in B.

10. Two matrices, A and B, can be multiplied if the number of columns in A equals the number of rows in B.

## 3.2 Statistics

List of various statistics how they are calculated in the program for coefficients and analysis of variance table. The arithmetic mean or the mean of a set of N numbers $x_1$, $x_2$, $x_3$, . . ., xn is denoted by $\bar{x}$ and is defined as

$$\bar{x} = X \text{ mean} = \frac{\Sigma x_i}{n} \tag{1}$$

and variance of x is

$$VAR(x) = \frac{\Sigma(x_i - \bar{x})^2}{(n - 1)} = \frac{\Sigma x_i^2 - (\bar{x}) \cdot n}{(n - 1)} . \tag{2}$$

The arithmetic mean or the mean of a set of N numbers $y_1$, $y_2$, $y_3$, . . ., $y_N$ is denoted by $\bar{y}$ and is defined as

$$\bar{y} = y \text{ mean} = \frac{\Sigma y_i}{n} \tag{3}$$

variance of y is

$$VAR(y) = \frac{\Sigma(y_i - \bar{y})^2}{(n - 1)} = \frac{\Sigma y_i^2 - (\bar{y})^2 \cdot n}{(n - 1)} \tag{4}$$

covariance is

$$COV(x,y) = \frac{\Sigma xy}{V(\Sigma x^2)(\Sigma y^2)} \text{ or } = \frac{\Sigma xy - n\bar{x}\bar{y}}{(n - 1)\sqrt{VAR(x) \cdot VAR(y)}} \tag{5}$$

where

$x = (x - \bar{x})$

$y = (y - \bar{y})$.

$$\text{R-SQUARE} = \frac{\text{TOTAL SS} - (\text{RESIDUAL})^2}{\text{TOTAL SS}}$$

$$= \frac{(\Sigma y^2 - \bar{Y}\Sigma Y_i) - \Sigma(\text{RESIDUAL})^2}{(\Sigma y^2 - \bar{y}\Sigma y_i)} \tag{6}$$

where

$$\text{TOTAL SS} = (\Sigma Y^2 - \bar{Y}\Sigma Y_i)$$

$$(\text{RESIDUAL})^2 = (Y - Y\ \text{EST})^2.$$

In the analysis of variance table for selected degree, R-Square, a measure of the closeness of the fit, is also displayed.

$0 \leq$ R-Square $\leq 1$, perfect fit$\rightarrow$R-Square = 1. The AOV table provides a test of the significance of the presently selected degree.

<u>Estimate of y for x and RESIDUALS</u>

Y ESTIMATE is calculated from given value of x and y and calculated coefficients for a certain degree of polynomial.

We begin by fitting a first degree polynomial. Y estimate for a first degree polynomial would be

$$Y\ \text{EST} = \hat{a}_0 + \hat{a}_1 x \tag{7}$$

for a second degree polynomial would be

$$Y\ \text{EST} = \hat{a}_0 + \hat{a}_1 x + \hat{a}_2 x^2 \tag{8}$$

and a third degree polynomial would be

$$Y\ EST = \hat{a}_0 + \hat{a}_1 x + \hat{a}_2 x^2 + a_3 x^3 \tag{9}$$

and so on

where $\hat{a}_0$, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, . . . are calculated coefficients for a given set of data.

Residuals are calculated from given value of y minus Y EST (estimated value of Y).

RESIDUAL = Y - Y ESTIMATE

The program will plot the residuals.

## 4. POLYNOMIAL REGRESSION MODEL

Let us suppose that we have evidence that a quantity y, is functionally related to a quantity x, by $y = f(x)$. The general procedure is to collect data (various values of y and x) and estimate or test hypothesis about the parameters in $f(x)$. The reader should be familiar with degree of the function that fits the data well and make decision accordingly. A general model is discussed without proof. For simplicity, a linear model is discussed with minimal proof showing how different statistics are arrived and a numerical example illustrates the theory in the next section.

### 4.1 Generalized Polynomial Model

A polynomial model can be written as

$$y = \beta_0 + \beta_1 x_j + \beta_2 x_j^2 + \dots + \beta_p x_j^P . \tag{10}$$

There are two situations in which an experimenter may want to use a polynomial model:

1. Where he knows, theoretically or otherwise, that his data fits a polynomial of Degree P or less, and he wishes to find the maximum likelihood or least squares estimate of the $\beta_i$, set confidence intervals on the $\beta_i$, or test hypotheses about the $\beta_i$.

2. Where it is not known what function fits the data; a search must be made to find a polynomial of low degree that adequately describes them.

3. A simple discussion of how to select a degree of regression is given. The generalized polynomial model uses normal equations.

The normal equation $X'X\hat{\beta} = X'Y$, where $\hat{\beta}$ can be solved by finding the inverse of $X'X$, and get $\hat{\beta} = (X'X)^{-1} X'Y$. This equation plays an important part in estimation and testing hypothesis about the parameter in the model.

Various methods of solving this equation are: where X and Y are known variables and $\hat{\beta}$ an unknown parameter. Therefore, $\hat{\beta}$ will have the necessary calculated coefficients.

## 4.2 Estimating and Testing Coefficients in a Polynomial Model

Consider the Model

$$y_j = \beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j} + \beta_3 x_{3j} + ej \qquad (11)$$

where ej is the measurement error

$j = 1, 2, \ldots, n$

if we let $x_{1j} = x_j$, $x_{2j} = x_j^2$, $x_{3j} = x_j^3$, we get the particular model

$$y_j = \beta_0 + \beta_1 x_j + \beta_2 x_j^2 + \beta_3 x_j^3 \; ej \; . \qquad (12)$$

The pertinent matrices are

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & x_n^3 \end{bmatrix} \qquad X'X = \begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 & \Sigma x_i^5 \\ \Sigma x_i^3 & \Sigma x_i^4 & \Sigma x_i^5 & \Sigma x_i^6 \end{bmatrix} \qquad (13)$$

$$X'Y = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \\ \Sigma x_i^2 y_i \\ \Sigma x_i^3 y_i \end{bmatrix} \quad (14)$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot\ \cdot\ \cdot \\ y_n \end{bmatrix} \quad \hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \cdot \\ \cdot \\ \hat{\beta}_n \end{bmatrix} \quad (15)$$

Where X and Y denote a matrix, and X',Y' will denote the transpose of X and Y, then the polynomial model is exactly like the linear model discussed below.

## 4.3 A Simple Linear Model

A simple linear model is $Y_i = \beta_1 + \beta_2 x_i + e_i$, $i = 1, 2, \ldots, n$ where $\beta_1$ and $\beta_2$ are unknown scalar constants and $x_i$ and $y_i$ are known scalar constants, $e_i$ is the measurement error. Matrices for the linear equation are.

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \cdot & \cdot\ \cdot \\ 1 & x_n \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot\ \cdot\ \cdot \\ y_n \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} . \quad (16)$$

It is easy to see that

$$S = X'X = \begin{bmatrix} n & \Sigma x_i \\ \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \quad (17)$$

$$S^{-1} = \frac{1}{n\Sigma(x_i - \bar{x})^2}\begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix} \quad (18)$$

and

$$X'Y = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}. \quad (19)$$

$$\text{Thus, } \beta = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = S^{-1} XY = \frac{1}{n\Sigma(x_i - \bar{x})^2}\begin{bmatrix} \Sigma x_i^2 \, \Sigma y_i - \Sigma x_i \Sigma x_i y_i \\ -\Sigma x_i \Sigma y_i + n\Sigma y_i x_i \end{bmatrix}$$

$$\text{or } \hat{\beta}_2 = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}, \quad \hat{\beta}_1 = \bar{y} - \hat{\beta}_2\bar{x} \quad (20)$$

where $\beta_2$ is the slope of the line, or, in other words, it is the change in E(y) per unit change in x, and $\beta_1$ is the value of E(y) when x = 0.

Since the $COV(\hat{\beta}) = S^{-1}\sigma^2$, we see that

$$COV(\hat{\beta}_1, \hat{\beta}_2) = -\frac{\sigma^2 \Sigma x_i}{n(\Sigma x_i - \bar{x})^2} \quad (21)$$

$$VAR(\hat{\beta}_1) = \frac{\sigma^2 \Sigma x_i^2}{n\Sigma(x_i - \bar{x})^2} \quad (22)$$

and

$$VAR(\hat{\beta}_2) = \frac{\sigma^2}{\Sigma(x_i - \bar{x})^2}. \quad (23)$$

To minimize $VAR(\hat{\beta}_2)$, we choose our $x_i$ such that $\Sigma(x_i - \bar{x})^2$ is as large as possible. To minimize $VAR(\hat{\beta}_1)$, we choose the $x_i$ such that $\Sigma x_i^2/\Sigma(x_i - \bar{x})^2$ is as small as possible. Since $\Sigma(x_i - \bar{x})^2 \leq \Sigma x_i^2$, the $VAR(\hat{\beta}_1)$ is minimum if $x_i$ are chosen such that $\bar{x} = 0$. This also makes the $COV(\hat{\beta}_1, \hat{\beta}_2) = 0$. Note: It is assumed that n is fixed. To estimate $\sigma^2$, we note that any of the following formulas can be used:

$$\hat{\sigma}^2 = \frac{1}{(n - 2)} (Y'Y - Y'XS^{-1}X'Y) = \frac{1}{(n - 2)} (Y'Y - \hat{\beta}'X'Y) \tag{24}$$

$$= \frac{1}{(n - 2)} (Y - X\hat{\beta})'(Y - X\hat{\beta}) \tag{25}$$

$$= \frac{1}{(n - 2)} (Y'Y - \hat{\beta}'S\hat{\beta}) \tag{26}$$

$$= \frac{1}{(n - 2)} \left\{ \Sigma(y_i - \bar{y})^2 - \frac{[\Sigma(x_i - \bar{x})(y_i - \bar{y})]^2}{\Sigma(x_i - \bar{x})^2} \right\} . \tag{27}$$

### 4.4 Procedure to Find the Polynomial That Best Fits the Data

For example, if the reader is not familiar with the degree of polynomial that best fits the data, he shall begin by fitting a first degree polynomial $y = \beta_1 + \beta_2 x + e$, as shown in Table 1 and then test the hypothesis. If by using the F test we decide that $\beta_2 = 0$, we conclude that the line $y = \beta_1 + e$ fits the data adequately. To look for the value of F refer to Table C-1 of Appendix C. Proceed downward under column headed $r_2 = n - p$ until n-p is reached. Then proceed right to the column head $r_1 = p - 1$, the result is the required value of F where n the number of independent observations in the sample (sample size) and p the parameters that must be estimated. If instead we decide that $\beta_2 \neq 0$, we fit the second degree polynomial, by looking at $Y = \beta_1 + \beta_2 x + \beta_3 x^2 + e$ as in Table 2, if we conclude that $\beta_2 \neq 0$, we next fit the cubic $Y = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3 + e$. If F is not significant, we conclude that $\beta_4 = 0$ and a second degree polynomial adequately fits the data. If F is significant, we conclude that $\beta_4 = 0$ and that a second degree polynomial adequately fits the data. If F is significant, we conclude that $\beta_3 \neq 0$ and proceed to fit a fourth degree polynomial of the data.

To arrive at the quantities in Table 1, we use the normal equations below

$$\begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix} \quad (28)$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} x'y = \begin{bmatrix} \Sigma y_1 \\ \Sigma x_i y_i \end{bmatrix}, \; y'y = \Sigma y^2 \quad (29)$$

$$\Sigma(\text{RESIDUAL})^2 = [y - (\hat{\beta}_1 + \hat{\beta}_2 x)]^2 \quad (30)$$

or

$$= [Y - Y\ \text{EST}]^2 \quad (31)$$

TABLE 1. ANALYSIS OF VARIANCE FOR LINEAR POLYNOMIAL

| SV | DF | SS | MS | F |
|---|---|---|---|---|
| TOTAL | n - 1 | $yy' - \bar{y}\Sigma y_i$ | | |
| REG | p - 1 | TOTAL SS - $(\text{RESI})^2$ | $\frac{\text{TOTAL SS} - (\text{RESI})^2}{(p - 1)}$ | $\frac{(\text{TOTAL SS} - (\text{RESI})^2}{(p - 1)} \Big/ \frac{(\text{RESI})^2}{(n - p)}$ |
| RESID | n - p* | $(\text{RESIDUAL})^2$ | $\frac{(\text{RESIDUAL})^2}{(n - p)} = s^2$ | |

where

p = 2

1 degree of freedom for $\beta_1$

1 degree of freedom for $\beta_2$

Reg = regression

RESI = residual

DF = degree of freedom

SS = sum of squares

MS = mean of sum of squares

F = tests for significance of the highest order term.

To arrive at the quantities in Table 2, we use the normal equations below

$$\begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix} \qquad (32)$$

We therefore continue in this fashion until we arrive at the first nonsignificant result; if we obtain significant results for linear, quadratic . . ., up to a p - 1 degree polynomial and then obtain nonsignificance for a polynomial of degree p, we conclude that a p - 1 degree polynomial fits the data. However, in most cases this is not desirable. More desirable is a low degree polynomial that represents the data.

TABLE 2. ANALYSIS OF VARIANCE FOR QUADRATIC POLYNOMIAL

| SV | DF | SS | MS | F |
|---|---|---|---|---|
| TOTAL | n - 1 | $Y'Y - \bar{Y}\Sigma y_i$ | | |
| REG | p - 1 | TOTAL SS - $(\text{RESI})^2$ | $\frac{\text{TOTAL SS} - (\text{RESI})^2}{(p - 1)}$ | $\frac{\text{TOTAL SS} - (\text{RESI})^2}{(p - 1)} \Big/ \frac{(\text{RESIDUAL})^2}{(n - p)}$ |
| RESID | n - p* | $(Y'Y - \hat{\beta}x'Y)$ or $(\text{RESI})^2$ | $\frac{(\text{RESIDUAL})^2}{n - p}$ | |

*p = 3

1 degree of freedom for $a_0$

1 degree of freedom for $a_1$

1 degree of freedom for $a_2$

$$\hat{\beta} = \begin{bmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}, \quad x'Y = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i \; x_i \\ \Sigma y_i \; x_i^2 \end{bmatrix}, \quad Y'Y = \Sigma y^2$$

or

$$(\text{RESIDUAL})^2 = [Y - (\hat{a}_0 + \hat{a}_1 x + \hat{a}_2 \; x \;)]^2 \; .$$

$$= [Y - Y \; \text{EST}]^2$$

Whenever we conclude that a certain degree polynomial fits the data, we can then estimate the coefficient and thus determine a least square curve.

Two things in the above procedure that need discussing are:

1. The nonsignificance of a result does not imply that the data actually came from any specified degree of polynomial. It is merely a procedural criterion for establishing what polynomial is adequate.

   Suppose it is decided that the quadratic model

   $$y = \beta_1 + \beta_2 x + \beta_3 x^2 + e_2 \tag{33}$$

   is an adequate representation of the data. Examination of the linear model $y = \beta_1 + \beta_2 x + e_1$, and the term $e_1$, which we assumed took on the aspect of a random variable, has a strong component of $x^2$ in it. This may introduce a bias in the error sum of squares for linear. Similar remarks hold when we decide that $p^{th}$ degree polynomial fits the data. The remainder sum of squares for the lower degree may be biased.

   Some statisticians recommended that two consecutive nonsignificant results appear before a decision is made on the degree of polynomial. Suppose, for example, that the first two consecutive nonsignificant results are found when testing $\beta_3 = 0$ and when testing $\beta_4 = 0$; then we conclude that a linear polynomial fits the data.

2. The computing involved is not difficult for a low-degree polynomial (Table 1 requires no difficult computations; Table 2 requires that we solve a system of three equations with three unknowns) when obtaining the sum of squares because of $\beta_1$, $\beta_2$, $\beta_3$. If, however,

the data fit a polynomial of degree 3 or more, the computation can become quite troublesome, if hand calculated but program is capable of handling up to 25 degrees.

If the x values are in arithmetic progression (or equally spaced), we can greatly reduce the computations by using orthogonal polynomials. Orthogonal polynomial is not discussed in this report. The program automatically uses orthogonal polynomials when appropriate.

The coefficient table displays the coefficients of the fit and the standard error for each coefficient. The standard error may then be used to set confidence intervals for the coefficients. If the fit for the present selected degree is not perfect, the T statistics for each coefficient will be displayed. The t-test is a test that the coefficient is equal to zero.

Referring to Table C-2 of Appendix C proceed downward under column headed $V = n - 1$ until entry $n - 1$ is reached then proceed right to column headed $t_{\cdot 995}$ or $t_{\cdot 95}$. The result is the required value.

## 5. NUMERICAL EXAMPLE OF CALCULATING QUANTITIES

This section calculates a simple numerical model for a linear equation. It could be extended to 2nd and Kth order. For the higher order, matrix theory, is used in the program.

A simple numerical linear model is

$$y_i = a_0 + a_1 x_i \quad i = 1, 2, . . . n \ . \tag{34}$$

The points used for this example are

$$X = 1, 3, 4, 6, 8, 9, 11, 14 \text{ and } Y = 1, 2, 4, 4, 5, 7, 8, 9 \ . \tag{35}$$

These constants can be expressed as matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \\ 1 & 9 \\ 1 & 11 \\ 1 & 14 \end{bmatrix}, \ Y = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 5 \\ 7 \\ 8 \\ 9 \end{bmatrix}, \ \hat{\beta} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} . \tag{36}$$

The transpose of matrix x and y denoted by x' and y' and are expressed by

$$x' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 & 8 & 9 & 11 & 14 \end{bmatrix} \text{ and } y' = (1\ 2\ 4\ 4\ 5\ 7\ 8\ 9) \tag{37}$$

$$S = x'x = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 & 8 & 9 & 11 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \\ 1 & 9 \\ 1 & 11 \\ 1 & 14 \end{bmatrix} = \begin{bmatrix} 8 & 56 \\ 56 & 524 \end{bmatrix} \tag{38}$$

where $n = 8$; $\Sigma x_i = 56$, $\Sigma x_i^2 = 524$

$$S^{-1} = \frac{1}{(4192-3136)} \begin{bmatrix} 524 & -56 \\ -56 & 8 \end{bmatrix} = \begin{bmatrix} 524/1056 & -56/1056 \\ -56/1056 & 8/1056 \end{bmatrix}$$

$$= \begin{bmatrix} 0.49621212 & -0.05303030 \\ -0.05303030 & 0.00757575 \end{bmatrix} \tag{39}$$

$$x'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 & 8 & 9 & 11 & 14 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 40 \\ 364 \end{bmatrix} \tag{40}$$

$$\begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix} = \begin{bmatrix} 40 \\ 364 \end{bmatrix} \tag{41}$$

The matrix equation $X'\ x\hat{\beta} = X'Y$ are called the normal equation,

$$\hat{\beta} = \left[\frac{1}{x'x}\right]\left[X'Y\right] = \left[S^{-1}\right](X1Y) \tag{42}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \left[S^{-1}\right](X'Y) \tag{43}$$

$\hat{\beta}$ will have the calculated scaler constants for $a_0$ and $a_1$. Substituting the values obtained in Equations (39) and (40) in Equation (43)

$$\hat{\beta} = \begin{bmatrix} 0.49621212 & -0.05303030 \\ -0.05303030 & 0.00757575 \end{bmatrix} \begin{bmatrix} 40 \\ 364 \end{bmatrix} = \begin{bmatrix} 0.545455600 \\ 0.63636100 \end{bmatrix} \tag{44}$$

where $(\hat{\beta}^1)$ and $(\hat{\beta}^2)$ are known scaler constants (NEW $a_0$ and NEW $a_1$) calculated from $x_i$ and $y_i$ are shown as

$$\hat{\beta}_1 = \hat{a}_0 = 0.545455600 \quad (45)$$

$$\hat{\beta}_2 = \hat{a}_1 = 0.63636100 \quad (46)$$

where $\hat{\beta}_1$ is the slope of the line, or in other words, it is the change in E(y) per unit change in x, and $\hat{\beta}_2$ is the value of E(y) when x = 0, is y-intercept (estimated value of y is Y EST = 0.545455600 + 0.636361000x).

To estimate $\hat{\sigma}^2$, the mean sum of squares for residuals is

$$\hat{\sigma}^2 = \frac{1}{(n - 2)} [(y'y) - (y'xS^{-1}x'y)] \quad (47)$$

substituting the numerical values in Equation (47)

$$= \left(\frac{1}{6}\right) \left[256 - \begin{pmatrix} 40 \\ 364 \end{pmatrix} \begin{pmatrix} 0.545455600 \\ 0.636361000 \end{pmatrix}\right] = \left[\frac{1}{6} \left(256\right) - \begin{pmatrix} 21.818224 \\ 231.6354040 \end{pmatrix}\right] \quad (48)$$

$$= \frac{1}{6} (256 - 253.4536280) = \frac{1}{6} (2.546372) = 0.42439533 \quad (49)$$

where

$$Y'Y = \Sigma Y_i^2 = (1\ 2\ 4\ 4\ 5\ 7\ 8\ 9) \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 5 \\ 7 \\ 8 \\ 9 \end{bmatrix} = 256$$

$y'x$ and $S^{-1}x'y$ are calculated in Equations (40) and (44). Also,

$$y'x = x'y = \begin{pmatrix} \Sigma y \\ \Sigma x_i y_i \end{pmatrix} . \quad (50)$$

Statistics

$$XMean = \frac{\Sigma x_i}{n} = \frac{56}{8} = 7 \quad (51)$$

$$VAR(x) = \frac{\Sigma(x_i - \bar{x})^2}{(n - 1)} = \frac{132}{7} = 18.85714285 \quad (52)$$

$$YMEAN = \frac{\Sigma y_i}{n} = \frac{40}{8} = 5 \quad (53)$$

$$VAR(y) = \frac{(y_i - \bar{y})^2}{n - 1} = \frac{56}{7} = 8 \quad (54)$$

$$COV(x,y) = \frac{[\Sigma(x_i - \bar{x})(y_i - \bar{y})]}{\sqrt{(x_i - \bar{x})(y_i - \bar{y})}} = \frac{\Sigma xy - n\bar{x}\bar{y}}{(n-1)VAR(x)VAR(y)} \quad (55)$$

$$= \frac{364-8*7*5}{7\sqrt{(18.85714285)*8}} = \frac{364-280}{7\sqrt{150.8571428}} \quad (56)$$

$$= \frac{84}{7*12.28239157} - \frac{84}{85.97674099} \quad (57)$$

$$= 0.977008422 \quad (58)$$

where

n = 8

x = 7

y = 5

$\Sigma xy = 364$

$VAR(x) = 18.857/4285$

$VAR(y) = 8.$

$$R\text{-}SQUARE = \frac{(256-200) - (2.54545)}{(256-200)} = \frac{53.45455}{56}$$

$$= 0.95454554 \qquad (59)$$

where

$\Sigma y^2 = y'y = 256$

$\bar{y}\Sigma y_i = 5*40 = 200$

Y EST = 0.545455600 + 0.636361000x

Y EST = Estimated value of y for every change in x.

| X | Y | Y EST | (Y - Y EST) RESIDUAL | $(Y - Y\ EST)^2$ $(RESIDUAL)^2$ |
|---|---|---|---|---|
| 1.000000 | 1.000000 | 1.181818 | -0.181818 | 0.03305778 |
| 3.000000 | 2.000000 | 2.454545 | -0.454545 | 0.20661115 |
| 4.000000 | 4.000000 | 3.090909 | 0.909091 | 0.82644645 |
| 6.000000 | 4.000000 | 4.363636 | -0.363636 | 0.13223114 |
| 8.000000 | 5.000000 | 5.636364 | -0.636364 | 0.40495914 |
| 9.000000 | 7.000000 | 6.272727 | 0.727273 | 0.5289602 |
| 11.000000 | 8.000000 | 7.545455 | 0.454545 | 0.20661116 |
| 14.000000 | 9.000000 | 9.454545 | -0.454545 | 0.20661116 |

$$(RESIDUAL)^2 = (Y - Y\ EST)^2 = 2.54545. \qquad (60)$$

Analysis of variance for linear polynomial. Total sum of square for n - 1 degree of freedom is

$$\text{TOTAL SS} = yy' - \bar{y}\Sigma y_i = \Sigma y_i^2 - \bar{y}\Sigma y_i = 256 - 5*40$$

$$= 256-200 = 56. \tag{61}$$

$$\text{Regression SS} = 56-2.54545 = 53.45455. \tag{62}$$

$$\text{Regression MS} = \frac{\text{Regression SS}}{(P - 1)} \tag{63}$$

where SS = Sum of squares.

$$\text{Regression MS} = \frac{53.45455}{1} = 53.45455 \tag{64}$$

where MS = Mean sum

for linear Equation (p) = 2, one degree of freedom for $a_0$ and one for $a_1$, for quadratic Equation (p) will be equal 3

$$F = \frac{\text{Regression MS}}{(\text{Residual})^2/(n - p)} = \frac{(53.45455)(6)}{2.54545} = \frac{320.7273}{2.54545} = 126.00024 \tag{65}$$

$$\text{RESIDUAL} = \text{Residual degree of freedom} = n - p = 8 - 2 = 6 \tag{66}$$

$$\text{RESIDUAL SS} = \text{Residual sum of squares} = (\text{Residual})^2 = (2.54545) \tag{67}$$

Mean sum of squares for RESIDUAL (68)

$$\sigma^2 = \frac{(\text{Residual})^2}{\text{D}\cdot\text{F}} = \frac{2.54545}{6}$$

$$= 0.42424167. \tag{69}$$

# 6. POLYNOMIAL REGRESSION OPERATING INSTRUCTIONS

Operating instructions follow the hardware and data tape structure requirements.

Questions requiring a yes or no response are ended with a colon (:) or a question mark (?). The yes or no may be shortened to Y or N. When a number value is needed, the equal sign and the question mark (=?) will be displayed. Enter the value and press RETURN. The bell is activated to gain attention when operator action is necessary. When the bell rings, the requirement will be identified by a displayed message.

## 6.1 Program Loading

Insert the program and data tape for regression analysis. Next, place the overlay card on user definable keys. The program may be loaded automatically by pressing AUTO LOAD; or manually by typing FIND8, OLD, and RUN or pressing KEY 15 (INITIALIZE). The RETURN key must be pressed after each of the three typed entries when in manual code.

The polynomial regression program is contained on 6 tape files, number 8 through 13. The operating system, utility subroutine and user definable key assignments are contained in File 8, which always resides in memory during program execution. Files 9 through 13 contain specific portions of the program and are called by pressing the user definable keys.

Only one of these files may be in memory at any time. If the required portion of the program is not in memory, the operating system on File 8 will summon the correct file to perform the action requested.

## 6.2 Program Execution

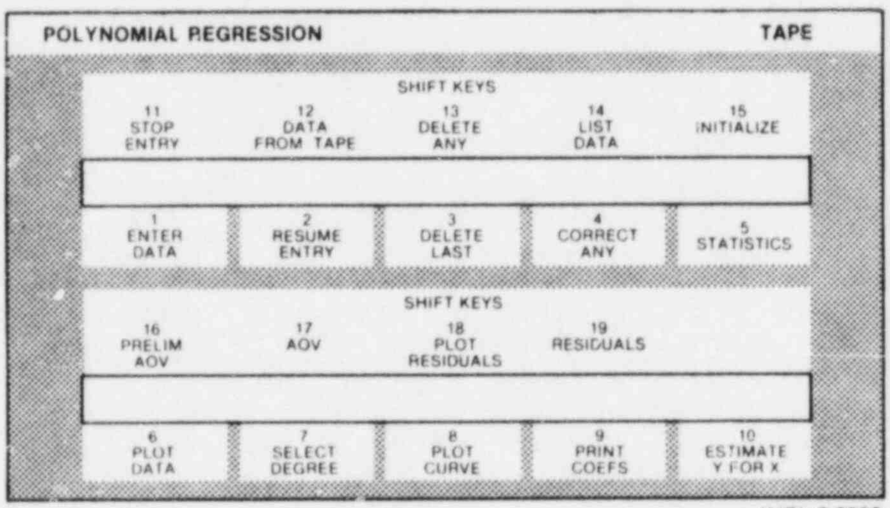
When the program has been loaded, the I/O light will go out. Data may now be entered from the keyboard or from existing magnetic tape files.

If ENTER DATA is pressed, the number of the first data file must be entered when requested, and a scratch file must be designated. DATA FROM

TAPE will cause the program to request the numeric designation of the first data file, the total number of data files, and the designation of a scratch file.

When the file information has been entered, program execution is begun by using the user definable keys. Subsequent data entries will be stored on consecutive magnetic tape files.

Interactivity between operator and machine is featured throughout this program, and keys may be used in any sequence. Following is a list of user definable keys. See Figure 1 for example of key card.




INEL 2 2368

Figure 1. Key card example.

USER DEFINABLE KEYS DESCRIPTION

| Key | Key No. | Function |
|---|---|---|
| ENTER DATA | 1 | Initializes the program and allows entry of data from the keyboard. |
| RESUME ENTRY | 2 | Permits data entry from the keyboard to be resumed. Program initialization is simplified, and data entries continue from the stopping point (refer to STOP ENTRY). |
| DELETE LAST | 3 | Deletes the last data point entered and requests reentry. Repetitive use permits deletion of more than one point. |

USER DEFINABLE KEYS DESCRIPTION (continued)

| Key | Key No. | Function |
|---|---|---|
| CORRECT ANY | 4 | Enables any data point to be corrected. |
| STATISTICS | 5 | Displays the mean, variance, minimum, maximum, and correlation for x and y. |
| PLOT DATA | 6 | Plots the data previously entered. |
| SELECT DEGREE | 7 | Allows selection of the degree of regression. Any degree which does not exceed that selected during initialization may be designated. |
| PLOT CURVE | 8 | Plots the data as does the PLOT DATA key, then graphs the present selected degree polynomial. |
| PRINT COEFS | 9 | Displays the coefficients, standard error, and T-statistic for the present selected degree regression. |
| ESTIMATE Y FOR X | 10 | Displays the y-estimate for any designated x value. The estimate is based on the current selected degree. |
| STOP ENTRY | 11 | Permits storage of data so the machine may be turned off without loss. RESUME ENTRY may be used later to continue data entry. |
| DATA FROM TAPE | 12 | Initializes program and allows entry of data from magnetic tape. Additional data may then be added from the keyboard. |
| DELETE ANY | 13 | Allows deletion of any data point. |
| LIST DATA | 14 | Displays a data table. |
| INITIALIZE | 15 | Initializes the Polynomial Regression program. |
| PRELIMINARY AOV | 16 | Displays the preliminary analysis of variance table to assist in selection of the degree of regression. |
| AOV | 17 | Displays the analysis of variance table for the present selected degree. |
| PLOT RESIDUALS | 18 | Produces a plot of x or y against the standardized residuals for the present selected degree. |
| RESIDUALS | 19 | Displays the table of x and y values, the y estimate and RESIDUALS. |

### 6.2.1 Output Options

Output options are as shown in Appendix A.

May be selected from the following list:

1. Descriptive statistics for x and y.

2. Analysis of variance table for each degree regression selected.

3. Regression curve plotted against the data.

4. List and plot of residuals.

5. Coefficients for each degree regression with standard error and t-test.

6. Estimated y for input x.

The degree of regression may be changed without reentering the data.

### 6.2.2 Hardware Requirements

A Tektronix 4050 series graphic system with 16K memory is required. 64K memory will permit handling of larger degree regressions. The peripheral device (the hard copy unit must be attached to exercise all the options available in the software). The program will execute without this device, but permanent copies of the output data will not be available.

### 6.2.3 Data Tape Structure

All data stored on tape must be in binary form. Data may be stored on one tape file or several, but files must be consecutive.

Data on tape must be arranged in the following manner in each data file:

FILE NUMBER OF POINTS FIRST SECOND ..............LAST

NUMBER ON THIS FILE POINT POINT ...............POINT

A point consists of an x, y pair. The program resides on File 1 and Files 8 through File 13. A scratch file, required for storage of normal equations and other variables, is most effective when it immediately precedes the first data file.

The most efficient program operation requires data storage on the program tape. Premarked files are available on the program tape as shown below:

| File Number | Number of Bytes | Recommended Use |
|---|---|---|
| 14* | 2,600 | Scratch File |
| 15 | 220 | Data File |
| 16 | 220 | Data File |
| -- | -- | -- |
| -- | -- | -- |
| -- | -- | -- |
| 84 | 2,200 | Data File |

If data files are stored on separate data tapes rather than on the program tape, tapes must occasionally be removed and replaced during execution. When another tape must be inserted, the bell sounds and a message is displayed to indicate the necessary tape change.

### 6.2.4 Internal Data Storage

The number of data points that may be stored internally is dependent upon the value of variable D(15). D(15) is located on line 3180 of file Number 9, and is set to 100. The variable may be increased to a larger value by altering the code in that line. If your system is not larger than 16K, it cannot be altered.

# 7. CONCLUSION

Polynomial regression for the Tektronix 4052 computer performs with a variety of options. Displays data on the screen that may be used to verify the accuracy of entries at any time. Simple statistics for X and Y are displayed, but caution is advised. Minimum and maximum are recalculated when data is plotted; not before. Because calculations are not accomplished prior to the plot, incorrect values will be shown if deletions have occurred since the last plot was displayed. Preliminary analysis of variance initiates and displays several statistics that are helpful in deciding which degree of regression to select. From the coefficient of the fit, and the standard error for each coefficient, standard error then may be used to set confidence intervals for the coefficients. If the fit for the present selected degree is not perfect, the T-statistics for each coefficient will also be displayed. R-square, a measure of the closeness of fit, is also calculated and displayed. The residuals are displayed and the difference plotted between the actual Y and the estimated Y value. All the results are hand calculated and verified.

# 8. REFERENCES

1. F. A. Graybill, An Introduction to Linear Statistical Models, Vol. 1, New York: McGraw-Hill, 1961.

2. F. A. Graybill, Introduction to Matrices with Applications in Statistics, Belmont California: Wadsworth Publishing Co., 1969.

3. R. W. Kopitzke, "Unpublished Notes."

4. R. W. Koptizke, T. J. Boardman, F. A. Graybill, "Least Squares Programs--A Look at the Square Root Procedure," The American Statistician, Vol. 29, pp. 64-66.

5. S. H. Wilkinson, The Algebraic Eigenvalue Problem, London: Oxford Press, 1965.

APPENDIX A

EXAMPLES OF OUTPUT

APPENDIX A

EXAMPLES OF OUTPUT

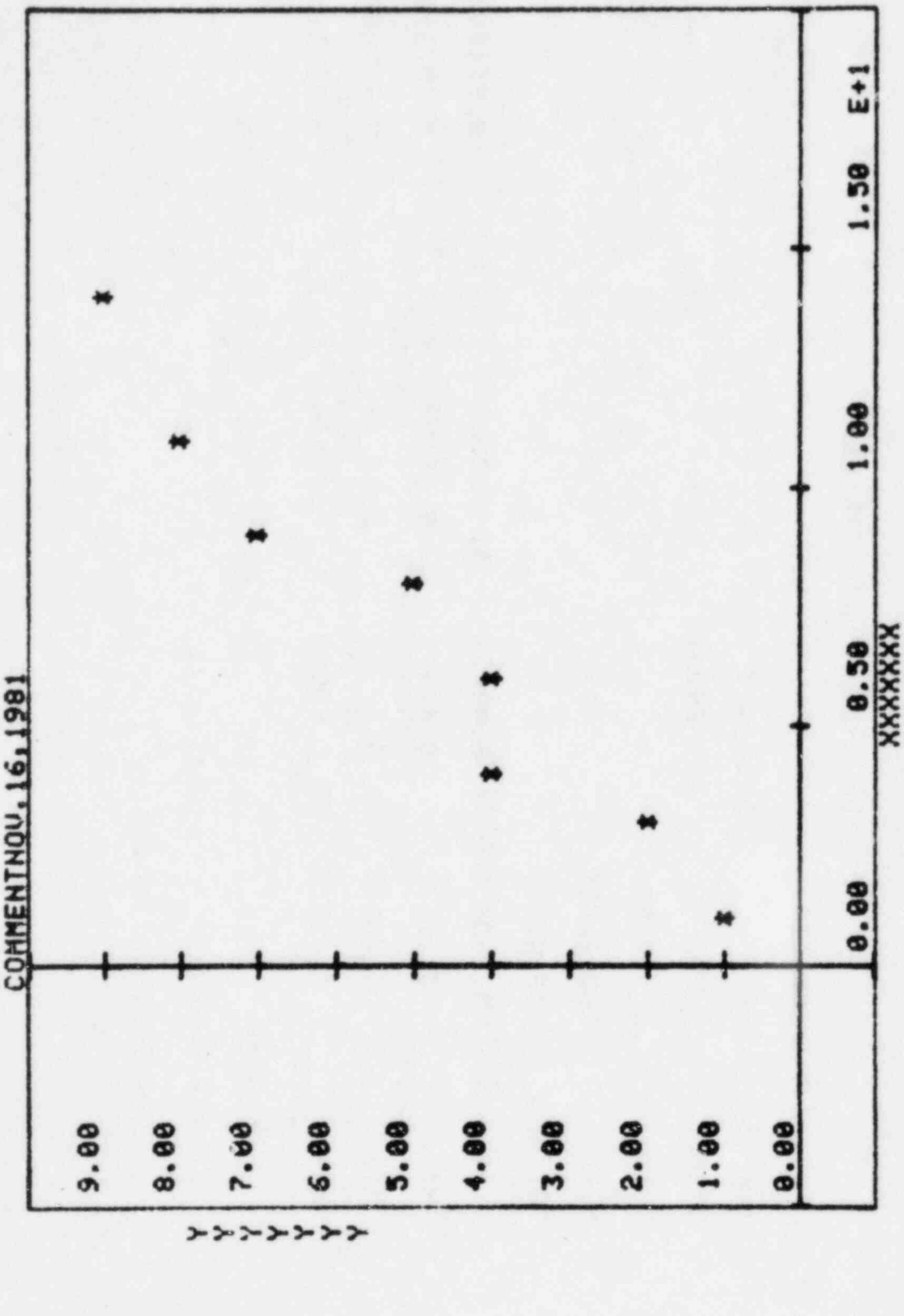
The following are examples that were used in this report.

***DATA***

COMMENT:NOV.16,1981

XI = XXXXXXX , YI = YYYYYYY

| I | XI | YI |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 3 | 2 |
| 3 | 4 | 4 |
| 4 | 6 | 4 |
| 5 | 8 | 5 |
| 6 | 9 | 7 |
| 7 | 11 | 8 |
| 8 | 14 | 9 |


COMMENTNOV.16,1981
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
YYYYYYY
0.00
0.50
1.00
1.50
E+1
XXXXXXX

***PRELIMINARY AOV TABLE***

COMMENT:NOV.16,1981

X-AXIS = XXXXXXX , Y-AXIS = YYYYYYY

| SOURCE | SS | RES ERROR | F | DF | R-SQUARE |
|---|---|---|---|---|---|
| TOTAL | 256.00000 | | | | |
| MEAN | 200.00000 | | | | |
| TOT ADJ | 56.00000 | | | | |
| X↑1 | 53.45455 | 0.42424 | 126.00000 | (1,6) | 0.954545 |
| X↑2 | 0.17170 | 0.47474 | 0.36184 | (1,5) | 0.9576[illegible]3 |
| X↑3 | 0.08120 | 0.57312 | 0.14169 | (1,4) | 0.959063 |
| X↑4 | 0.46360 | 0.60962 | 0.76048 | (1,3) | 0.967342 |

```
COMMENT:NOV.16,1981

MAX. DEGREE = 4 , X = XXXXXXX  ,  Y = YYYYYYY

XMIN = 1            YMIN = 1
XMAX = 14           YMAX = 9

N = 8

X MEAN = 7
VAR(X) = 18.8571428571

Y MEAN = 5
VAR(Y) = 8

COR(X,Y) = 0.977008420918
```

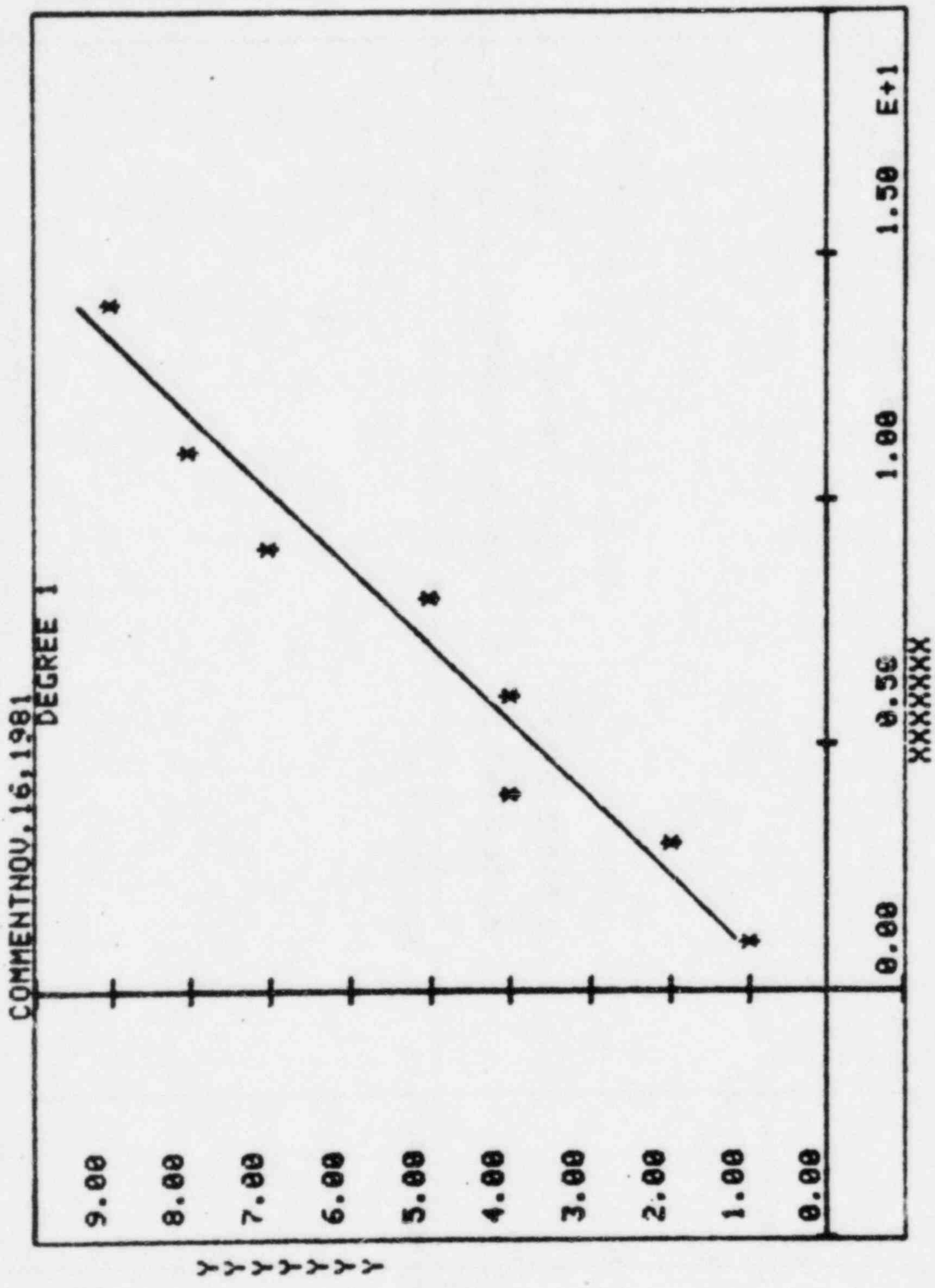
***ANALYSIS OF VARIANCE***

COMMENT:NOV.16,1981

X = XXXXXXX , Y = YYYYYYY

| SOURCE | DF | SS | MS | F |
|---|---|---|---|---|
| TOTAL | 7 | 56 | | |
| REG | 1 | 53.4545454545 | 53.4545454545 | 126 |
| RESID | 6 | 2.54545454546 | 0.424242424243 | |

R-SQUARE = 0.954545454545


COMMENTNOV.16,1981
DEGREE 1
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
YYYYYYY
0.00
0.50
1.00
1.50
E+1
XXXXXXX

***COEFFICIENTS***

COMMENT:NOV.16,1981

X = XXXXXXX , Y = YYYYYYY

| I | C0(I) | STD ERROR | T |
|---|---|---|---|
| 0 | 0.545454545455 | 0.458818360901 | 1.18882473603 |
| 1 | 0.636363636364 | 0.0566917785875 | 11.2249721603 |

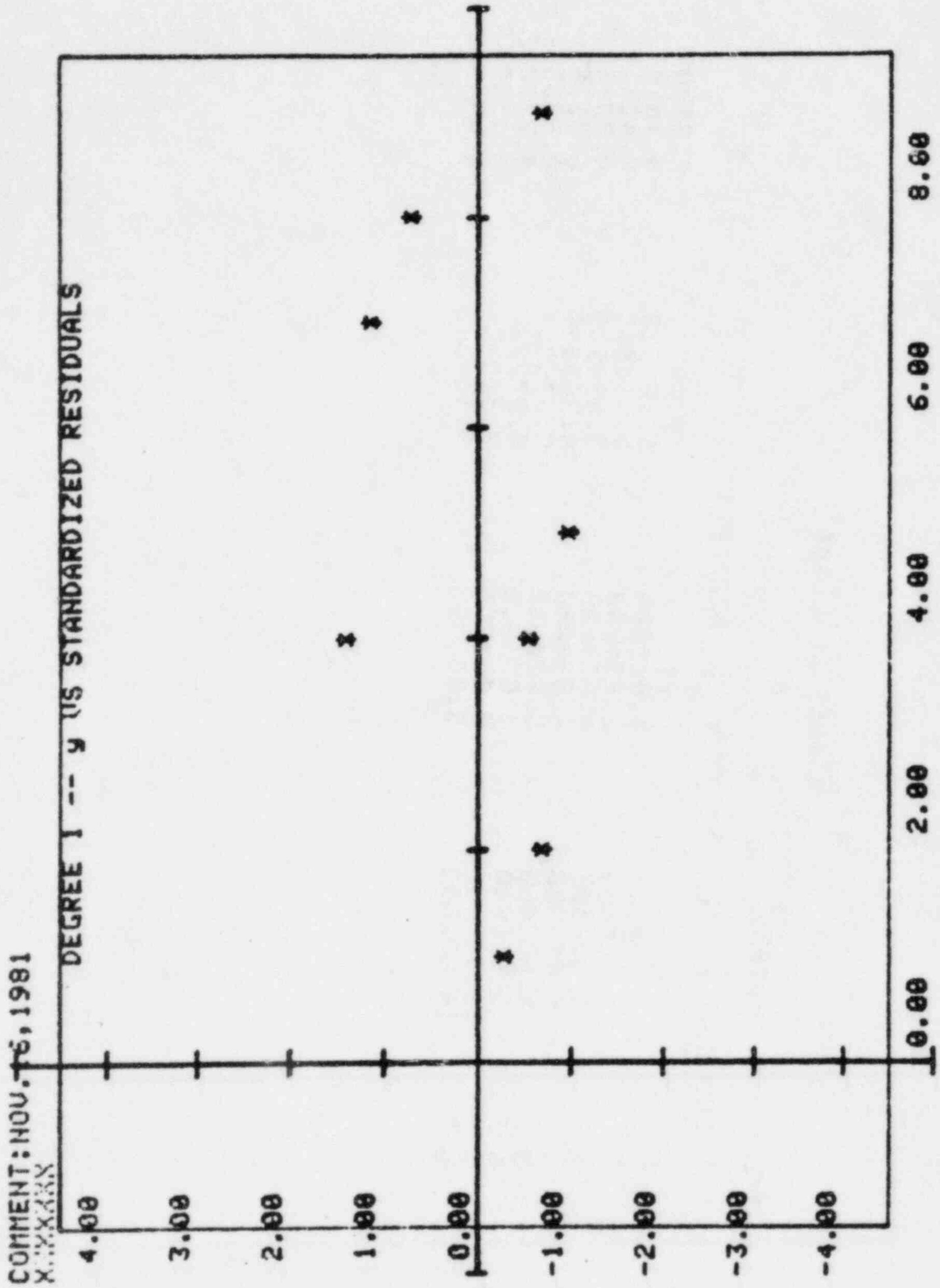
8 DATA POINTS    MAX DEG = 4

***RESIDUALS***

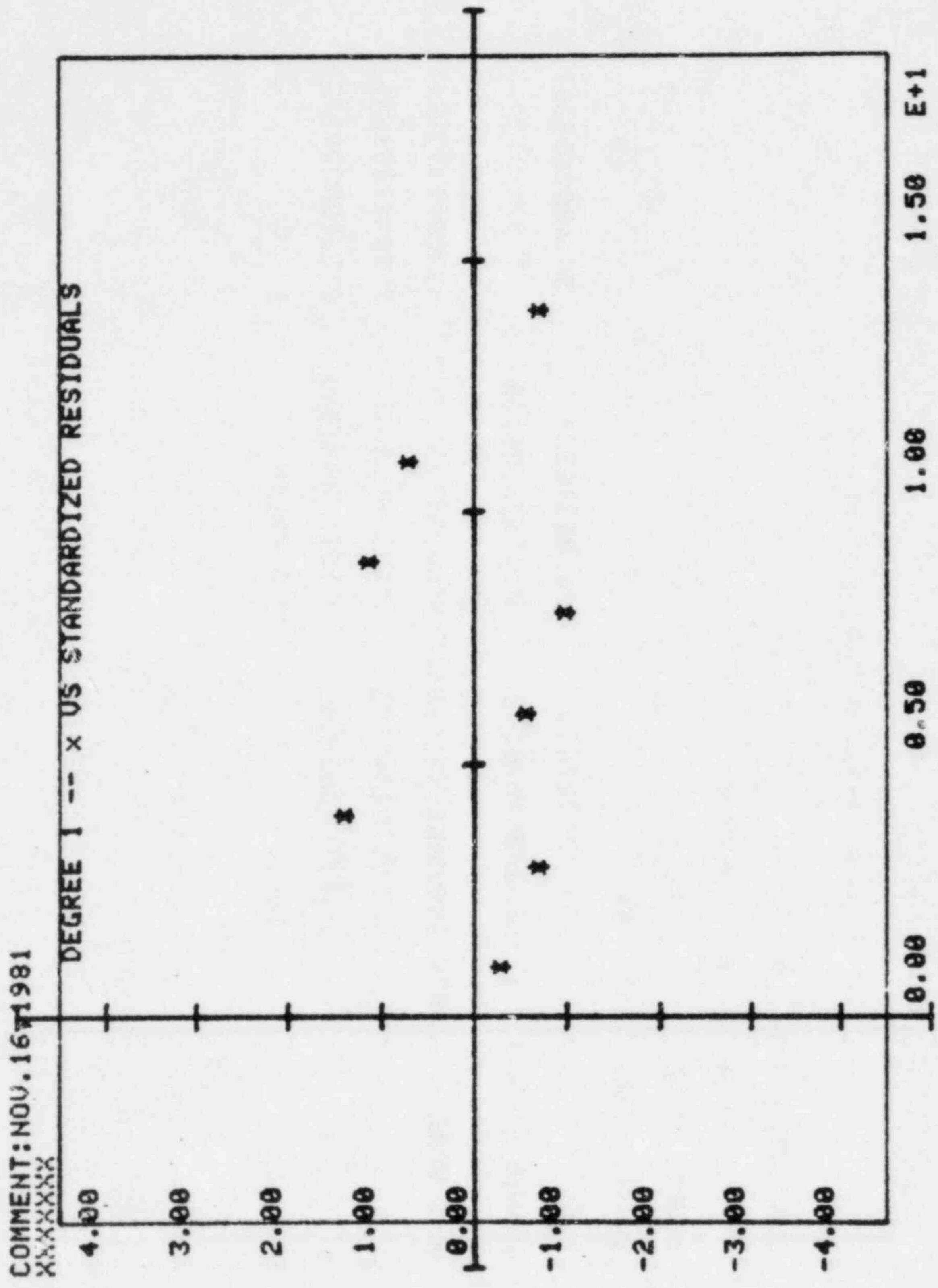
COMMENT:NOV.16,1981

XI = XXXXXXX , YI = YYYYYYY , DEGREE 1

| I | XI | YI | Y EST | RESIDUAL |
|---|---|---|---|---|
| 1 | 1.000000 | 1.000000 | 1.181818 | -0.181818 |
| 2 | 3.000000 | 2.000000 | 2.454545 | -0.454545 |
| 3 | 4.000000 | 4.000000 | 3.090909 | 0.909091 |
| 4 | 6.000000 | 4.000000 | 4.363636 | -0.363636 |
| 5 | 8.000000 | 5.000000 | 5.636364 | -0.636364 |
| 6 | 9.000000 | 7.000000 | 6.272727 | 0.727273 |
| 7 | 11.000000 | 8.000000 | 7.545455 | 0.454545 |
| 8 | 14.000000 | 9.000000 | 9.454545 | -0.454545 |



COMMENT:NOV. 16,1981
X:XXXXX
DEGREE I -- y VS STANDARDIZED RESIDUALS
4.00
3.00
2.00
1.00
0.00
-1.00
-2.00
-3.00
-4.00
0.00
2.00
4.00
6.00
8.00



COMMENT:NOV.16, 1981
XXXXXXX
DEGREE 1 -- X VS STANDARDIZED RESIDUALS
4.00
3.00
2.00
1.00
0.00
-1.00
-2.00
-3.00
-4.00
0.00
0.50
1.00
1.50
E+1

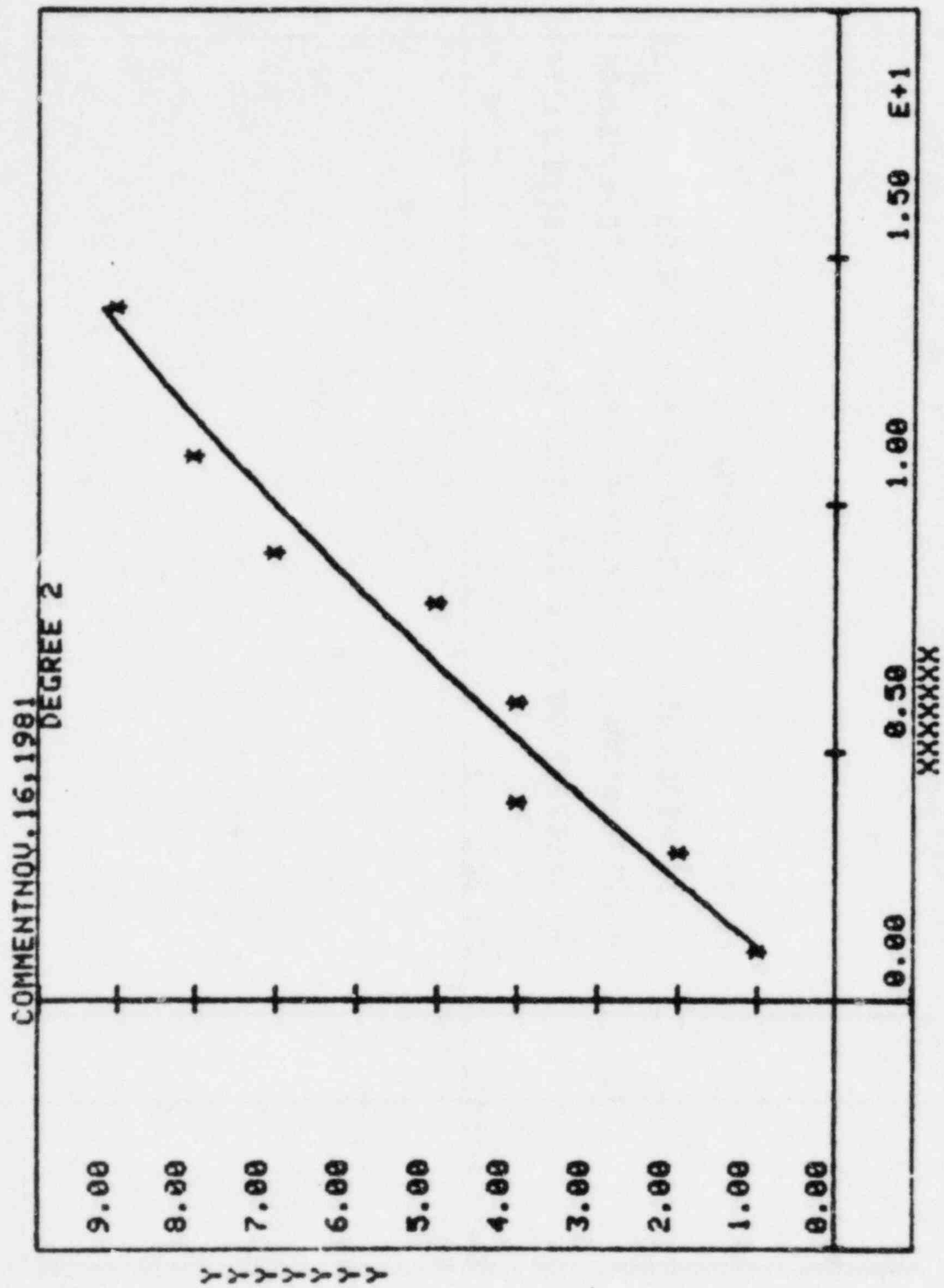
***ANALYSIS OF VARIANCE***

COMMENT:NOV.16,1981

X = XXXXXXX , Y = YYYYYYY

| SOURCE | DF | SS | MS | F |
|---|---|---|---|---|
| TOTAL | 7 | 56 | | |
| REG | 2 | 53.626324695 | 26.8131623475 | 56.4802656261 |
| RESID | 5 | 2.37367530502 | 0.474735061003 | |

R-SQUARE = 0.957612940982


COMMENTNOV.16,1981
DEGREE 2
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
YYYYYYY
0.00
0.50
1.00
1.50
E+1
XXXXXXX

***COEFFICIENTS***

COMMENT:NOV.16,1981

X = XXXXXXX , Y = YYYYYYY

| I | C0(I) | STD ERROR | T |
|---|---|---|---|
| 0 | 0.194808086205 | 0.758529033864 | 0.256823506429 |
| 1 | 0.772286845062 | 0.233782708526 | 3.30343527086 |
| 2 | -0.00917267788761 | 0.0152488277063 | -0.601533315496 |

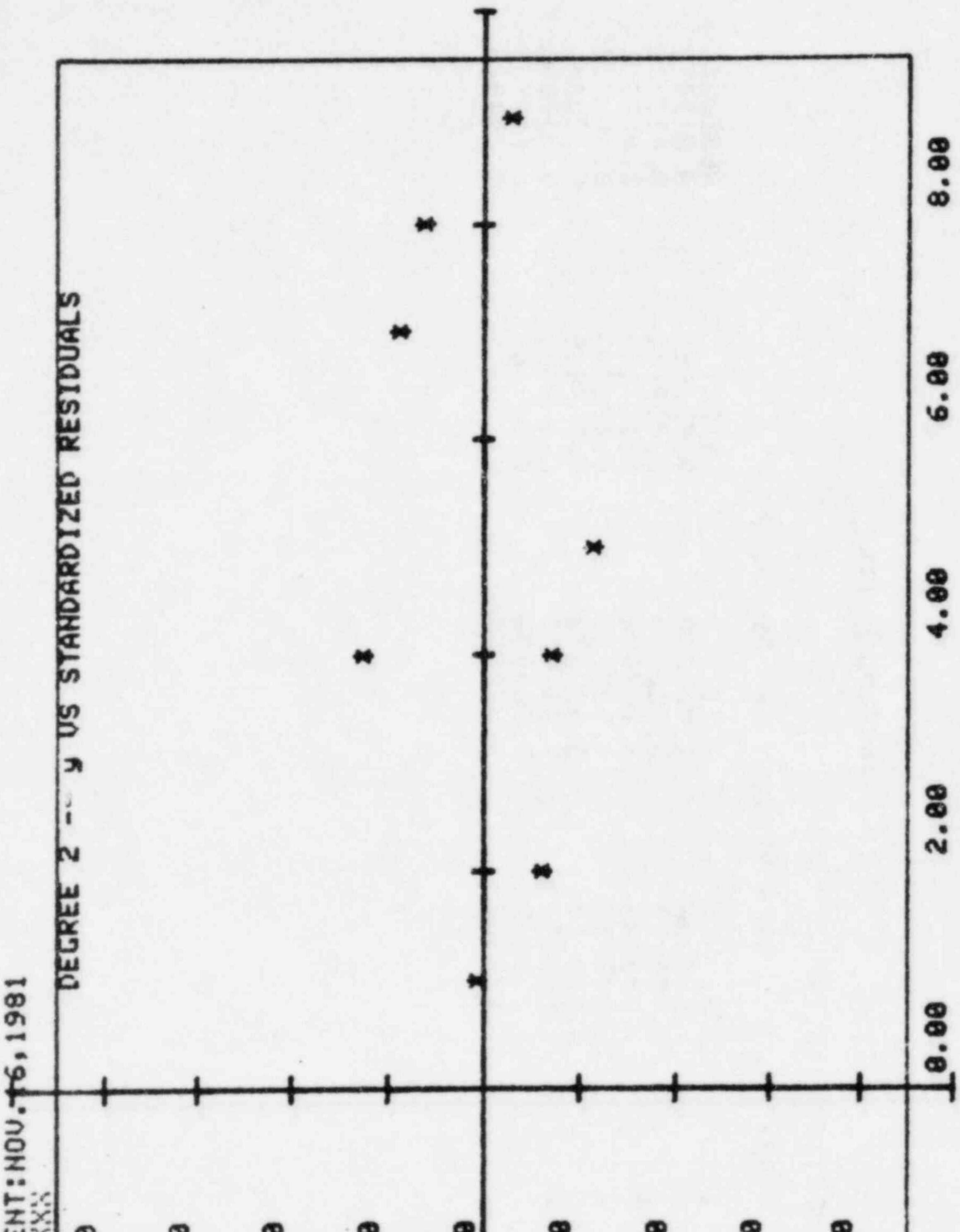
8 DATA POINTS    MAX DEG = 4

***RESIDUALS***

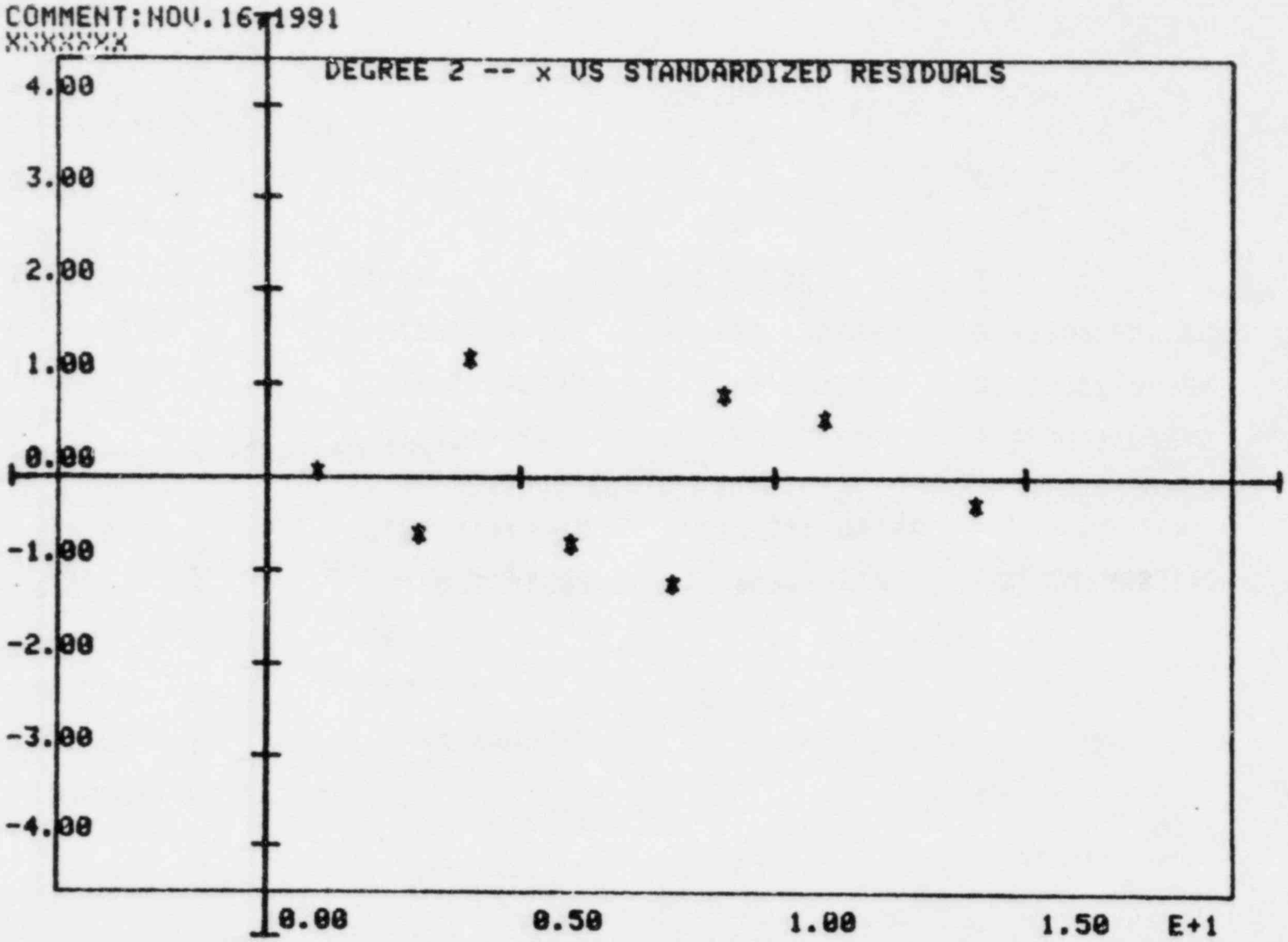
COMMENT:NOV.16,1981

XI = XXXXXXX , YI = YYYYYYY , DEGREE 2

| I | XI | YI | Y EST | RESIDUAL |
|---|---|---|---|---|
| 1 | 1.000000 | 1.000000 | 0.957921 | 0.042079 |
| 2 | 3.000000 | 2.000000 | 2.429112 | -0.429112 |
| 3 | 4.000000 | 4.000000 | 3.137189 | 0.862811 |
| 4 | 6.000000 | 4.000000 | 4.498308 | -0.498308 |
| 5 | 8.000000 | 5.000000 | 5.786045 | -0.786045 |
| 6 | 9.000000 | 7.000000 | 6.402396 | 0.597604 |
| 7 | 11.000000 | 8.000000 | 7.580061 | 0.419939 |
| 8 | 14.000000 | 9.000000 | 9.208968 | -0.208968 |



COMMENT:NOV.
6,1981
XXXXXX
DEGREE 2 - y VS STANDARDIZED RESIDUALS
4.00
3.00
2.00
1.00
0.00
-1.00
-2.00
-3.00
-4.00
0.00
2.00
4.00
6.00
8.00


COMMENT:NOV.16,1991
XXXXXXX
DEGREE 2 -- x VS STANDARDIZED RESIDUALS
4.00
3.00
2.00
1.00
0.00
-1.00
-2.00
-3.00
-4.00
0.00
0.50
1.00
1.50
E+1

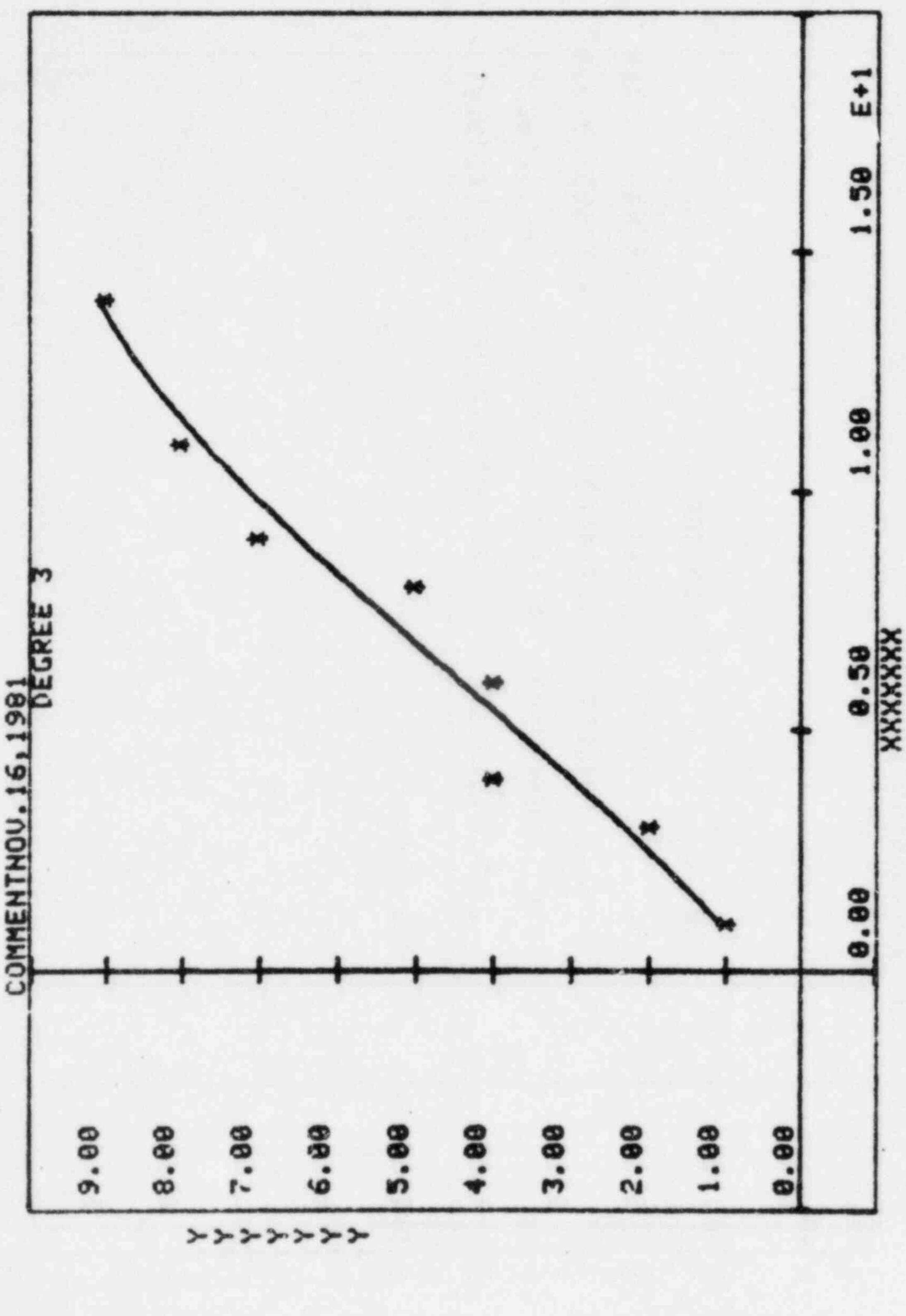
***ANALYSIS OF VARIANCE***

COMMENT:NOV.16,1981

X = XXXXXXX , Y = YYYYYYY

| SOURCE | DF | SS | MS | F |
|---|---|---|---|---|
| TOTAL | 7 | 56 | | |
| REG | 3 | 53.7075291602 | 17.9025097201 | 31.2370555113 |
| RESID | 4 | 2.29247083978 | 0.573117709945 | |

R-SQUARE = 0.959063020718


COMMENTNOV.16,1981
DEGREE 3
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
YYYYYYY
0.00
0.50
1.00
1.50
E+1
XXXXXXX

***COEFFICIENTS***

COMMENT:NOV.16,1981

X = XXXXXXX , Y = YYYYYYY

| I | C0(I) | STD ERROR | T |
|---|---|---|---|
| 0 | 0.542811599183 | 1.24472400042 | 0.436089927566 |
| 1 | 0.521171945097 | 0.714862225067 | 0.729052294024 |
| 2 | 0.0316209457267 | 0.109661261468 | 0.288351103237 |
| 3 | -0.00179542981502 | 0.00476980244526 | -0.376415970184 |

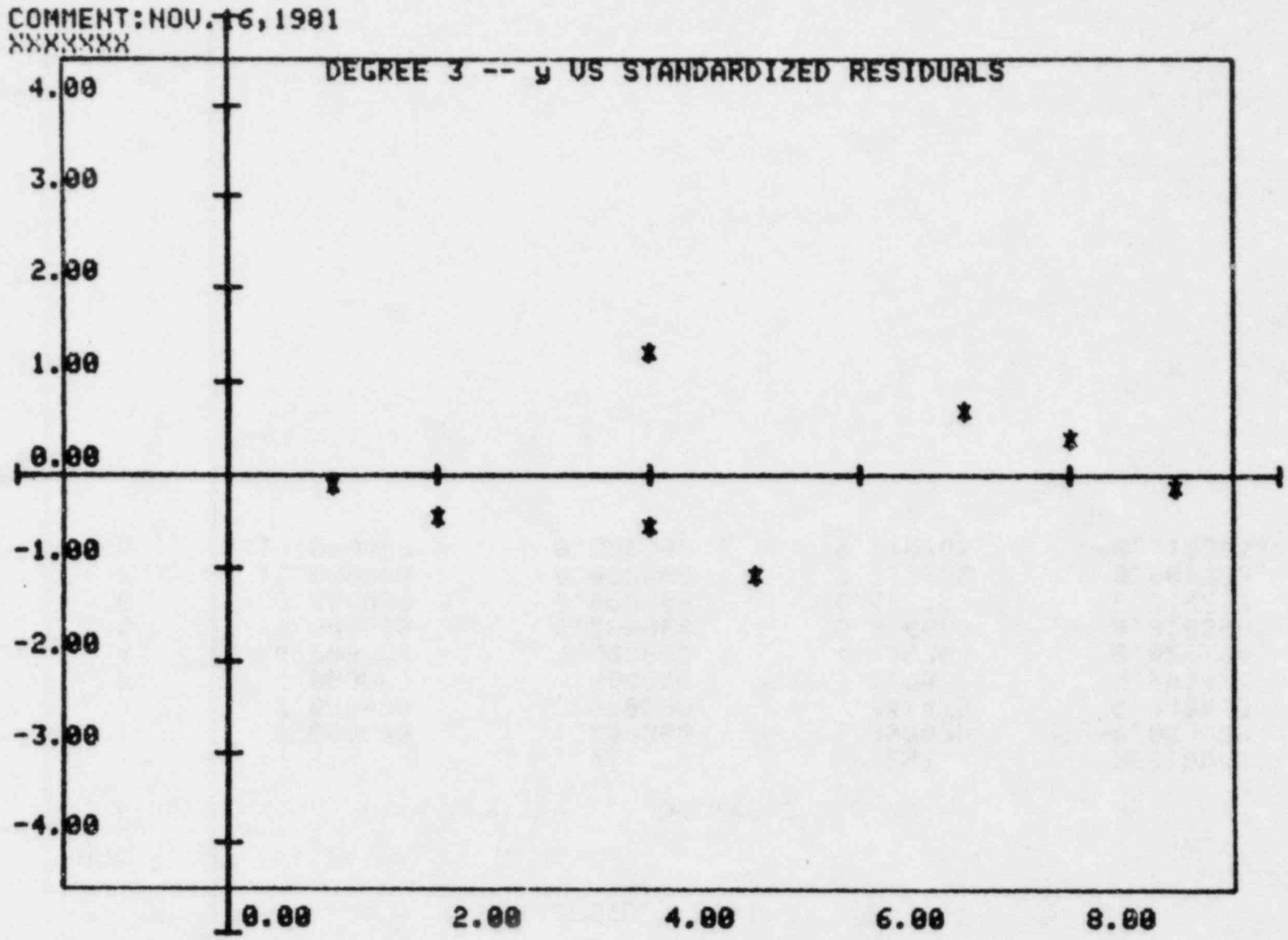
8 DATA POINTS MAX DEG = 4

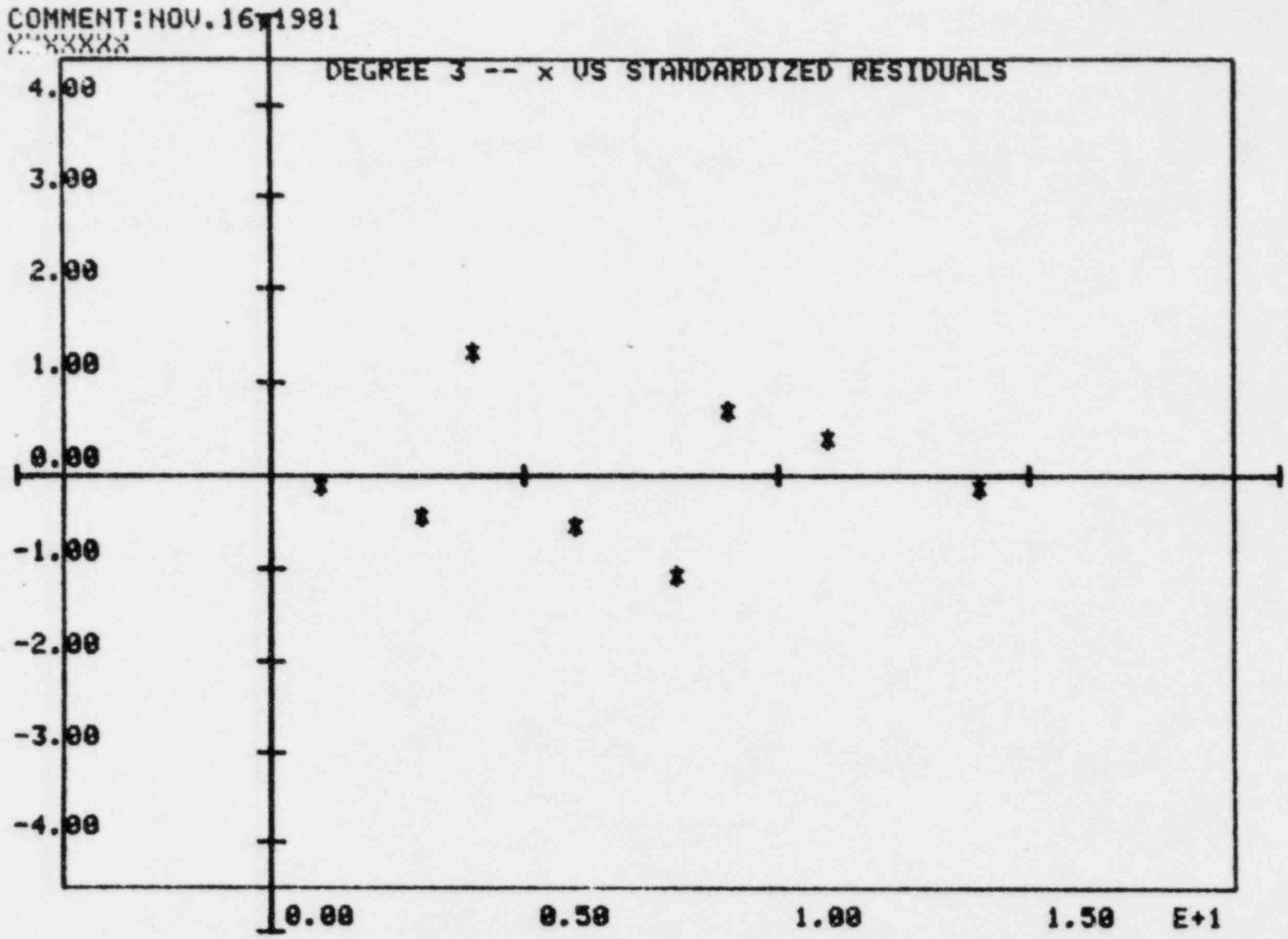
***RESIDUALS***

COMMENT:NOV.16,1981

XI = XXXXXXX , YI = YYYYYYY , DEGREE 3

| I | XI | YI | Y EST | RESIDUAL |
|---|---|---|---|---|
| 1 | 1.000000 | 1.000000 | 1.093809 | -0.093809 |
| 2 | 3.000000 | 2.000000 | 2.342439 | -0.342439 |
| 3 | 4.000000 | 4.000000 | 3.018527 | 0.981473 |
| 4 | 6.000000 | 4.000000 | 4.420384 | -0.420384 |
| 5 | 8.000000 | 5.000000 | 5.816668 | -0.816668 |
| 6 | 9.000000 | 7.000000 | 6.485787 | 0.514213 |
| 7 | 11.000000 | 8.000000 | 7.712120 | 0.287880 |
| 8 | 14.000000 | 9.000000 | 9.110265 | -0.110265 |


COMMENT:NOV.16,1981
XXXXXXX
DEGREE 3 -- y VS STANDARDIZED RESIDUALS
4.00
3.00
2.00
1.00
0.00
-1.00
-2.00
-3.00
-4.00
0.00
2.00
4.00
6.00
8.00



COMMENT:NOV.16,1981
DEGREE 3 -- x VS STANDARDIZED RESIDUALS
4.00
3.00
2.00
1.00
0.00
-1.00
-2.00
-3.00
-4.00
0.00
0.50
1.00
1.50
E+1

# APPENDIX B

## PROGRAM LISTING FOR 4052 TEKTRONIX MACHINE

## APPENDIX B

## PROGRAM LISTING FOR 4052 TEKTRONIX MACHINE

This appendix is used to perform polynomial regression analysis and is written in Basic for the Tektronix 4052 computer. The polynomial regression program is contained on 6 tape files, numbered 8 through 13. Program can be automatic loaded, or by find 8, OLD, and RUN. Files 9 through 13 contain specific portions of the program. Only one of these files may be in memory at any time. File 2 through 7 are dummy files.

```
100 INIT
150 PRINT "                    PROGRAM          TITLEJ"
160 PRINT
170 PRINT "                    POLYNOMIAL REGRESSIONJJJ"
180 PRINT
190 REM ***  FILE 2 THRU 7 ARE DUMMY FILES.
200 PRINT
210 FIND 8
220 OLD

  1 PRINT "LJPOLYNOMIAL  REGRESSIONIIJJIPRESS KEY #1, 2, OR 12"
  2 END
  4 F9=-1
  5 GO TO 2000
  8 F9=3
  9 GO TO 2000
  12 F9=4
  13 GO TO 2000
  16 F9=5
  17 GO TO 2000
  20 F9=12
  21 GO TO 2000
  24 F9=23
  25 GO TO 2000
  28 F9=24
  29 GO TO 2000
  32 F9=26
  33 GO TO 2000
  36 F9=27
  37 GO TO 2000
  40 F9=32
  41 GO TO 2000
  44 F9=7
  46 GO TO 2000
  48 F9=-2
  49 GO TO 2000
  52 F9=6
  53 GO TO 2000
  56 F9=8
  57 GO TO 2000
  60 GO TO 86
  64 F9=13
  65 GO TO 2000
  68 F9=25
  69 GO TO 2000
  72 F9=33
  73 GO TO 2000
  76 F9=34
  77 GO TO 2000
  80 F9=35
  81 GO TO 2000
  86 PRINT "LJINSERT PROGRAM TAPE AND PRESS ""RETURN"" GG";
  87 INPUT F$
  88 FIND 8
```

```
89 OLD
90 REM  **OPERATING SYSTEM**
100 IF D(17) THEN 150
110 IF D(17)=W0 THEN 150
120 PRINT "GGLINSERT PROGRAM TAPE AND PRESS ""RETURN"". ";
130 INPUT F$
140 D(17)=W0
150 GO TO INT(F9*0.1)+W1 OF 180,200,220,260
160 FIND @D(24):9
170 GO TO 280
180 FIND @D(24):10
190 GO TO 280
200 FIND @D(24):11
210 GO TO 280
220 IF F9<24 THEN 240
230 IF D(9)<>W1 THEN 200
240 FIND @D(24):12
250 GO TO 280
260 IF D(10)<W0 THEN 220
270 FIND @D(24):13
280 DELETE 2001,9998
290 APPEND @D(24):2000
300 GO TO 2000
310 C6=W1
320 RETURN
330 REM  **READ IN NORMAL EQUATIONS**
340 IF D(9)=W0 THEN 640
350 IF D(1)>W0 THEN 390
360 PRINT "JERROR:  REQUESTED OPERATION IS IMPOSSIBLE SINCE"
370 PRINT "         A SCRATCH BLOCK WAS NOT PROVIDED FOR X'X"
380 END
390 GOSUB 850
400 PRINT "JRESTORING NORMAL EQUATIONS FROM SCRATCH FILE #";D(W1)
410 FIND @D(25):D(W1)
420 DIM F0(W8)
430 C6=W0
440 FOR C9=W5 TO W8
450 F0(C9)=D(C9)
460 NEXT C9
470 F0(W4)=D(16)
480 D2=D(17)
490 READ @D(25):X$,Y$,Z$,N,D
500 D(17)=D2
510 D2=D(W4)+W1
520 D3=D2+W1
530 DELETE C,F,C0
540 DIM C(W4),F(D2,D3),C0(D2),D0(D(15)+W1,W2)
550 READ @D(25):F
560 IF F0(W4)=W0 THEN 600
570 FOR C9=W5 TO W8
580 D(C9)=F0(C9)
590 NEXT C9
600 IF NOT(C6) THEN 640
610 PRINT "JERROR IN READING FROM SCRATCH FILE -- THERE"
```

```
620 PRINT 'WAS NOT ENOUGH DATA IN THAT TAPE FILE._GGGGG'
630 END
640 RETURN
650 REM  **READ DATA POINTS ONE-BY-ONE**
660 IF F1>N OR D(23) THEN 680
670 IF D(W3)=W1 THEN 820
680 IF C5 THEN 790
690 C5=W1
700 D1=-D(W2)
710 C4=D(W2)-W1
720 C4=C4+W1
730 IF C4<D(W2)+D(W3) THEN 750
740 RETURN
750 FIND @D(25):C4
760 IF TYP(0)<>W3 THEN 720
770 READ @D(25):C6
780 C6=W0
790 READ @D(25):D6,D7
800 IF C6 THEN 720
810 RETURN
820 D6=D0(F1,W1)
830 D7=D0(F1,W2)
840 RETURN
850 IF D(17)<>W0 THEN 890
860 PRINT 'GGLIINSERT DATA TAPE AND PRESS ''RETURN''. ';
870 INPUT F$
880 D(17)=0.25
890 RETURN
2000 REM  INITIALIZE CONSTANTS
2010 PAGE
2020 DIM D(26),D$(20),F$(1),F0(8),X$(20),Y$(20),Z$(45)
2030 D=0
2035 D(18)=32
2040 D(24)=33
2045 D(26)=51
2050 PRINT
2060 SET KEY
2070 W0=0
2080 W1=1
2090 W2=2
2100 W3=3
2110 W4=4
2120 W5=5
2130 W6=6
2140 W7=7
2150 W8=8
2160 W9=9
2170 ON EOF (0) THEN 310
2180 C6=W0
2190 F0=W0
9999 GO TO 100
2000 REM  *********FILE #9 - ENTER % LOAD********************
2010 GO TO -F9 OF 2040,2190
2020 GO TO 100
2030 REM  **************************************************
```

```
2040 PRINT "L***ENTER DATA***J"
2050 D=W0
2060 PRINT "JYOUR DATA AND THE NORMAL EQUATIONS WILL BE AUTOMATICALLY";
2070 PRINT " STORED ON_TAPE.  YOU MAY STORE THEM ON THE STAT. VOL. 3";
2080 PRINT " TAPE ITSELF (FILES_#14 TO 84 ARE AVAILABLE)   OR  ON ";
2090 PRINT "YOUR OWN DATA TAPE.  NOTE: THE_PROGRAM WILL RUN FASTER ";
2100 PRINT "IF YOU STORE THEM ON THE STAT. VOL. 3 TAPE."
2110 PRINT "JDO YOU WANT TO STORE THEM ON THE STAT. VOL. 3 TAPE: ";
2120 INPUT F$
2130 IF F$="N" THEN 2160
2140 IF F$<>"Y" THEN 2110
2150 D(17)=W1
2160 GOSUB 2830
2170 GO TO 100
2180 REM  ************************************
2190 PRINT "L***LOAD DATA FROM TAPE***"
2200 D=W0
2210 PRINT "JIS THE POLYNOMIAL REGRESSION PROGRAM AVAILABLE"
2220 PRINT "          ON YOUR DATA TAPE IN FILES #8 TO #13 :  ";
2230 INPUT F$
2240 IF F$="N" THEN 2270
2250 IF F$<>"Y" THEN 2210
2260 D(17)=W1
2270 GOSUB 2830
2280 PRINT "JHOW MANY DIFFERENT DATA FILES DO YOU HAVE = ";
2290 INPUT F1
2300 D(W3)=INT(F1)
2310 IF D(W3)<W1 THEN 2280
2320 C5=W0
2330 GOSUB 850
2340 F1=1.0E+100
2350 C7=W1
2360 GOSUB 660
2370 IF C6 THEN 2610
2380 REM  ******************************************
2390 REM  *****FORM SKELETON OF NORMAL EQUATIONS*******
2400 C3=W1
2410 D(12)=D(12)+D7*D7
2420 FOR C2=W1 TO D2
2430 F(W1,C2)=F(W1,C2)+C3
2440 F(C2,D3)=F(C2,D3)+C3*D7
2450 C3=C3*D6
2460 NEXT C2
2470 FOR C2=W2 TO D2
2480 F(C2,D2)=F(C2,D2)+C3
2490 C3=C3*D6
2500 NEXT C2
2510 D(W5)=D(W5) MIN D6
2520 D(W6)=D(W6) MAX D6
2530 D(W7)=D(W7) MIN D7
2540 D(W8)=D(W8) MAX D7
2550 N=N+W1
2560 IF D(W3)>W1 THEN 2360
2570 IF N>D(15) THEN 2360
```

```
2580 DO(N,W1)=D6
2590 DO(N,W2)=D7
2600 GO TO 2360
2610 PAGE
2620 IF N<=D(15) THEN 2640
2630 D(23)=W1
2640 IF D(W1)<=WO THEN 2740
2650 GOSUB 850
2660 PRINT "JSTORING NORMAL EQUATIONS ON TAPE FILE #";D(W1)
2670 FIND @D(25):D(W1)
2672 IF F9<>2 THEN 2680
2673 READ @D(25):X$,Y$,Z$
2674 FIND @D(25):D(W1)
2675 C6=WO
2676 GO TO 2690
2680 C6=WO
2682 PRINT "ENTER X-AXIS LABEL & UNITS (20 CHAR.) = ";
2683 INPUT X$
2684 PRINT "ENTER Y-AXIS LABEL & UNITS (20 CHAR.) = ";
2685 INPUT Y$
2686 PRINT "ENTER COMMENT(TEST ID,DATE,SER# ETC.(45 CHAR.)) = ";
2687 INPUT Z$
2690 WRITE @D(25):X$,Y$,Z$,N,D,F
2700 CLOSE
2710 IF NOT(C6) THEN 2740
2720 PRINT "JJFATAL ERROR:  THIS FILE IS MARK'ED TOO SMALL!GGGGGG"
2730 END
2740 PRINT "JDO YOU WISH TO ENTER DATA BY HAND NOW: ";
2750 INPUT F$
2760 PRINT
2770 IF F$="Y" THEN 100
2780 IF F$<>"N" THEN 2640
2790 C8=N
2800 END
2810 REM  ****************************************
2820 REM  ****COMMON INITIALIZATION**************
2830 PRINT "JJ***INITIALIZATION***"
2840 D(24)=33
2850 D(25)=33
2860 IF D(24)=D(25) THEN 2880
2870 D(17)=W2
2880 D1=WO
2890 DELETE C,F,CO,DO,FO
2900 D(W5)=1.0E+200
2910 D(W6)=-D(W5)
2920 D(W7)=D(W5)
2930 D(W8)=D(W6)
2940 N=WO
2950 D3=(INT(MEMORY/8000)+W2)*W8
2960 D2=(D3<20)*12+(D3>20)*25
2970 PRINT "JNOTE: ALWAYS LEAVE A TAPE (PROGRAM OR DATA) IN"
2980 PRINT "       THE MACHINE UNTIL TOLD TO CHANGE TAPES!!!GGGG"
2990 PRINT "       JALSO, WHEN THE ""BUSY"" LIGHT IS ON YOU SHOULD NOT";
3000 PRINT " INTERRUPT_       THE PROGRAM  (BY PRESSING A ""USER ";
```

```
3010 PRINT " DEFINABLE KEY"" OR THE_        ""BREAK"" KEY)  UNLESS ";
3020 PRINT "THE BLINKING ? IS VISIBLE!!GGGG"
3030 PRINT "JENTER THE MAX. DEGREE FIT DESIRED = ";
3040 INPUT F1
3050 D(W4)=INT(F1)
3060 IF D(W4)>W0 AND D(W4)<=D2 THEN 3110
3070 PRINT "JERROR:  WITH YOUR ";D3;"K MACHINE THE MAX. DEGREE MUST";
3080 PRINT " BE IN THE RANGE:_I              1 <= MAX. DEGREE <= ";
3090 PRINT D2;"GGGGCG"
3100 GO TO 3030
3110 D2=D(W4)+W1
3120 D3=D2+W1
3130 D(10)=-W1
3140 D(13)=W1
3150 D(16)=W1
3160 PRINT
3170 F9=-F9
3180 D(15)=1000
3190 D(18)=32
3200 D(21)=100
3210 DIM C(W4),F(D2,D3),C0(D2),D0(D(15)+W1,W2),F0(W8)
3220 F=W0
3230 PRINT "JYOUR SCRATCH TAPE FILE MUST BE AT LEAST";
3240 PRINT 11*(26+D2*D3);" BYTES LONG."
3250 PRINT "YOUR DATA FILES MUST BE AT LEAST ";11*W2*D(15);
3260 PRINT " BYTES LONG."
3270 IF D(17)<>W1 THEN 3300
3280 PRINT "JDEFAULT FOR SCRATCH  FILE IS #14."
3290 PRINT "DEFAULT FOR 1ST DATA FILE IS #15."
3300 PRINT "J  ENTER THE SCRATCH  FILE # = ";
3310 INPUT F1
3320 D(W1)=INT(F1)
3330 PRINT "J  ENTER THE 1ST DATA FILE # = ";
3340 INPUT F1
3350 D(W2)=INT(F1)
3360 IF D(W2)<W1 THEN 3330
3370 IF D(W2)=D(W1) THEN 3330
3380 RETURN
2000 REM  **FILE #10 - ENTRY,CORRECT, ETC.**
2010 IF F9<W0 THEN 100
2020 IF D(13) OR F9<W7 THEN 2080
2030 C1=W0
2040 GOSUB 2290
2050 GOSUB 4700
2060 IF F9<>W7 THEN 2080
2070 END
2080 GO TO F9 OF 2780,2780,2100,4370,3290,3290,4650,4250
2090 GO TO 100
2100 IF D(16) THEN 2200
2110 PRINT "L***RESUME DATA ENTRY***"
2120 PRINT "JINSERT THE TAPE WITH YOUR DATA AND ENTER THE SCRATCH";
2130 PRINT " FILE # _II       (DEFAULT = 14)  = ";
2140 INPUT F1
2150 IF F1<W1 THEN 2120
```

```
2160 FIND F1
2170 READ @33:X$,Y$,Z$,N,D
2180 GOSUB 510
2190 D1=W0
2200 GOSUB 340
2210 IF D1=D(W2)+D(W3)-W1 THEN 2820
2220 C1=W0
2230 IF D(13) THEN 2250
2240 GOSUB 2290
2250 D1=D(W2)+D(W3)-W1
2260 GOSUB 2450
2270 GO TO 2820
2280 REM  **DUMPS DATA BUFFER ONTO TAPE BLOCK #D1**
2290 IF D1<D(W2) OR D1=>D(W2)+D(W3) THEN 2430
2300 IF C8>D(15) OR NOT(D9) THEN 2430
2310 GOSUB 850
2320 D(W1)=(D(W1)<>D1)*D(W1)
2330 PRINT "JSTORING ";C8;" DATA POINTS ON TAPE FILE #";D1
2340 FIND @D(25):D1
2350 C6=W0
2360 WRITE @D(25):C8
2370 IF C8<=0 THEN 2410
2380 DIM D0(C8,W2)
2390 WRITE @D(25):D0
2400 DIM D0(D(15)+W1,W2)
2410 CLOSE
2420 IF C6 THEN 4830
2430 RETURN
2440 REM  **READ ALL OF BLOCK #D1 INTO BUFFER**
2450 GOSUB 850
2460 IF D1<D(W2) THEN 2570
2470 PRINT "JREADING DATA POINTS FROM TAPE FILE #";D1
2480 FIND @D(25):D1
2490 C6=W0
2500 READ @D(25):C8
2510 IF C8<=W0 THEN 2560
2520 C8=C8 MIN D(15)
2530 DIM D0(C8+W1,W2)
2540 READ @D(25):D0
2550 DIM D0(D(15)+W1,W2)
2560 D9=C6
2570 RETURN
2580 REM  **FORM SKELETON OF NORMAL EQUATIONS**
2590 C3=C7
2600 D(12)=D(12)+C7*D7*D7
2610 D(13)=W0
2620 FOR C2=W1 TO D2
2630 F(W1,C2)=F(W1,C2)+C3
2640 F(C2,D3)=F(C2,D3)+C3*D7
2650 C3=C3*D6
2660 NEXT C2
2670 FOR C2=W2 TO D2
2680 F(C2,D2)=F(C2,D2)+C3
2690 C3=C3*D6
2700 NEXT C2
```

```
2710 IF C7<W0 THEN 2760
2720 D(W5)=D(W5) MIN D6
2730 D(W6)=D(W6) MAX D6
2740 D(W7)=D(W7) MIN D7
2750 D(W8)=D(W8) MAX D7
2760 RETURN
2770 REM  **BEGIN ENTRY BY HAND**
2780 PRINT "L***DATA ENTRY FROM THE KEYBOARD***"
2790 D1=-W1
2800 C8=W0
2810 REM  **RESUME ENTRY BY HAND**
2820 C1=W0
2830 C7=W1
2840 GOSUB 850
2841 REM ***  X-AXIS LABEL & Y-AXIS LABEL ***
2842 PRINT "ENTER X-AXIS LABEL & UNITS (20 CHAR. ) = ";
2843 INPUT X$
2845 PRINT "ENTER Y-AXIS LABEL & UNITS (20 CHAR. ) = ";
2846 INPUT Y$
2847 PRINT "ENTER COMMENT: TEST ID,DATE,SER#,ETC(45 CHAR. )) = ";
2848 INPUT Z$
2850 PRINT "JARE THE XI'S EVENLY SPACED: ";
2860 INPUT F$
2870 D(14)=W2
2880 IF F$="N" THEN 2930
2890 IF F$<>"Y" THEN 2850
2900 D(14)=W1
2910 PRINT "JENTER INITIAL X AND STEP SIZE = ";
2920 INPUT D(20),D(19)
2930 IF C8=W0 THEN 2960
2940 IF C8<D(15) THEN 3040
2950 GOSUB 2290
2960 GOSUB 3070
2970 D9=W1
2980 D1=D(W2)+D(W3)
2990 D(23)=D(W3)
3000 D(W3)=D(W3)+W1
3010 D0(W1,W1)=D6
3020 D0(W1,W2)=D7
3030 C8=W1
3040 GOSUB 3070
3050 GO TO 2940
3060 REM  **ENTER ONE POINT BY HAND**
3070 PRINT "J";N1W1)
3080 GO TO D(14) OF 3090,3200
3090 PRINT "   X = ";D(20),"  ENTER Y = ";
3100 INPUT D$
3110 D$=D$&",, 1E288"
3120 D6=VAL(D$)
3130 IF D6<>-1.0E+288 THEN 3160
3140 D(20)=D(20)+D(19)
3150 GO TO 3070
3160 D0(C8+W1,W2)=D6
3170 D0(C8+W1,W1)=D(20)
```

```
3180 D(20)=D(20)+D(19)
3190 GO TO 3220
3200 PRINT "  ENTER X,Y = ";
3210 INPUT D0(C8+W1,W1),D0(C8+W1,W2)
3220 D6=D0(C8+W1,W1)
3230 D7=D0(C8+W1,W2)
3240 GOSUB 2590
3250 C8=C8+W1
3260 N=N+W1
3270 RETURN
3280 REM  **CORRECT/DELETE ANY POINT**
3290 C1=W0
3300 PAGE
3310 IF F9=W6 THEN 3340
3320 PRINT "***CORRECT";
3330 GO TO 3350
3340 PRINT "***DELETE";
3350 PRINT " ANY POINT***"
3360 GOSUB 340
3370 PRINT "JENTER X, XEPS = ";
3380 INPUT D6,F6
3390 PRINT "ENTER Y, YEPS = ";
3400 INPUT D7,F7
3410 F6=ABS(F6)
3420 F7=ABS(F7)
3430 F4=W1
3440 GOSUB 3780
3450 IF F4=W0 THEN 3370
3460 IF NOT(D9) THEN 3550
3470 IF F9=W6 THEN 3530
3480 PRINT "ENTER NEW X,Y = ";
3490 INPUT D6,D7
3500 C7=W1
3510 GOSUB 2590
3520 GO TO 3540
3530 N=N W1
3540 IF D9 THEN 3570
3550 PRINT "JCAN'T CORRECT OR DELETE THIS PAIR SINCE TAPE_FILE #";
3560 PRINT ABS(D1);"  HAS MORE THAN ";D(15);" X,Y PAIRS ON IT."
3570 D1=W0
3580 GO TO 3370
3590 IF F9=W6 THEN 3630
3600 D0(F3,W1)=D6
3610 D0(F3,W2)=D7
3620 GO TO 3720
3630 IF F3=C8 THEN 3700
3640 F5=F3
3650 FOR F2=F3+W1 TO C8
3660 D0(F5,W1)=D0(F2,W1)
3670 D0(F5,W2)=D0(F2,W2)
3680 F5=F2
3690 NEXT F2
3700 C8=C8-W1
3710 PRINT "I           DELETEDJ"
3720 C7=-W1
```

```
3730 D6=F6
3740 D7=F7
3750 GOSUB 2590
3760 GO TO 3370
3770 REM  **LOOK FOR (X,Y) = (D6,D7)**
3780 IF D1<D(W2) THEN 3840
3790 GOSUB 4010
3800 IF F3<=C8 THEN 3960
3810 IF D(W3)<W2 THEN 3920
3820 IF D(13) THEN 3840
3830 GOSUB 2290
3840 F4=-F4
3850 D1=D(W2)
3860 GOSUB 2450
3870 GOSUB 4010
3880 IF F3<=C8 THEN 3960
3890 D1=D1+W1
3900 IF D1<D(W2)+D(W3) THEN 3860
3910 D1=D1-W1
3920 PRINT "JNO DATA POINT IS EQUAL TO THE ENTERED"
3930 PRINT "  X,Y VALUES:  ";D6;D7;"   TRY AGAIN."
3940 F4=W0
3950 RETURN
3960 F6=D6
3970 F7=D7
3980 PRINT "JACTUAL X,Y VALUES FOUND = ";D6;D7
3990 RETURN
4000 REM  **SEARCH BUFFER FOR D6,D7**
4010 IF C8<=W0 THEN 4060
4020 FOR F3=W1 TO C8
4030 IF ABS(D0(F3,W1)-D6)>F6 THEN 4050
4040 IF ABS(D0(F3,W2)-D7)<=F7 THEN 4210
4050 NEXT F3
4060 F3=C8+W1
4070 IF C8<D(15) OR D9 THEN 4190
4080 F3=D(15)+W1
4090 GOSUB 2450
4100 IF ABS(D0(F3,W1)-D6)>F6 THEN 4120
4110 IF ABS(D0(F3,W2)-D7)<=F7 THEN 4200
4120 F3=F3+W1
4130 READ @D(25):C2,C3
4140 IF C6 THEN 4190
4150 IF ABS(C2-D6)>F6 OR ABS(C3-D7)>F7 THEN 4120
4160 D6=C2
4170 D7=C3
4180 C8=F3
4190 RETURN
4200 C8=F3
4210 D6=D0(F3,W1)
4220 D7=D0(F3,W2)
4230 RETURN
4240 REM  **PRINT USER DATA**
4250 IF NOT(D(23)) AND D(W3)<W2 THEN 4270
4260 GOSUB 850
4270 PRINT "LI***DATA***";"J"
```

```
4272 PRINT "COMMENT:";Z$;"J"
4275 PRINT " XI = ";X$;"  ,  YI = ";Y$;"J"
4282 PRINT " JII XII YI"
4290 C5=W0
4300 FOR F1=W1 TO N
4310 GOSUB 660
4320 PRINT F1,D6,D7
4330 NEXT F1
4340 PRINT "JJ"
4350 END
4360 REM  **DELETE LAST **
4370 GOSUB 340
4380 IF D1=D(W2)+D(W3)-W1 THEN 4470
4390 GOSUB 2290
4400 D1=D(W2)+D(W3)-W1
4410 C6=W0
4420 GOSUB 2450
4430 IF D9 THEN 4470
4440 IF N=W0 THEN 2960
4450 PRINT "JCAN'T DELETE LAST PAIR SINCE THE TAPE FILE IS TOO LONG.GG"
4460 END
4470 IF C8>W0 THEN 4510
4480 IF N<W1 THEN 2930
4490 D(W3)=D(W3)-W1
4500 GO TO 4400
4510 D6=D0(C8,W1)
4520 D7=D0(C8,W2)
4530 C7=-W1
4540 GOSUB 2590
4550 C7=W1
4560 C8=C8-W1
4570 N=N-W1
4580 D(20)=D(20)-D(19)
4590 PRINT "***PAIR ";N+W1;" DELETED***"
4600 IF C8>W0 THEN 3040
4610 D(W3)=D(W3)-W1
4620 D1=-W1
4630 GO TO 2930
4640 REM  **PREPARE FOR STOP**
4650 C1=W0
4660 GOSUB 2290
4670 GOSUB 4700
4680 END
4690 REM  **STORE NORMAL EQUATIONS**
4700 IF D(13) THEN 4820
4710 IF D(W1)<=W0 THEN 4810
4720 GOSUB 850
4730 PRINT "JYOUR DATA IS ON ";D(W3);" TAPE FILES (#";D(W2);
4740 PRINT " TO #";D(W2)+D(W3)-W1;")."
4750 PRINT "JSTORING NORMAL EQUATIONS ON TAPE FILE #";D(W1)
4760 FIND @D(25):D(W1)
4770 C6=W0
4780 WRITE @D(25):X$,Y$,Z$,N,D,F
4790 CLOSE
4800 IF C6 THEN 4830
```

```
4810 D(13)=W1
4820 RETURN
4830 PRINT "JJFATAL ERROR:  THIS FILE IS MARK'ED TOO SMALL!GGGGGG"
4840 END
9999 GO TO 100
2000 REM  *****FILE #11 - STAT, PRELIM AOV, REDUCTIN*****
2010 IF F9=23 THEN 100
2020 IF F9<W3 THEN 100
2030 IF F9=W8 THEN 100
2040 IF F9=W7 THEN 2080
2050 IF F9>W7 THEN 2090
2060 GOSUB 340
2070 GO TO 100
2080 END
2090 IF F9=12 THEN 3358
2100 GOSUB 2150
2110 IF F9<>13 THEN 100
2120 GO TO 2910
2130 REM  ****************************************************
2140 REM  ***FILL X'X, REDUCE, AND INVERT***********************
2150 IF D(W9)=W1 THEN 2880
2160 GOSUB 340
2170 PRINT "LJI***""INVERTING"" MATRIX***J"
2180 REM  FILL NORMAL EQUATIONS
2190 F5=D(W4)
2200 IF F5<W2 THEN 2280
2210 F4=W1
2220 FOR F1=W2 TO F5
2230 FOR F2=F1 TO F5
2240 F(F1,F2)=F(F4,F2+W1)
2250 NEXT F2
2260 F4=F1
2270 NEXT F1
2280 C(W1)=F(W1,W2)
2290 C(W2)=F(W2,W2)
2300 C(W3)=F(W1,D3)
2310 C(W4)=F(W2,D3)
2320 D(W' )=-W1
2330 D( .0)=-W1
2340 D(11)=W0
2350 REM  PERFORM 'SQUARE-ROOT' REDUCTION
2360 F(W1,W1)=SQR(F(W1,W1))
2370 F3=W1/F(W1,W1)
2380 FOR F1=W2 TO D3
2390 F(W1,F1)=F(W1,F1)*F3
2400 NEXT F1
2410 C6=D(12)-F(W1,D3)^W2
2420 F4=W1
2430 FOR F1=W2 TO D2
2440 FOR F2=F1 TO D3
2450 FOR F3=W1 TO F4
2460 F(F1,F2)=F(F1,F2)-F(F3,F1)*F(F3,F2)
2470 NEXT F3
2480 NEXT F2
2490 F3=F(F1,F1)
```

```
2500 IF F3<1.0E-10 THEN 2620
2510 IF C6<W0 THEN 2620
2520 C6=C6-F(F1,D3)^W2/F3
2530 F(F1,F1)=SQR(F3)
2540 F3=W1/F(F1,F1)
2550 FOR F2=F1+W1 TO D3
2560 F(F1,F2)=F(F1,F2)*F3
2570 NEXT F2
2580 F4=F1
2590 NEXT F1
2600 GO TO 2740
2610 REM  REDUCE DEGREE - BECAUSE OF INSTABILITY
2620 F5=F4-W1
2630 D2=F4
2640 D(W4)=F5
2650 FOR F2=W1 TO D2
2660 F(F2,F1)=F(F2,D3)
2670 NEXT F2
2680 D3=F1
2690 DIM C0(F4)
2700 PRINT "IJMAX. DEGREE REDUCED TO ";F5;"GGGGGG"
2710 FOR F2=W1 TO W5*225
2720 NEXT F2
2730 REM INVERSION
2740 FOR F2=D2 TO W2 STEP -W1
2750 F(F2,F2)=W1/F(F2,F2)
2760 F4=F2
2770 FOR F3=F2-W1 TO W1 STEP -W1
2780 C2=W0
2790 FOR F1=F4 TO F2
2800 C2=C2-F(F3,F1)*F(F1,F2)
2810 NEXT F1
2820 F(F3,F2)=C2/F(F3,F3)
2830 F4=F3
2840 NEXT F3
2850 NEXT F2
2860 F(W1,W1)=W1/F(W1,W1)
2870 D(W9)=W1
2880 RETURN
2890 REM  ***********************************************************
2900 REM  *********PRELIM. AOV TABLE***************************
2910 PRINT "LI ***PRELIMINARY AOV TABLE***J"
2913 PRINT "COMMENT:";Z$;"J"
2915 PRINT "X-AXIS = ";X$;"  ,  Y-AXIS = ";Y$;"J"
2920 PRINT "SOURCE          SS          RES ERROR          F          DF";
2930 PRINT "          R-SQUARE"
2940 F4=F(W1,D3)
2950 F4=F4*F4
2960 PRINT "JTOTAL  ";
2970 F2=D(12)
2980 GOSUB 3290
2990 PRINT "J_MEAN   ";
3000 F2=F4
3010 GOSUB 3290
3020 F3=D(12)-F4
```

```
3030 PRINT "┘_TOT ADJ";
3040 F2=F3
3050 GOSUB 3290
3060 PRINT
3070 F5=W0
3080 FOR F1=W2 TO D2
3090 F2=F(F1,D3)
3100 F2=F2*F2
3110 F5=F5+F2
3120 C2=N-F1
3130 C3=W0
3140 F4=W0
3150 IF C2=W0 THEN 3180
3160 F4=(F3-F5)/C2
3170 C3=F2/F4
3180 PRINT " ┘X^";F1-W1;"    ";
3190 GOSUB 3290
3200 F2=F4
3210 GOSUB 3290
3220 F2=C3
3230 GOSUB 3290
3240 PRINT USING "3X,""(1,"",FD,"")""":C2
3250 PRINT USING "60X,""Ŀ"",-D.6D":F5/F3
3260 NEXT F1
3270 PRINT "┘┘"
3280 END
3290 IF ABS(F2)=>1000 OR ABS(F2)<0.01 AND F2<>W0 THEN 3320
3300 PRINT USING "3X,-3D.5D,S":F2
3310 RETURN
3320 PRINT USING "3X,2E,S":F2
3330 RETURN
3340 REM  ***********************************************************
3350 REM  **************PRINT SIMPLE STATISTICS************************
3358 PRINT "LCOMMENT:";Z$;"┘"
3360 PRINT "MAX. DEGREE = ";D(W4);" , X = ";X$;"  ,  Y = ";Y$;"┘"
3370 PRINT "XMIN = ";D(W5),"YMIN = ";D(W7)
3380 PRINT "XMAX = ";D(W6),"YMAX = ";D(W8)
3390 PRINT "┘N = ";N
3400 IF D(W9)<>W0 THEN 3470
3410 C(W1)=F(W1,W2)
3420 C(W4)=F(W2,D3)
3430 C(W3)=F(W1,D3)
3440 C(W2)=F(W1,W3)
3450 IF D3>W3 THEN 3470
3460 C(W2)=F(W2,W2)
3470 F4=C(W1)/N
3480 C2=N-W1
3490 F3=(C(W2)-F4*N*F4)/C2
3500 PRINT "┘X MEAN = ";F4
3510 PRINT "VAR(X) = ";F3
3520 F5=C(W3)/N
3530 F2=(D(12)-F5*N*F5)/C2
3540 PRINT "┘Y MEAN = ";F5
3550 PRINT "VAR(Y) = ";F2
```

```
3560 F1=(C(W4)-F4*N*F5)/SQR(F2*F3)/C2
3570 PRINT "JCOR(X,Y) = ";F1;"JJ"
3580 END

2000 REM  *****FILE #12 - AOV, COEF, PLOT DATA, ETC*****
2010 IF F9=23 THEN 3000
2020 IF F9>23 THEN 2090
2030 IF F9<W3 THEN 100
2040 IF F9>W7 THEN 100
2050 IF F9<>W7 THEN 2070
2060 END
2070 GOSUB 340
2080 GO TO 100
2090 IF D(W9)=W1 THEN 2120
2100 GOSUB 340
2110 GO TO 100
2120 IF D(10)=>W0 THEN 2150
2130 IF F9=24 THEN 2190
2140 GOSUB 2230
2150 IF F9>30 THEN 100
2160 GO TO F9-23 OF 2190,2530,3510,2780
2170 REM  *******************************************
2180 REM  ******SELECT DEGREE************************
2190 GOSUB 2230
2200 END
2210 REM  *******************************************
2220 REM  ***ENTER DEGREE, CALC. COEF, AND CALC. SIGMA*
2230 PAGE
2240 PRINT "SELECT DEGREE (<=";D(W4);") OF REGRESSION = ";
2250 INPUT F1
2260 D(10)=-W1
2270 D(11)=W0
2280 D4=INT(ABS(F1))+W1
2290 PRINT
2300 IF D4>D2 THEN 2240
2310 D5=W0
2320 FOR F1=W1 TO D4
2330 F3=W0
2340 FOR F2=F1 TO D4
2350 F3=F3+F(F1,F2)*F(F2,D3)
2360 NEXT F2
2370 CO(F1)=F3
2380 D5=D5+F(F1,D3)^W2
2390 NEXT F1
2400 IF D4<=W1 THEN 2440
2410 C2=F(W1,D3)^W2
2420 PRINT "JR-SQUARE FOR DEGREE ";D4-W1;" FIT = ";(D5-C2)/(D(12)-C2)
2430 PRINT "J"
2440 C2=D(12)-D5
2450 D5=W0
2460 IF C2<D(12)*1.0E-10 THEN 2490
2470 IF D4=>N THEN 2490
2480 D5=SQR(C2/(N-D4))
2490 D(10)=D4-W1
```

```
2500 RETURN
2510 REM  *****************************************************************
2520 REM  *********CALC. AND PRINT AOV TABLE*********************************
2530 PRINT "LI***ANALYSIS OF VARIANCE***JJ"
2540 IF D(10) THEN 2574
2550 PRINT "J  AN AOV TABLE DOES NOT EXIST WHEN THE SELECTED ";
2560 PRINT "DEGREE IS ZERO.JJ"
2570 END
2574 PRINT "COMMENT:";Z$;"J"
2575 PRINT "X = ";X$;"  ,  Y = ";Y$;"J"
2580 PRINT "SOURCE   DFI  SSI  MSI  F"
2590 F5=D(12)-F(W1,D3)^W2
2600 PRINT "JTOTAL     ";N-W1,F5
2610 F4=W0
2620 IF D4<W2 THEN 2660
2630 FOR F1=W2 TO D4
2640 F4=F4+F(F1,D3)^W2
2650 NEXT F1
2660 F3=D5*D5
2670 F2=F4/D(10)
2680 C2=W0
2690 IF F3=W0 THEN 2710
2700 C2=F2/F3
2710 PRINT "JREG       ";D(10),F4,F2,C2
2720 PRINT "JRESID     ";N-D4,F5-F4,F3
2730 PRINT "JJR-SQUARE = ";F4/F5
2740 PRINT "JJ"
2750 END
2760 REM  **************************************************************
2770 REM  ***********PRINT COEFS.***************************
2780 PRINT "LI***COEFFICIENTS***";"J"
2784 PRINT "COMMENT:";Z$;"J"
2785 PRINT "X = ";X$;"  ,  Y = ";Y$;"J"
2790 PRINT "JJII CO(I)I STD ERROR";
2800 IF D5=W0 THEN 2830
2810 PRINT "I T"
2820 GO TO 2840
2830 PRINT
2840 FOR F1=W1 TO D4
2850 F5=W0
2860 FOR F2=F1 TO D4
2870 F5=F5+F(F1,F2)^W2
2880 NEXT F2
2890 F5=SQR(F5)*D5
2900 PRINT "J";F1-W1,CO(F1),F5,"";
2910 IF F5=W0 THEN 2940
2920 PRINT CO(F1)/F5
2930 GO TO 2950
2940 PRINT
2950 NEXT F1
2960 PRINT "JJ";N;" DATA POINTSIMAX DEG = ";D(W4);"JJ"
2970 END
2980 REM  ***************************************************
2990 REM  ************PLOT DATA******************************
```

```
3000 GOSUB 3040
3002 GOSUB 5005
3003 RETURN
3025 REM  ************************************************
3030 REM  ***SET WINDOW, DRAW  AXIS, PLOT DATA, AND RETURN*
3040 WINDOW 0,130,0,100
3045 VIEWPORT 5,125,5,95
3050 REM  DRAW  BOX
3060 IF NOT(D(23)) AND D(W3)<W2 THEN 3080
3070 GOSUB 850
3080 PAGE
3090 MOVE @D(18):0,0
3100 DRAW @D(18):130,0
3110 DRAW @D(18):130,100
3120 DRAW @D(18):0,100
3130 DRAW @D(18):0,0
3140 REM  SET WINDOW
3150 F5=W5
3160 GOSUB 3770
3170 F0(W1)=F2
3180 F0(W2)=F3
3190 F5=W7
3200 GOSUB 3770
3210 F0(W5)=F2
3220 F0(W6)=F3
3230 F5=W1
3240 GOSUB 3850
3250 F5=W5
3260 GOSUB 3850
3270 WINDOW F0(W1),F0(W2),F0(W5),F0(W6)
3280 REM  MAKE AXIS AND LABEL IT
3290 GOSUB 4130
3300 REM  PLOT * AT EACH DATA POINT
3310 C5=W0
3320 F2=0.34/72*(F0(W1)-F0(W2))
3330 F3=0.4/34*(F0(W5)-F0(W6))
3340 D(W5)=1.0E+200
3350 D(W6)=-D(W5)
3360 D(W7)=D(W5)
3370 D(W8)=D(W6)
3380 FOR F1=W1 TO N
3390 GOSUB 660
3400 MOVE @D(18):D6+F2,D7+F3
3410 PRINT @D(18):"*";
3420 D(W5)=D(W5) MIN D6
3430 D(W6)=D(W6) MAX D6
3440 D(W7)=D(W7) MIN D7
3450 D(W8)=D(W8) MAX D7
3460 NEXT F1
3470 HOME @D(18):
3480 RETURN
3490 REM  ***************************************************
3500 REM  ******PLOT DATA AND REGRESSION*******************
3510 GOSUB 3040
```

```
3520 PRINT @D(18):"                                    JJDEGREE ";D(10)
3530 F6=D(W5)
3540 GOSUB 3700
3550 MOVE @D(18):F6,F7
3560 F5=(D(W6)-D(W5))/D(21)
3570 C2=D(21)
3580 IF D4>W2 THEN 3610
3590 F5=D(W6)-D(W5)
3600 C2=W1
3610 FOR F1=W1 TO C2
3620 F6=F6+F5
3630 GOSUB 3700
3640 DRAW @D(18):F6,F7
3650 NEXT F1
3660 HOME @D(18):
3665 GOSUB 3002
3670 END
3680 REM  ****************************************
3690 REM  ************EVALUATE POLY. AT F6 ***********
3700 F7=W0
3710 FOR F8=D4 TO W1 STEP -W1
3720 F7=F7*F6+C0(F8)
3730 NEXT F8
3740 RETURN
3750 REM  ****************************************
3760 REM  ********INCREASE WINDOW BY 1.16*************
3770 F2=1.2*D(F5)-0.2*D(F5+W1)
3780 F3=1.1*D(F5+W1)-0.1*D(F5)
3790 IF F2<>F3 THEN 3820
3800 F3=F3+W1
3810 F2=F2-W1
3820 RETURN
3830 REM  ********************************************
3840 REM  ******ADJUST THE WINDOW FOR NEAT TICS************
3850 F1=(F0(F5+W1)-F0(F5))/(F5+W3)
3860 F2=10^INT(LGT(F1))
3870 F1=F1/F2
3880 IF F1<1.41421356 THEN 3960
3890 IF F1=>3.16227766 THEN 3920
3900 F2=W2*F2
3910 GO TO 3960
3920 IF F1=>7.071067812 THEN 3950
3930 F2=W5*F2
3940 GO TO 3960
3950 F2=10*F2
3960 F1=INT(F0(F5)/F2)
3970 F3=F2*(F1+W2)
3980 IF F3<F0(F5) THEN 4010
3990 F3=F3-F2
4000 GO TO 3980
4010 F0(F5)=F3
4020 F5=F5+W1
4030 F1=INT(F0(F5)/F2)
4040 F3=F2*(F1-W2)
4050 IF F0(F5)<F3 THEN 4080
```

```
4060 F3=F3+F2
4070 GO TO 4050
4080 F0(F5)=F3
4090 F0(F5+W1)=F2
4100 RETURN
4110 REM  **************************************************
4120 REM  *********MAKE AXIS*******************************
4130 AXIS @D(18):F0(W3),F0(W7)
4140 F5=W8
4150 GOSUB 4210
4160 F5=W4
4170 GOSUB 4210
4180 RETURN
4190 REM  ************************************************
4200 REM  *********LABEL AN AXIS**************************
4210 F4=F0(F5-W1)
4220 F0(W4)=F0(W1)
4230 C2=(F0(W6)-F0(W5))/138
4240 F0(W8)=F0(W5)
4250 F3=ABS(F0(F5-W3)+F4) MAX ABS(F0(F5-W2)-F4)
4260 F3=INT(LGT(F3)+1.0E-8)
4270 F2=10^-F3
4280 F1=F0(F5-W2)-F4/W2
4290 F0(F5)=F0(F5)+F4
4300 IF F0(F5)>F1 THEN 4340
4310 MOVE @D(18):F0(W4),F0(W8)+C2
4320 PRINT @D(18): USING "-D.2D,S":F0(F5)*F2
4330 GO TO 4290
4340 IF F3=W0 THEN 4380
4350 F0(F5)=F1
4360 MOVE @D(18):F0(W4),F0(W8)+C2
4370 PRINT @D(18): USING "2A,+FD,S":" E";F3
4380 RETURN
5000 REM ** ROUTINE TO PRINT X$,Y$,Z$- X AXIS, Y AXIS LABELS AND
5001 REM ** A COMMENT LINE AT THE TOP.  WE WILL HAVE TO ADJUST THE
5002 REM ** WINDOW AND VIEWPORT TO MAKE IT FIT NICELY.
5005 WINDOW 0,130,0,100
5008 VIEWPORT 5,125,5,95
5010 MOVE 50,0
5020 PRINT "J";X$;
5030 HOME
5040 PRINT "JJJJJJJ"
5050 PRINT " ";
5060 FOR L=1 TO LEN(Y$)
5070 A$=SEG(Y$,L,1)
5080 PRINT A$;"HJ";
5090 NEXT L
5094 HOME
5096 PRINT "J                    COMMENT";Z$
5100 HOME
5110 RETURN
5900 IF D(10)<5 THEN 5950
5910 PRINT "JDERIVATIVES NOT AVAILABLE ABOVE THE 4th DEGREE."
5920 GO TO 6400
5950 PRINT "DEGREE SELECTED WAS ";D(10);"JJ"
```

```
6000 PRINT USING 6600:"f(x)=";CO(1);"+(";CO(2);")X";
6010 IF D(10)=1 THEN 6300
6020 PRINT USING 6620:"+(";CO(3);")X^2";
6030 IF D(10)=2 THEN 6300
6040 PRINT USING 6620:"+(";CO(4);")X^3";
6050 IF D(10)=3 THEN 6300
6060 PRINT USING 6620:"+(";CO(5);")X^4";
6300 PRINT "J"
6310 PRINT USING 6630:"dy/dx=(";CO(2);")X^0";
6315 IF D(10)=1 THEN 6400
6330 PRINT USING 6620:"+(";2*CO(3);")X^1",
6340 IF D(10)=2 THEN 6400
6350 PRINT USING 6620:"+(";3*CO(4);")X^2";
6360 IF D(10)=3 THEN 6400
6370 PRINT USING 6620:"+(";4*CO(5);")X^3";
6400 RETURN
6600 IMAGE 5a,4d.4d,2a,4d.4d,2a,s
6620 IMAGE 2a,4d.4d,4a,s
6630 IMAGE 7a,4d.4d,4a,s

2000 REM  *****FILE #13 - ESTIMATES, RESIDUALS, ETC*****
2010 IF F9<31 THEN 100
2020 IF D(W9)<>W1 THEN 100
2030 IF D(10)<W0 THEN 100
2040 GO TO F9-31 OF 2070,2930,2320,2700
2050 REM  ********************************************
2060 REM  **********ESTIMATES**************************
2070 PRINT "L***ESTIMATES***"
2080 PRINT "JFOR EITHER SINGLE X VALUES OR A TABLE OF";
2090 PRINT " X VALUES (ENTER S OR T): ";
2100 INPUT F$
2110 IF F$="S" THEN 2250
2120 IF F$<>"T" THEN 2080
2130 PRINT "JENTER INITIAL X, FINAL X, AND STEP SIZE = ";
2140 INPUT F6,F2,F5
2150 GOSUB 3580
2160 PRINT "JX = ";F6,"Y EST = ";F7
2170 F6=F6+F5
2180 IF SGN(F6-F2)<>SGN(F5) THEN 2150
2190 IF F6=F2 OR F6=F2+F5 THEN 2220
2200 F6=F2
2210 GO TO 2150
2220 PRINT "JJ"
2230 END
2240 REM  SINGLE X ESTIMATES
2250 PRINT "JX = ";
2260 INPUT F6
2270 GOSUB 3580
2280 PRINT "I         KY EST = ";F7
2290 GO TO 2250
2300 REM  *********************************************
2310 REM  *********PRINT RESIDUALS***********************
2320 IF NOT(D(23)) AND D(W3)<W2 THEN 2340
2330 GOSUB 850
2340 PRINT "LI         ***RESIDUALS***J"
```

```
2342 PRINT "COMMENT:";Z$;"J"
2345 PRINT " XI = ";X$;"  ,  YI = ";Y$;"  ,  DEGREE ";D(10);"J"
2350 PRINT "    I              XI                    YI                    Y EST";
2360 PRINT "              RESIDUAL"
2370 C5=W0
2380 F3=W0
2390 F4=W0
2400 FOR F1=W1 TO N
2410 GOSUB 660
2420 F6=D6
2430 GOSUB 3580
2440 F2=D7-F7
2450 PRINT USING "5D,S":F1
2460 GOSUB 2630
2470 D6=D7
2480 GOSUB 2630
2490 D6=F7
2500 GOSUB 2630
2510 D6=F2
2520 GOSUB 2630
2530 PRINT
2540 F4=F4+F2*F2
2550 IF F1=1 THEN 2580
2560 F3=F3+(F2-F5)^W2
2570 F5=F2
2580 NEXT F1
2590 D(22)=F3/F4
2600 D(11)=W1
2610 PRINT "JJ"
2620 END
2630 IF ABS(D6)=>1000000 OR ABS(D6)<1.0E-3 AND D6<>W0 THEN 2660
2640 PRINT USING "-8D.6D,S":D6
2650 RETURN
2660 PRINT USING "3X,5E,S":D6
2670 RETURN
2680 REM  ***********************************************
2690 REM  *********PRINT DURBIN-WATSON*********************
2700 PRINT "L***DURBIN-WATSON***J"
2710 IF D(11)<>W0 THEN 2890
2720 IF NOT(D(23)) AND D(W3)<W2 THEN 2740
2730 GOSUB 850
2740 C5=W0
2750 F3=W0
2760 F4=W0
2770 FOR F1=W1 TO N
2780 GOSUB 660
2790 F6=D6
2800 GOSUB 3580
2810 F2=D7-F7
2820 F4=F4+F2*F2
2830 IF F1=W1 THEN 2860
2840 F3=F3+(F2-F5)^W2
2850 F5=F2
2860 NEXT F1
2870 D(22)=F3/F4
```

```
2880 D(11)=W1
2890 PRINT 'JDURBIN-WATSON STATISTIC = ';D(22);'JJ'
2900 END
2910 REM  ***********************************************
2920 REM  *********PLOT RESIDUALS**************************
2930 PRINT 'L***PLOT STANDARDIZED RESIDUALS***'
2935 PRINT '     XI = ';X$;'  ,  YI = ';Y$;'  ,  DEGREE ';D(10);'J'
2940 IF D5<>W0 THEN 2970
2950 PRINT 'JPERFECT FIT -- ALL RESIDUALS ARE 0'
2960 END
2970 PRINT 'JPLOT XI VS. STANDARDIZED RESIDUALS OR'
2980 PRINT '      YI VS. STANDARDIZED RESDUALS (ENTER X OR Y) : ';
2990 INPUT F$
3000 C1=W0
3010 IF F$='X' THEN 3040
3020 IF F$<>'Y' THEN 2940
3030 C1=W2
3040 WINDOW 0,130,0,100
3045 VIEWPORT 5,125,5,95
3050 D$=F$
3060 REM  MAKE THE BOX
3070 IF NOT(D(23)) AND D(W3)<W2 THEN 3100
3080 GOSUB 850
3090 F$=D$
3100 PAGE
3110 MOVE @D(18):0,0
3120 DRAW @D(18):130,0
3130 DRAW @D(18):130,100
3140 DRAW @D(18):0,100
3150 DRAW @D(18):0,0
3160 HOME @D(18):
3162 GOSUB 4230
3178 PRINT @D(18):'                    JDEGREE ';D(10);' -- ';F$;
3180 PRINT @D(18):' VS STANDARDIZED RESIDUALS'
3190 REM  ADJUST HORIZONTAL WINDOW
3200 F0(W5)=-W5
3210 F0(W6)=W5
3220 F0(W7)=W1
3230 F0(W1)=D(W5+C1)*1.2-0.2*D(W6+C1)
3240 F0(W2)=D(W6+C1)*1.1-0.1*D(W5+C1)
3250 IF F0(W1)<>F0(W2) THEN 3280
3260 F0(W1)=F0(W1)-W1
3270 F0(W2)=F0(W2)+W1
3280 F5=W1
3290 GOSUB 3650
3300 WINDOW F0(W1),F0(W2),F0(W5),F0(W6)
3310 GOSUB 3930
3320 C2=0.34/72*(F0(W1)-F0(W2))
3330 C3=0.4/34*(F0(W5)-F0(W6))
3340 C5=W0
3350 F3=W0
3360 F4=W0
3370 FOR F1=W1 TO N
3380 GOSUB 660
3390 F6=D6
```

```
3400 GOSUB 3580
3410 F2=D7-F7
3420 F4=F4+F2*F2
3430 IF F1=W1 THEN 3460
3440 F3=F3+(F2-F5)^W2
3450 F5=F2
3460 IF C1=W0 THEN 3480
3470 D6=D7
3480 F2=F2/D5
3490 MOVE @D(18):D6+C2,F2+C3
3500 PRINT @D(18):'*';
3510 NEXT F1
3520 D(22)=F3/F4
3530 D(11)=W1
3540 HOME @D(18):
3550 END
3560 REM  ****************************************************
3570 REM  **********EVALUATE POLY. AT F6**********************
3580 F7=W0
3590 FOR F8=D4 TO W1 STEP -W1
3600 F7=F7*F6+C0(F8)
3610 NEXT F8
3620 RETURN
3630 REM  ****************************************************
3640 REM  **********ADJUST WINDOW FOR NEAT TICS**************
3650 F1=(F0(F5+W1)-F0(F5))/(F5+W3)
3660 F2=10^INT(LGT(F1))
3670 F1=F1/F2
3680 IF F1<1.414213562 THEN 3760
3690 IF F1=>3.16227766 THEN 3720
3700 F2=W2*F2
3710 GO TO 3760
3720 IF F1=>7.071067812 THEN 3750
3730 F2=W5*F2
3740 GO TO 3760
3750 F2=10*F2
3760 F1=INT(F0(F5)/F2)
3770 F3=F2*(F1+W2)
3780 IF F3<F0(F5) THEN 3810
3790 F3=F3-F2
3800 GO TO 3780
3810 F0(F5)=F3
3820 F5=F5+W1
3830 F1=INT(F0(F5)/F2)
3840 F3=F2*(F1-W2)
3850 IF F0(F5)<F3 THEN 3880
3860 F3=F3+F2
3870 GO TO 3850
3880 F0(F5)=F3
3890 F0(F5+W1)=F2
3900 RETURN
3910 REM  ****************************************************
3920 REM  *********MAKE AXIS*********************************
3930 AXIS @D(18):F0(W3),F0(W7)
```

```
3940 F5=W8
3950 GOSUB 4010
3960 F5=W4
3970 GOSUB 4010
3980 RETURN
3990 REM  ******************************************************
4000 REM  ********LABEL AN AXIS*****************************
4010 F4=FO(F5-W1)
4020 FO(W4)=FO(W1)
4030 C2=(FO(W6)-FO(W5))/138
4040 FO(W8)=FO(W5)
4050 F3=ABS(FO(F5-W3)+F4) MAX ABS(FO(F5-W2)-F4)
4060 F3=INT(LGT(F3)+1.0E-8)
4070 F2=10^-F3
4080 F1=FO(F5-W2)-F4/W2
4090 FO(F5)=FO(F5)+F4
4100 IF FO(F5)>F1 THEN 4140
4110 MOVE @D(18):FO(W4),FO(W8)+C2
4120 PRINT @D(18): USING "-D.2D,S":FO(F5)*F2
4130 GO TO 4090
4140 IF F3=W0 THEN 4180
4150 FO(F5)=F1
4160 MOVE @D(18):FO(W4),FO(W8)+C2
4170 PRINT @D(18): USING "2A,+FD,S":" E";F3
4180 RETURN
4190 REM  ***********************************************************
4230 WINDOW 5,130,0,100
4240 VIEWPORT 0,130,0,100
4250 MOVE 3,1
4260 PRINT "J";X$;
4270 HOME
4280 PRINT "JJJJJJJ"
4290 PRINT " ";
4300 FOR L=1 TO LEN(Y$)
4310 A$=SEG(Y$,L,1)
4320 NEXT L
4340 HOME
4350 PRINT "COMMENT:";Z$
4360 RETURN
```

APPENDIX C

POLYNOMIAL TABLES

## APPENDIX C
## POLYNOMIAL TABLES

This appendix contains Tables C-1 and C-2 that show the required values of polynomial regression.

This appendix also contains F distribution and "students" t distribution tables respectively. It is best to use F-test to decide what degree polynomial fits the data adequately, then estimate the coefficient and, thus, determine a least square curve. It is possible to define 95% to 99%, or other confidence intervals, by using the table t distribution in Appendix C-2.

TABLE C-1. THE F DISTRIBUTION*

$$\Pr(F \le f) = \int_0^f \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1 + r_1 w/r_2)^{(r_1 + 4_2)/2}} \, dw$$

| | n - p | $r_1 = P - 1$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pr(F ≤ f) | $r_2$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 |
| 0.95 | 1 | 161 | 200 | 216 | 225 | 230 | 234 | 237 | 239 | 241 | 242 | 244 | 246 |
| 0.975 | | 648 | 800 | 864 | 900 | 922 | 937 | 948 | 957 | 963 | 969 | 977 | 985 |
| 0.99 | | 4052 | 4999 | 5403 | 5625 | 5764 | 5859 | 5928 | 5982 | 6023 | 6056 | 6106 | 6157 |
| 0.95 | 2 | 18.5 | 19.0 | 19.2 | 19.2 | 19.3 | 19.3 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 |
| 0.975 | | 38.5 | 39.0 | 39.2 | 39.2 | 39.3 | 39.3 | 39.4 | 39.4 | 39.4 | 39.4 | 39.4 | 39.4 |
| 0.99 | | 98.5 | 99.0 | 99.2 | 99.2 | 99.3 | 99.3 | 99.4 | 99.4 | 99.4 | 99.4 | 99.4 | 99.4 |
| 0.95 | 3 | 10.1 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.70 |
| 0.975 | | 17.4 | 16.0 | 15.4 | 15.1 | 14.9 | 14.7 | 14.6 | 14.5 | 14.5 | 14.4 | 14.3 | 14.3 |
| 0.99 | | 34.1 | 30.8 | 29.5 | 28.7 | 28.2 | 27.9 | 27.7 | 27.5 | 27.3 | 27.2 | 27.1 | 26.9 |
| 0.95 | 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.86 |
| 0.975 | | 12.2 | 10.6 | 9.98 | 9.60 | 9.36 | 9.20 | 9.07 | 8.98 | 8.90 | 8.84 | 8.75 | 8.66 |
| 0.99 | | 21.2 | 18.0 | 16.7 | 16.0 | 15.5 | 15.2 | 15.0 | 14.8 | 14.7 | 14.5 | 14.4 | 14.2 |
| 0.95 | 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.62 |
| 0.975 | | 10.0 | 8.43 | 7.76 | 7.39 | 7.15 | 6.98 | 6.85 | 6.76 | 6.68 | 6.62 | 6.52 | 6.43 |
| 0.99 | | 16.3 | 13.3 | 12.1 | 11.4 | 11.0 | 10.7 | 10.5 | 10.3 | 10.2 | 10.1 | 9.89 | 9.72 |
| 0.95 | 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | 3.94 |
| 0.975 | | 8.81 | 7.26 | 6.60 | 6.23 | 5.99 | 5.82 | 5.70 | 5.60 | 5.52 | 5.46 | 5.37 | 5.27 |
| 0.99 | | 13.7 | 10.9 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 | 7.87 | 7.72 | 7.56 |
| 0.95 | 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | 3.57 | 3.51 |
| 0.975 | | 8.07 | 6.54 | 5.89 | 5.52 | 5.29 | 5.12 | 4.99 | 4.90 | 4.82 | 4.76 | 4.67 | 4.57 |
| 0.99 | | 12.2 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 | 6.62 | 6.47 | 6.31 |
| 0.95 | 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.35 | 3.28 | 3.22 |
| 0.975 | | 7.57 | 6.06 | 5.42 | 5.05 | 4.82 | 4.65 | 4.53 | 4.43 | 4.36 | 4.30 | 4.20 | 4.10 |
| 0.99 | | 11.3 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 | 5.81 | 5.67 | 5.52 |

TABLE C-1. (continued)

$$Pr(F \leq f) = \int_0^f \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1 + r_1 w/r_2)^{(r_1 + 4_2)/2}} dw$$

| $Pr(F \leq f)$ | n - p $r_2$ | $r_1$ = P - 1: 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.95 | 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.14 | 3.07 | 3.01 |
| 0.975 | | 7.21 | 5.71 | 5.08 | 4.72 | 4.48 | 4.32 | 4.20 | 4.10 | 4.03 | 3.96 | 3.87 | 3.77 |
| 0.99 | | 10.6 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 | 5.26 | 5.11 | 4.96 |
| 0.95 | 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 | 2.98 | 2.91 | 2.85 |
| 0.975 | | 6.94 | 5.46 | 4.83 | 4.47 | 4.24 | 4.07 | 3.95 | 3.85 | 3.78 | 3.72 | 3.62 | 3.52 |
| 0.99 | | 10.0 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 | 4.85 | 4.71 | 4.56 |
| 0.95 | 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.69 | 2.62 |
| 0.975 | | 6.55 | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.61 | 3.51 | 3.44 | 3.37 | 3.28 | 3.18 |
| 0.99 | | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 | 4.30 | 4.16 | 4.01 |
| 0.95 | 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.48 | 2.40 |
| 0.975 | | 6.20 | 4.77 | 4.15 | 3.80 | 3.58 | 3.41 | 3.29 | 3.20 | 3.12 | 3.06 | 2.96 | 2.86 |
| 0.99 | | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 | 3.80 | 3.67 | 3.52 |

* This table is abridged and adapted from "Tables of Percentage Points of the Inverted Beta Distribution," Biometrika, 33 (1943). It is published here with the kind permission of Professor E. S. Pearson on behalf of the authors, Maxine Merrington and Catherine M. Thompson, and of the Biometrika Trustees.

TABLE C-2. PERCENTILE VALUES ($t_p$) FOR STUDENTS t DISTRIBUTION WITH $\nu$ DEGREES OF FREEDOM (shaded area = $p$)

| $\nu$ (n - 1) | t0.995 | t0.99 | t0.975 | t0.95 | t0.90 | t0.80 | t0.75 | t0.70 | t0.60 | t0.55 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 63.66 | 31.82 | 12.71 | 6.31 | 3.08 | 1.376 | 1.000 | 0.727 | 0.325 | 0.158 |
| 2 | 9.92 | 6.96 | 4.30 | 2.92 | 1.89 | 1.061 | 0.816 | 0.617 | 0.289 | 0.142 |
| 3 | 5.84 | 4.54 | 3.18 | 2.35 | 1.64 | 0.978 | 0.765 | 0.584 | 0.277 | 0.137 |
| 4 | 4.60 | 3.75 | 2.78 | 2.13 | 1.53 | 0.941 | 0.741 | 0.569 | 0.271 | 0.134 |
| 5 | 4.03 | 3.36 | 2.57 | 2.02 | 1.48 | 0.920 | 0.727 | 0.559 | 0.267 | 0.132 |
| 6 | 3.71 | 3.14 | 2.45 | 1.94 | 1.44 | 0.906 | 0.718 | 0.553 | 0.265 | 0.131 |
| 7 | 3.50 | 3.00 | 2.36 | 1.90 | 1.42 | 0.896 | 0.711 | 0.549 | 0.263 | 0.130 |
| 8 | 3.36 | 2.90 | 2.31 | 1.86 | 1.40 | 0.889 | 0.706 | 0.546 | 0.262 | 0.130 |
| 9 | 3.25 | 2.82 | 2.26 | 1.83 | 1.38 | 0.883 | 0.703 | 0.543 | 0.261 | 0.129 |
| 10 | 3.17 | 2.76 | 2.23 | 1.81 | 1.37 | 0.879 | 0.700 | 0.542 | 0.260 | 0.129 |
| 11 | 3.11 | 2.72 | 2.20 | 1.80 | 1.36 | 0.876 | 0.697 | 0.540 | 0.260 | 0.129 |
| 12 | 3.06 | 2.68 | 2.18 | 1.78 | 1.36 | 0.873 | 0.695 | 0.539 | 0.259 | 0.128 |
| 13 | 3.01 | 2.65 | 2.16 | 1.77 | 1.35 | 0.870 | 0.694 | 0.538 | 0.259 | 0.128 |
| 14 | 2.98 | 2.62 | 2.14 | 1.76 | 1.34 | 0.868 | 0.692 | 0.537 | 0.258 | 0.128 |
| 15 | 2.95 | 2.60 | 2.13 | 1.75 | 1.34 | 0.866 | 0.691 | 0.536 | 0.258 | 0.128 |
| 16 | 2.92 | 2.58 | 2.12 | 1.75 | 1.34 | 0.865 | 0.690 | 0.535 | 0.258 | 0.128 |
| 17 | 2.90 | 2.57 | 2.11 | 1.74 | 1.33 | 0.863 | 0.689 | 0.534 | 0.257 | 0.128 |
| 18 | 2.88 | 2.55 | 2.10 | 1.73 | 1.33 | 0.862 | 0.688 | 0.534 | 0.257 | 0.127 |
| 19 | 2.86 | 2.54 | 2.09 | 1.73 | 1.33 | 0.861 | 0.688 | 0.533 | 0.257 | 0.127 |
| 20 | 2.84 | 2.53 | 2.09 | 1.72 | 1.32 | 0.860 | 0.687 | 0.533 | 0.257 | 0.127 |
| 21 | 2.83 | 2.52 | 2.08 | 1.72 | 1.32 | 0.859 | 0.686 | 0.532 | 0.257 | 0.127 |
| 22 | 2.82 | 2.51 | 2.07 | 1.72 | 1.32 | 0.858 | 0.686 | 0.532 | 0.256 | 0.127 |
| 23 | 2.81 | 2.50 | 2.07 | 1.71 | 1.32 | 0.858 | 0.685 | 0.532 | 0.256 | 0.127 |
| 24 | 2.80 | 2.49 | 2.06 | 1.71 | 1.32 | 0.857 | 0.685 | 0.531 | 0.256 | 0.127 |

TABLE C-2. (continued)

| ν (n - 1) | t0.995 | t0.99 | t0.975 | t0.95 | t0.90 | t0.80 | t0.75 | t0.70 | t0.60 | t0.55 |
|---|---|---|---|---|---|---|---|---|---|---|
| 25 | 2.79 | 2.48 | 2.06 | 1.71 | 1.32 | 0.856 | 0.684 | 0.531 | 0.256 | 0.127 |
| 26 | 2.78 | 2.48 | 2.06 | 1.71 | 1.32 | 0.856 | 0.684 | 0.531 | 0.256 | 0.127 |
| 27 | 2.77 | 2.47 | 2.05 | 1.70 | 1.31 | 0.855 | 0.684 | 0.531 | 0.256 | 0.127 |
| 28 | 2.76 | 2.47 | 2.05 | 1.70 | 1.31 | 0.855 | 0.683 | 0.530 | 0.256 | 0.127 |
| 29 | 2.76 | 2.46 | 2.04 | 1.70 | 1.31 | 0.854 | 0.683 | 0.530 | 0.256 | 0.127 |
| 30 | 2.75 | 2.46 | 2.04 | 1.70 | 1.31 | 0.854 | 0.683 | 0.530 | 0.256 | 0.127 |
| 40 | 2.70 | 2.42 | 2.02 | 1.68 | 1.30 | 0.851 | 0.681 | 0.529 | 0.255 | 0.126 |
| 60 | 2.66 | 2.39 | 2.00 | 1.67 | 1.30 | 0.848 | 0.679 | 0.527 | 0.254 | 0.126 |
| 120 | 2.62 | 2.36 | 1.98 | 1.66 | 1.29 | 0.845 | 0.677 | 0.526 | 0.254 | 0.126 |
| ∞ | 2.58 | 2.33 | 1.96 | 1.645 | 1.28 | 0.842 | 0.674 | 0.524 | 0.253 | 0.126 |

Source: R. A. Fisher and F. Yates, Statistical Tables for Biological, Agricultural and Medical Research (5th edition), Table III, Oliver and Boyd Ltd., Edinburgh, by permission of the authors and publishers.