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A Simplified Game Theory Model for the

Optimum Alarm Level from Nuclear Material

Accounting Data

M. Dresher, Consultant

M. Messinger, U.S. Nuclear Regulatory Commission

S. Moglewer, U.S. Nuclear Regulatory Commission

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A Simplified Game Theory Model for the Optimum Alarm Level from Nuclear Material Accounting Data

M. Dresher, M. Messinger and S. Moglewer

1. Introduction

Threshold discriminants are the standard procedure employed by the nuclear industry for detection of anomalies in the analysis of nuclear material accounting data in an attempt to provide assurance that no loss or diversion of material has occurred. The fundamental decision in this type of analysis is to determine a suitable value for the alarm level or threshold. The classical approach is generally based upon a statistical hypothesis testing model. Standard procedure is to arbitrarily establish an alarm level at some "reasonable" number, for example the upper limit of a 95% confidence interval based on the null hypothesis of no loss or diversion, and to accept the alternative hypothesis of nuclear material loss, i.e., reject the null, for realizations of the random variable which is usually inventory difference or MUF outside this arbitrary threshold. Type I and Type II error analysis and Operating Characteristic curves are then evaluated for sensitivity to different alternative hypotheses, i.e., statements of the amount lost or diverted [1].

Unfortunately there are a number of value judgments, or decision-maker utilities, that are implicit in such a model. Consequently these utilities may not be known to the decision-maker. This would reduce the validity of the classical approach for solution of real-world problems.

The theory of games, by requiring explicit formulation of the decision-maker utilities for establishment of a payoff function, can be useful in providing information on the utilities inherent in alternative alarm thresholds. This should provide the capability for more efficient auditing of accounting data. This paper will present a model that develops optimum alarm levels from nuclear material accounting data. The model will be concerned with detection of unauthorized diversion from inventory data.

2. Discussion of the Problem

The nuclear material accounting system keeps track of material inputs and outputs by taking inventory at regular intervals and locations. At the end of the inventory period a reconciliation between the physical inventory and the accounting records is effected. Due to measuring errors, recording errors, bias in individuals and instruments, process errors, and possible diversion, the inventory usually does not balance, leaving a quantity called ID (inventory difference). ID is a function of the realizations of the many errors involved and would include any diversion that may have occurred during the inventory period.

The decision-making problem is, given an ID reading, when should the alarm be sounded to verify a possible theft. If the alarm threshold is too large, there will be a low probability of alarming in case of a diversion. If the alarm is too small, there will be a high probability of false alarm with its consequent penalties. The decision-maker thus has a duality of objectives to consider for establishing an equilibrium. Furthermore the outcome is beyond the control of the decision-maker since the underlying statistics would be partially controlled by a hostile adversary, a diverter whose decisions would determine the mean of the distribution from which the inventory difference statistic would be obtained. The framework for developing a hypothesis test for diversion is shown in Figure 1.

This framework can be used to develop a payoff function for a game theoretic model that would solve for the optimum threshold. The rules for such a game would be:

Move 1. The diverter diverts an amount 0 or K of nuclear material.

Move 2. Nature or chance selects an amount of μ of ID from some given probability distribution for ID.

Move 3. The defender, knowing the outcome of Move 2, decides whether or not to alarm.

STATE OF AFFAIRS

RESULT OF INVENTORY	TRUE ID = 0	TRUE ID ≠ 0
ID ≤ THRESHOLD	OK	UNDETECTED LOSS/DIVERSION
ID > THRESHOLD	FALSE ALARM	OK

FIGURE 1
A FRAMEWORK FOR TEST OF HYPOTHESIS

Since each player has two choices available per move, there are four possible outcomes to the game and thus four possible payoff values. They can be summarized by the following 2 x 2 matrix, which describes the payoff to the defender as a function of these moves.

		DEFENDER	
		NO ALARM	ALARM
DIVERTER	DIVERT 0	0	M_1
	DIVERT K	M_0	M_2

This can be transformed into a regret matrix similar to Figure 1 by defining regret as the difference between the best and the actual payoff. With this definition $M_2 = 0$. (Regret is due to wrongly guessing the unknown state of affairs, See [2].) The matrix elements represent the payoff to the defender in terms of costs for each of the possible outcomes.

3. Formulation of Game

In order to analyze the above game, it is necessary to formulate the game in terms of the strategies of the players which take into account the chance move of the game. Thus, the game payoff will be described in terms of an expected value over the chance move.

First, it is clear that a strategy for the diverter is either $x = 0$ or $x = k$. The diverter has only two strategies.

Since the defender has information about the magnitude of ID, he must decide for each value of u whether or not to alarm. This is equivalent to choosing some z, where $-\infty < z < \infty$ such that if $u \leq z$, then the defender does not alarm, and if $u > z$ he does alarm. Thus, the defender has an infinite or a continuum of strategies $\{z\}$ such that $-\infty < z < \infty$.

We can now express the payoff of the game, which is an expected value, in terms of the strategies as follows:

$$M\,(x,\ z) = \begin{cases} M_1\,P(u > z \quad x = o) \\ M_o\,P(u \leq z \quad x = k) + M_2\,P(u > z \quad x = K) \end{cases}$$

where $P(u \leq z \quad x = k)$ is a normal probability function with mean k and unit standard deviation. It will be convenient to write the above payoff as

$$M(x,\ z) = \begin{cases} M(o,\ z) & \text{if } x = o \\ M(k,\ z) & \text{if } x = k \end{cases}$$

where

$$M(o,\ z) = \quad M_1\,P(u > z \quad x = o)$$

$$M(k,\ z) = \quad M_o\,P(u \leq z \quad x = k) + M_2\,P(u > z \quad x = k)$$

Since the payoff $M(x,\ z)$ involves four parameters, i.e., M_o, M_1, M_2, k, the solution of the game will depend on the values of these four parameters.

4. Computation of Max Min

We shall now show that this game has no saddle-point solution. We shall also obtain upper and lower bounds on the game value.

The lower bound on the game value is given by Max Min or

$$\max_x \min_z M(x, z) = \max \left[\min_z M(o, z), \min_z M(k, z) \right]$$

$$= \max \left[o, M_2 \right]$$

$$\max_x \min_z M(x, z) = 0$$

and the max min is assumed at x = o.

5. Computation of Min Max

In order to compute the upper bound to the game value, we need to compute $\min_z \max_x M(x, z)$. For any given z, we have

$$\max_x M(x, z) = \max \left[M(o, z), M(k, z) \right]$$

$$= \begin{cases} M(o, z) & \text{if } M(o, z) \geq M(k, z) \\ M(k, z) & \text{if } M(k, z) \geq M(o, z) \end{cases}$$

Suppose $M(o, z) \geq M(k, z)$, then

$$\operatorname*{Max}_{x} M((x, z) = M(o, z)$$

and this holds for a range of z's such that $-\infty < z \leq z_0$. We thus have

$$\operatorname*{Min}_{-\infty < z \leq z_0} \operatorname*{Max}_{x} M(x, z) = M(o, z_0)$$

Where the minimum is assumed at $z = z_0$.

Now suppose that

$$M(k, z) \geq M(o, z)$$

Then

$$\operatorname*{Max}_{x} M(x, z) = M(k, z)$$

This nolds for all z such that $z_0 \leq z \leq \infty$. In this case

$$\min_{z_0 \leq z \leq \infty} \max_{x} M(x, z) = M(k, z_0),$$

and the minimum is assumed at z_0.

Therefore, we obtain for the upper bound of the game value

$$\min_{z} \max_{x} M(x, z) = \min \left[\min_{\infty \leq z \leq z_0} M(o, z), \quad \min_{z_0 < z < \infty} M(k, z) \right]$$

$$= M(o, z_0) = M(k, z_0)$$

The value of z_0, where the minimum is assumed, is obtained by solving the following equation:

$$M(o, z) = M(k, z)$$

where $-\infty < z_0 < \infty$. Further, the solution is unique. It satisfies the following equation.

$$M_1 \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{\frac{-u^2}{2}} du = \frac{(M_o - M_2)}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(u-k)^2}{2}} du + M_2$$

6. Optimal Strategies

Having proven that the game does not have a saddle point, this implies that the optimal strategy of the diverter is given by

$$F^*(x) = \lambda\, I_o\,(x) + (1 - \lambda)\, I_k\,(x)$$

where $o < \lambda < 1$ and $I_k\,(x)$ is a step-function with a jump at $x = k$.

Suppose that the diverter uses $F^*(x)$, the diverter's optimal strategy, then the optimal strategy of the defender is such that the defender randomizes over those strategies of the defender for which

$$\min_z M(F^*, z) = V$$

Now

$$M(F^{*}, z) = \lambda M(o, z) + (1 - \lambda)M(k, z).$$

It can be shown that this function (of z) has a unique minimum. Hence, the defender's optimal strategy is a pure strategy z^{*} which minimizes $M(F^{*}, z) = \lambda M(o, z) \neq (1 - \lambda)M(k, z)$.

Note that $- \infty < z^{*} < \infty$. In particular, z^{*} may be negative, depending on the values of the four parameters M_o, M_1, M_2, and k. In such a case the defender would alarm even if an ID were negative.

The computation of z^{*} is relatively straight-forward. It is that z which minimizes $\underset{x}{\text{Max}}\, M(x, z)$ or $\underset{z}{\text{Min}}\, \underset{x}{\text{Max}}\, M(x, z) = M(o, z^{*}) = M(k, z^{*}) = V$

In order to obtain the value of λ, which will yield the diverter's optimal strategy, we first compute

$$\frac{\partial M(x, z)}{\partial z}$$

and evaluate this derivative at $x = o$, $z = z^{*}$ and at $x = k$, $z = z^{*}$, where z^{*} has been obtained above.

Let

$$\frac{\partial M(o, z^*)}{\partial z} = M'(o, z^*)$$

and

$$\frac{\partial M(k, z^*)}{\partial z} = M'(k, z^*)$$

In order to determine the value of λ, we solve the following equation for λ

$$\lambda M'(o, z^*) + (1 \;\; \lambda) M'(k, z^*) = 0$$

In summary we have the following results. The defenders optimum strategy Z* satisfies:

$$M_{,}[1-F(Z^*)] = (M_0 - M_2) F(Z^* - K) + M_2$$

The value of the game is given by:

$$V = M_1 [1 - F(Z^*)]$$

$$= (M_0 - M_2) F(Z^* - K) + M_2$$

and the optimum probability of not diverting is given by

$$\lambda = \frac{(M_0-M_2)\ f(Z^*-K)}{M_1 f(Z^*)+(M_0-M_2)f(Z^*-K)}$$

Where $f(Z) = \frac{d}{dz} F(Z)$

Figure 2 presents a graphical interpretation of the optimal solution to the game for the regret case where M_2 is set equal to 0. The optimum alarm threshold is observed to be obtained from a balance between the expected cost of a false alarm and the expected cost of a missed alarm. The shaded area to the left of z^* is Mo $F(z^* - k)$ = P(No Alarm/Diversion) (Cost to Defender Of No Alarm Given Diversion).

The shaded area to the right of z^* is M_1 $(1 - F(z^*)$ = P(False Alarm) (Cost To Defender If False Alarm).

As proved earlier, the optimum alarm threshold, z^* is given by

$$M_o\ F(z^* - k) = M_1\ (1 - F(z^*))$$

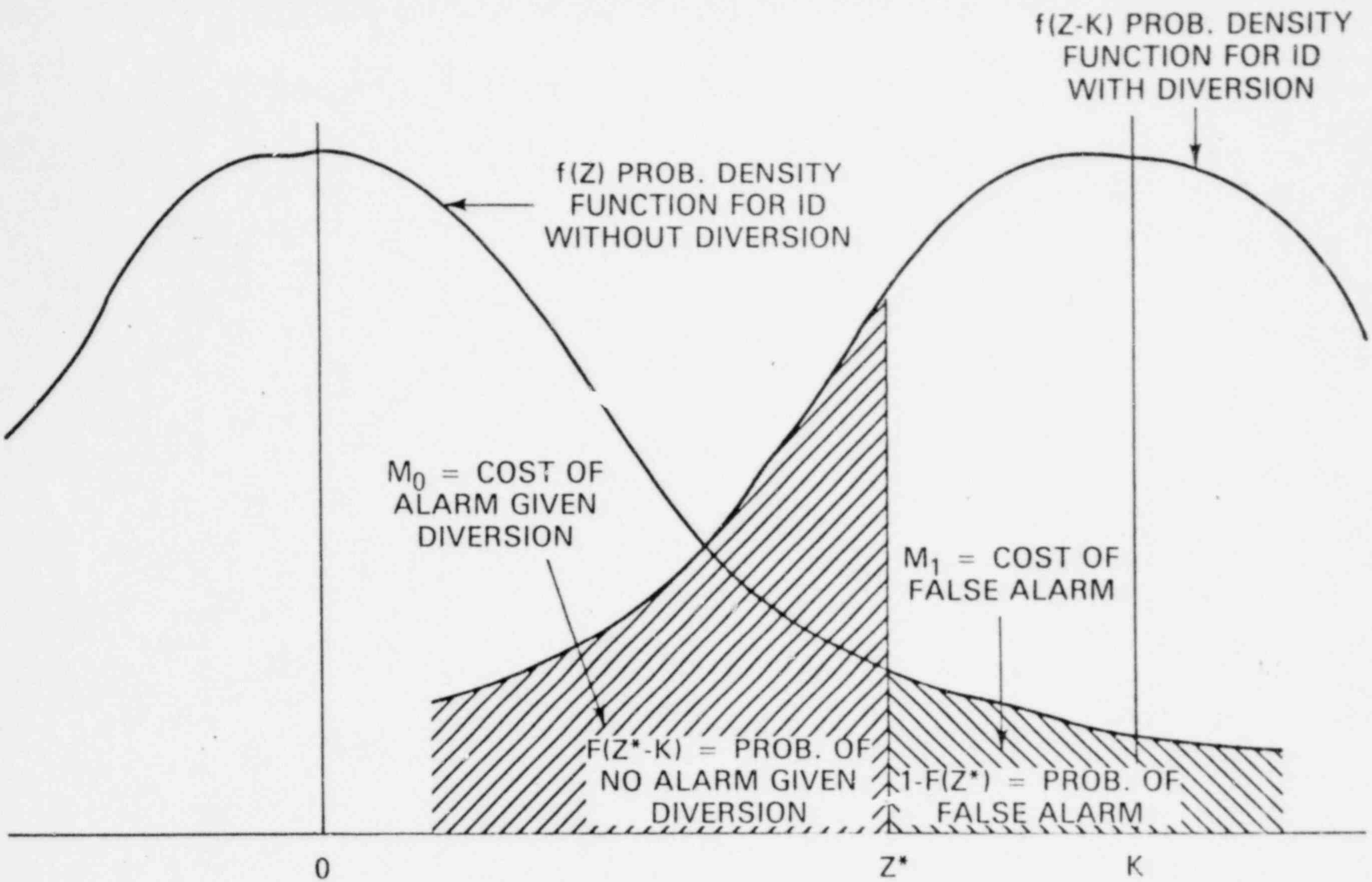

FIGURE 2
GRAPHICAL INTERPRETATION OF GAME SOLUTION

7. A Generalization

The game previously presented can be readily generalized to consider a general regret case where the diverter can choose between two possible levels of diversion. This is analagous to the classical hypothesis case where instead of specifying the Type I and Type II errors, two points of the power curve corresponding to two different values for the Type II error are specified. The regret matrix takes the following form.

DEFENDER

		NO ALARM (Type II error costs)	ALARM
DIVERTER	DIVERT K_1	M_3	M_1
	DIVERT K_2	M_o	M_2

where M_3 denotes the cost to the defender of a Type II error when K_1 is diverted

M_o denotes the cost to the defender of a Type II error when K_2 is diverted

M_1 denotes the cost to the defender of an alarm when K_1 is diverted

M_2 denotes the cost to the defender of an alarm when K_2 is diverted

By analogy with the solution to the previous case, the optimum alarm threshold satisfies

$$M_1 [1 - F(z^* - K_1)] + M_3 F(z^* - K_1)$$
$$= M_o F(z^* - K_2) + M_2 [1 - F(z^* - K_2)]$$

The value of the game, V, is given by

$$V = M_1 [1 - F(z^* - K_1)] + M_3 F(z^* - K_1)$$
$$= M_o F(z^* - K_2) + M_2 [1 - F(z^* - K_2)]$$

Letting λ denote the optimum probability by which the diverter chooses to divert K_1, one obtains

$$-\lambda M_1 f(z^* - K_1) + \lambda M_3 f(z^* - K1)$$
$$= f(1 - \lambda) [M_1 f(z^* - K_2) - M_2 f(z^* - K_2)] = 0$$

solving for λ, one obtains

$$\lambda = \frac{(M_o - M_2)\ f(z^* - K_2)}{(M_1 - M_3)\ f(z^* - k_1) + (M_1 - M_2)\ f(z^* - K_1)}$$

8. Numerical Results

The solution derived in Section 6 was evaluated numerically over a wide range of parameter and payoff element values. This was the case when $M_2 = 0$ (the regret matrix). Some of the results will be presented in this section. The solution in all cases assumed that f (z) was normally distributed with mean 0 or K and standard deviation σ. The results are presented in sigma (σ) units.

Figure 3 presents the payoff M as a function of the alarm level z for a specific case. The optimum alarm level z^* was unique and the optimum value of the payoff V occurred at a sharp point. (There was a severe penalty for excursions of the alarm level away from optimum). Other specific cases examined exhibited similar behaviorism.

Figure 4 shows the sensitivity of the optimum alarm z^* to the ratio of M_1/M_o, the ratio of value of false alarm to undetected loss (i.e., missed alarm). As false alarm increases in value relative to missed alarm, z^* tends to become larger. Also the false alarm rate (Type I error) and the undetected loss rate (Type II error) is illustrated for each of the alarm levels. As is to be expected, at small alarm levels Type I errors dominate and conversely at large

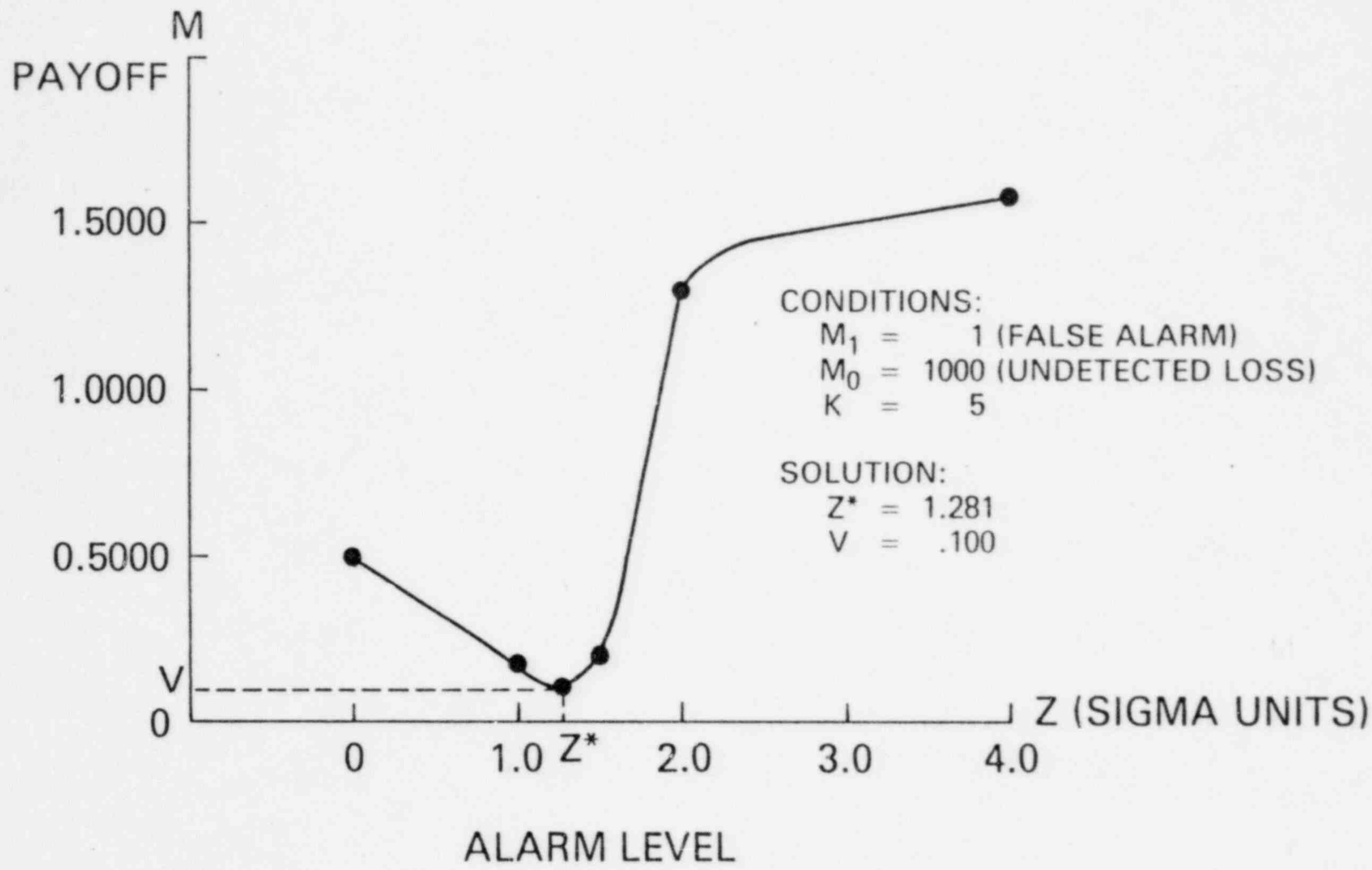

FIGURE 3
PAYOFF AS A FUNCTION OF ALARM LEVEL

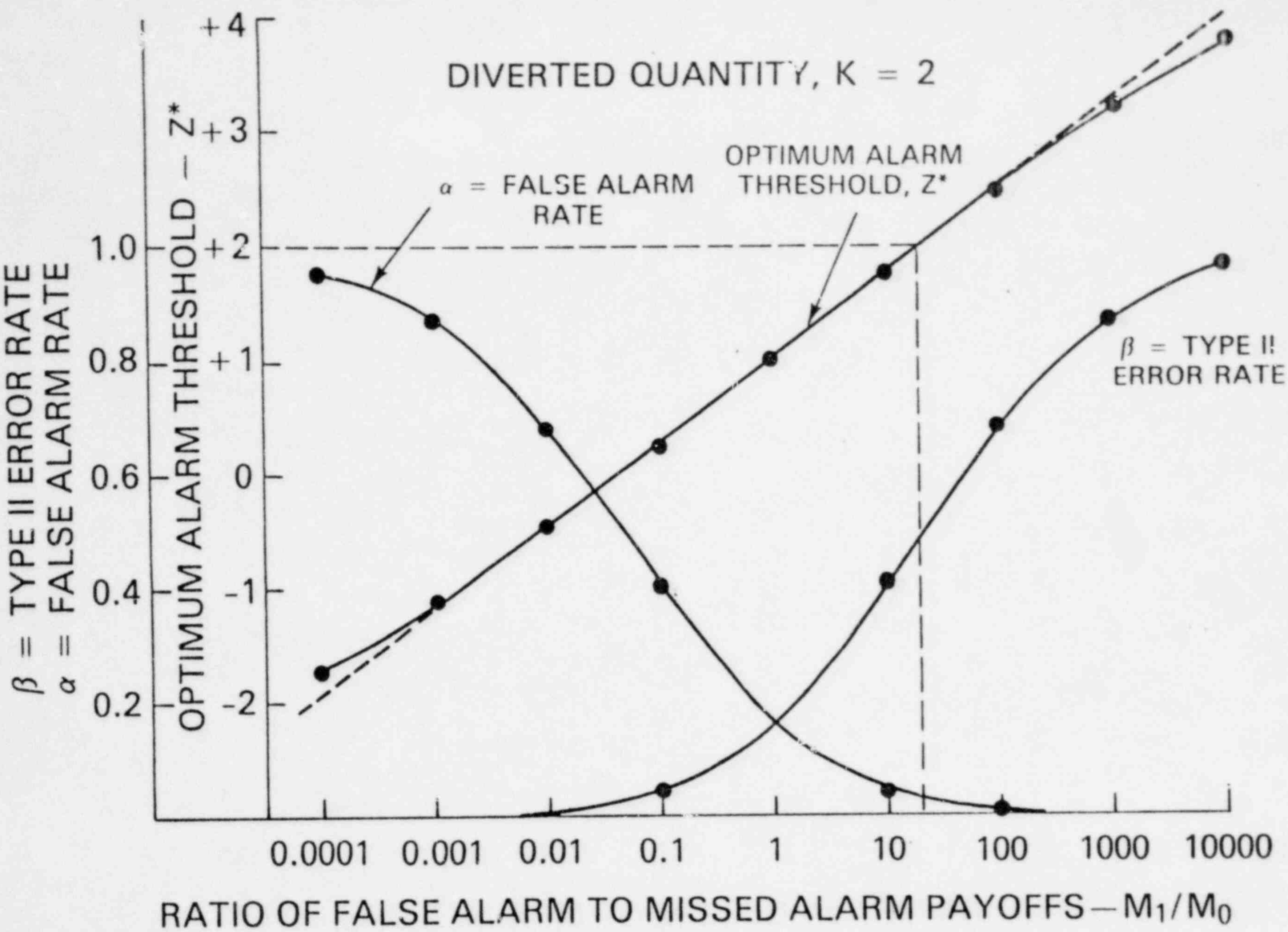

FIGURE 4
SENSITIVITY TO UTILITY RATIOS

alarm levels Type II errors dominate. However, these error rates are explicitly related in the figure to the underlying value judgments M_1/M_0.

Figure 5 and 6 present the sensitivity of the optimum alarm threshold and the error rates to the value of K, the amount diverted, for a specific utility ratio M_1/M_0. When false alarm rate drives the system, i.e., $M_1/M_0 = 100$, Type II errors dominate for smaller values of K, i.e., $K < 7$. When undetected losses drive the system, i.e., $M_1/M_0 = .01$, Type I errors dominate. The optimum alarm threshold increases almost linearly with increases in the value of K for a wide range of ratios M_1/M_0.

Figure 7 presents optimal alarm contours on M_1/M_0 - K coordinates. Since M_1/M_0 and K are the implicit judgments underlying establishment of alarm levels, this figure exposes the underlying value system for hypothesis testing models (of course within the constraints of the present model, i.e., normality and known σ). For example, when the alarm is established at $2\ \sigma$, the nominal 95% confidence level, the decision has implicitly accepted the values of M_1/M_0 and K along the $z^* = 2$ contour. For instance, for an alarm $z = 2$, and an amount of interest $K = 4$, the decision is implicit that the penalty of false alarm is equal to the penalty of undetected loss. The chart clearly illustrates that as false alarm becomes more important, that the optimum alarm will become larger.

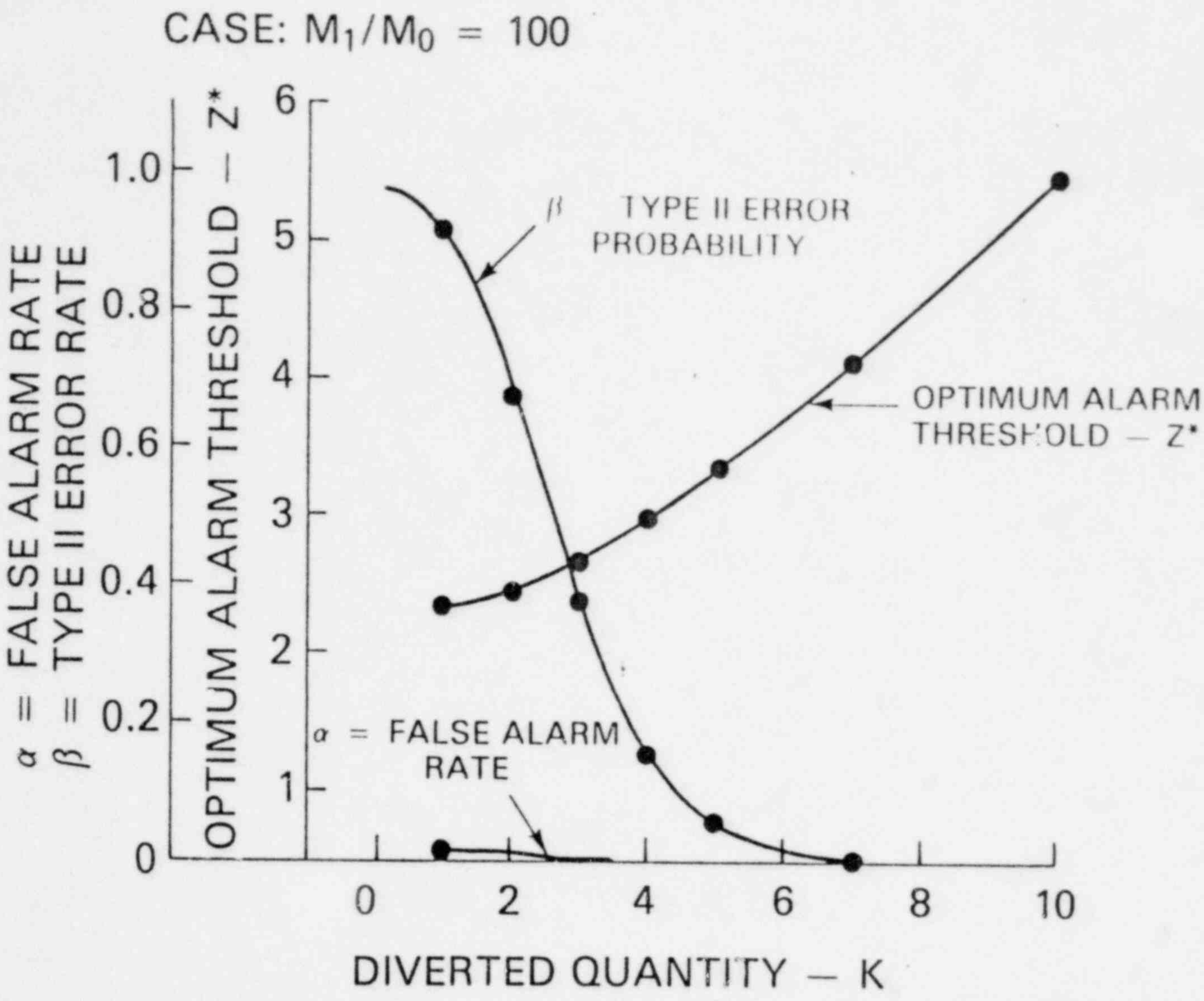

FIGURE 5
SENSITIVITY TO AMOUNT DIVERTED

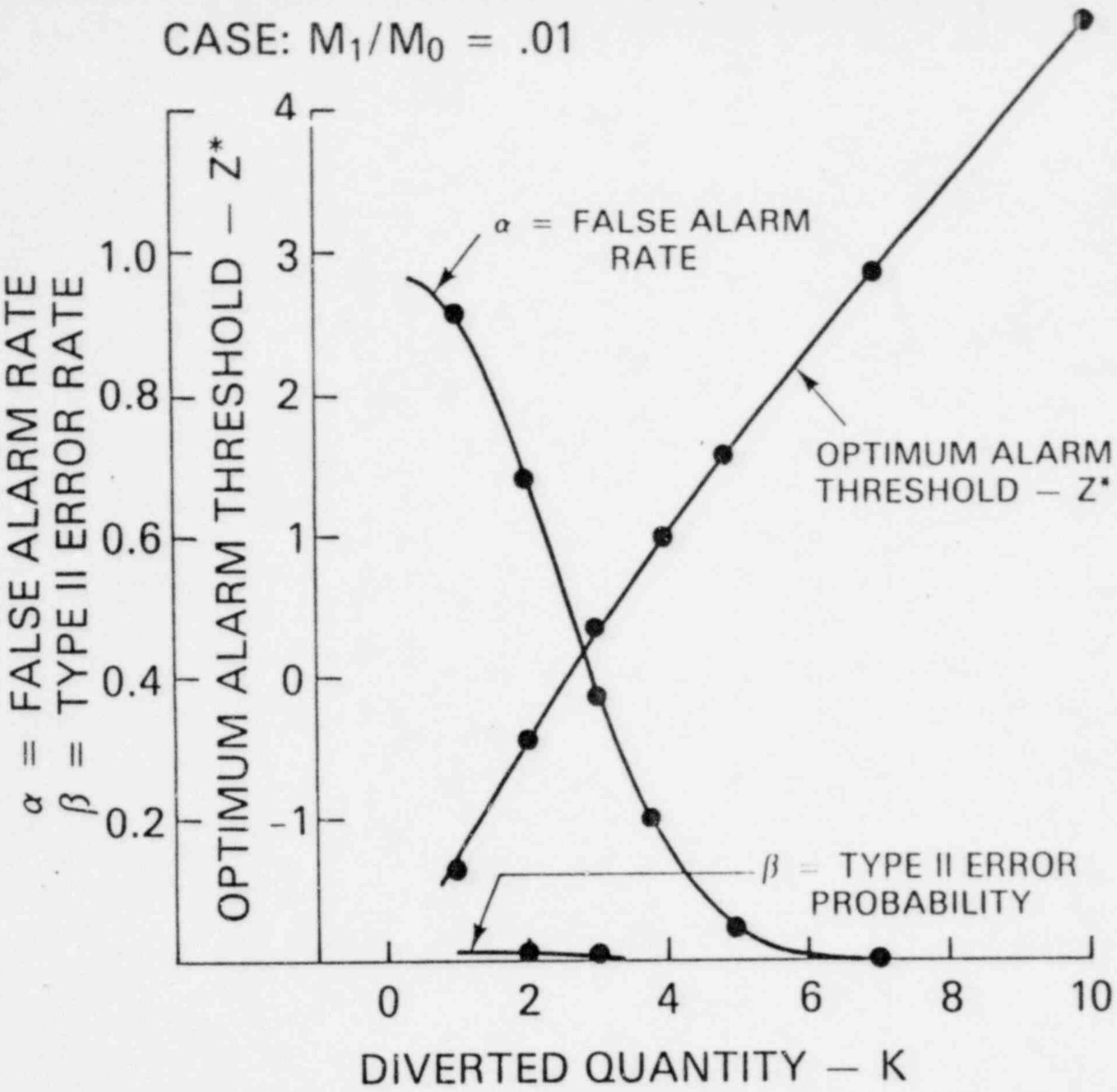

FIGURE 6
SENSITIVITY TO AMOUNT DIVERTED

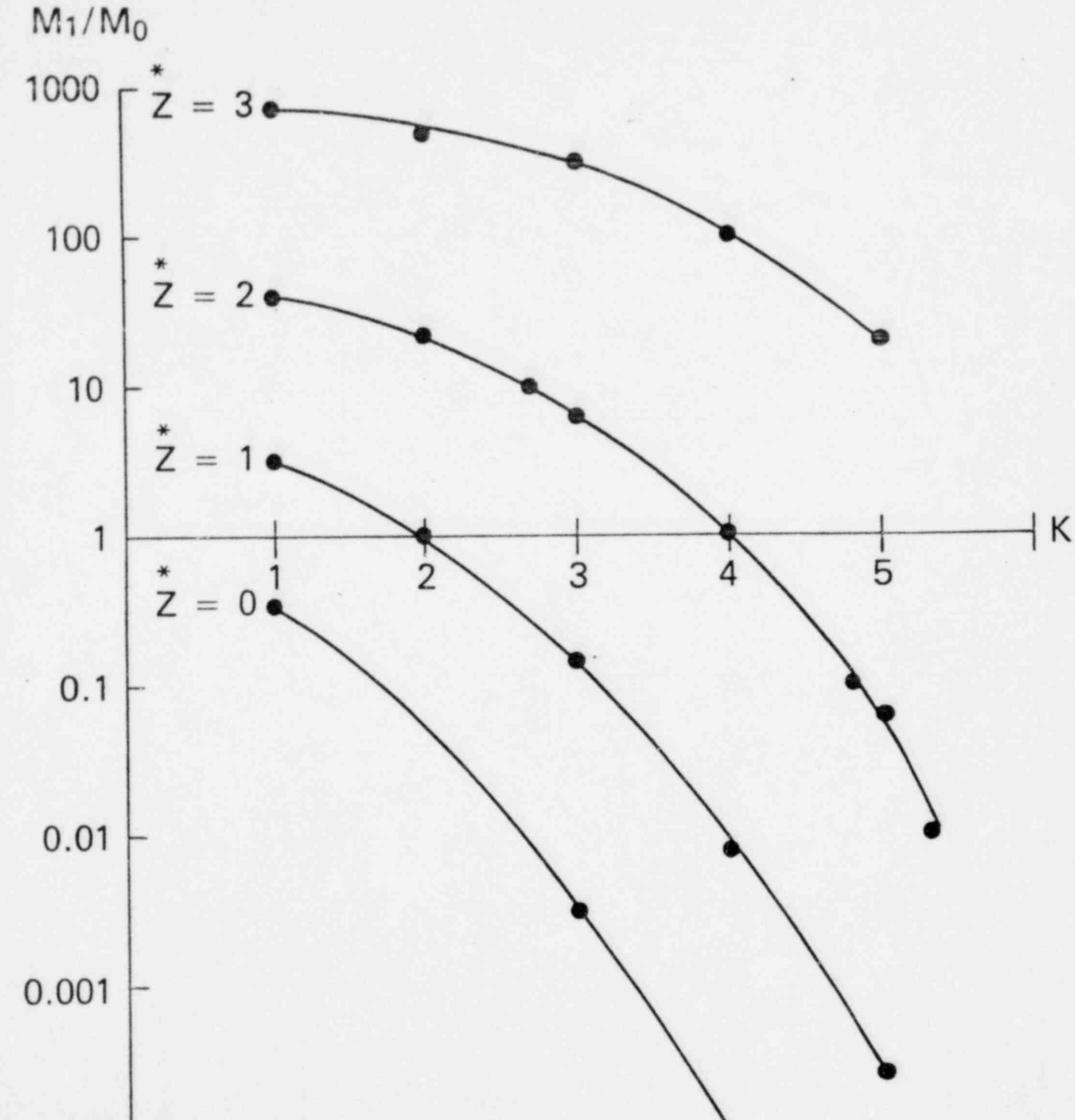

FIGURE 7
OPTIMAL ALARM CONTOURS

9. Summary and Conclusion

An approach based upon the theory of games has been presented that can be useful in exposing implicit value judgments that underlay the establishment of alarm thresholds for statistical hypothesis testing models. This approach has been illustrated by application to a nuclear material accounting problem. However, it should be applicable to other areas where auditing for fraud and theft is a factor.

Incorporation of the factor of competition and adversarial actions by game theory models should significantly improve the effectiveness of threshold discriminants. This would eliminate a major deficiency of the classical statistical approach. However, the two disciplines of game theory and statistics should be viewed as complementary, each discipline contributing that which it can capably do.

10. References

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