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Research Report

QUANTILE ESTIMATION WITH MORE OR LESS FLOOD-LIKE DISTRIBUTIONS

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Abstract: The desirable properties of an estimator relative to an hypothetical population may be irrelevent in practice unless the population at issue more or less resembles the hypothetical population. Evidence that floods are distributed with long, stretched upper tails suggests that use of the more common distributions results in a rather precise underestimation of the extreme quantiles and thereby in the underdesign of flood protection measures.

Introduction

Various distribution functions have been proposed for flood frequency analysis. The most recent distribution, introduced by Houghton [1977,1978], is Thomas' Wakeby distribution,

$$x = m + a\left[1-(1 - F)^{b}\right] - c\left[1-(1 - F)^{-d}\right] \qquad (1)$$

where $F \equiv F(x) = P[X \leq x]$. The Wakeby distribution is defined by five parameters and so a reasonably good fit to a sample might be expected. On the other hand, the more familar Gumbel distribution,

$$x = m - a \ln(-\ln F) \qquad (2)$$

is defined by only two parameters. And in contrast to Wakeby and other 'flood distributions', the Gumbel distribution has a unique value of skewness, $\gamma = 1.14$, and of kurtosis, $\lambda = 5.4$.

Whatever other appeal the Gumbel and Wakeby distributions may have, they serve as paradigms of simplicity and complexity, respectively, of flood distributions. There are several distributions of intermediate complexity; one in particular is the much-used Log-Normal distribution,

$$f(x) = \frac{1}{\sqrt{2\pi}\, b(x - a)} \exp\left\{\frac{-1}{2}\left[\frac{\ln(x - a) - m}{b}\right]^2\right\} \qquad (3)$$

where $f(x) = \frac{dF(x)}{dx}$. For the Log-Normal distribution, values of $\gamma \geq 0$ and $\lambda \geq 3$ are admissible, but such that γ and λ are related by virtue of both being functions of a single parameter, namely b (see e.g., Wallis *et al.*: 1974).

Empirical evidence, in relation to the condition of separation (see Matalas *et al.*: 1975), suggests that the distributions of floods are more nearly Wakeby-like with $b > 1$ and $d > 0$ (i.e., long stretched upper tails) than like any of the other more commomly suggested flood distributions (see Houghton: 1977 and Landwehr *et al.*: 1978). That the Wakeby distribution can satisfy the condition of separation does not imply that indeed floods are distributed as Wakeby. However, the Wakeby distribution provides a plausible description of flood sequences, and it also provides a means for representing the seemingly long, stretched upper tail structures of flood distributions, as well as the tail structures of the distributions of other hydrologic phenomena. Thus the Wakeby distribution provides a convenient analytical and a reasonable hydrologic basis for assessing the relative performances of alternative techniques of estimating the unknown quantiles of the distribution of hydrologic phenomena.

For the specific Wakeby populations considered by Landwehr *et al.* [1979*a,b*], the Wakeby, Log-normal, and Gumbel distributions, with alternative methods of fitting, were used to determine the biases and mean square errors of the estimates of the upper quantiles for each of the populations. Also, the expected underdesign losses associated with the estimates were assessed under the assumptions of linear and quadratic loss functions. The results provide an assessment of i) the relative performance of alternative techniques (i.e., choice of distribution and method of fitting) for estimating the quantiles of Wakeby distributions, and ii) the relative performance of the more common distributions (i.e., Log-Normal, and Gumbel) in estimating the unknown quantiles of flood distributions in hydrologic environments that are Wakeby-like.

Experimental Design

Landwehr *et al.* [1979*a,b*] considered six specific Wakeby distributions, each with lower bound $m = 0$. Values of the parameters (a, b, c, d) and of the statistical characteristics, mean, standard deviation, and coefficients of variation, skewness, and kurtosis, $(\mu, \sigma, C_v, \gamma, \lambda)$ for each of the distributions are given in Table 1. The distributions are depicted in Fig. 1, emphasizing differences in the left tails, and in Fig. 2, emphasizing differences in the right tails.

Figure 1 -- Cumulative distribution functions : WA-1, . . . WA-6

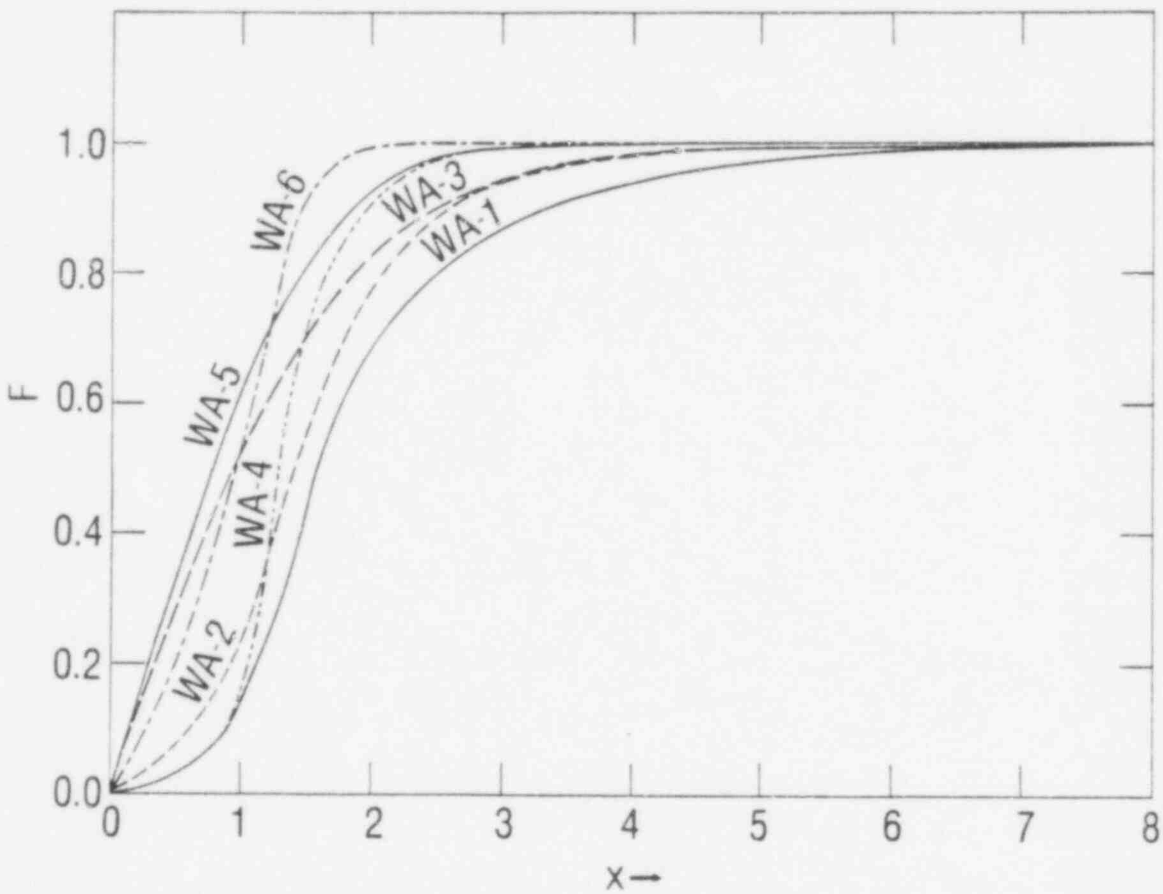
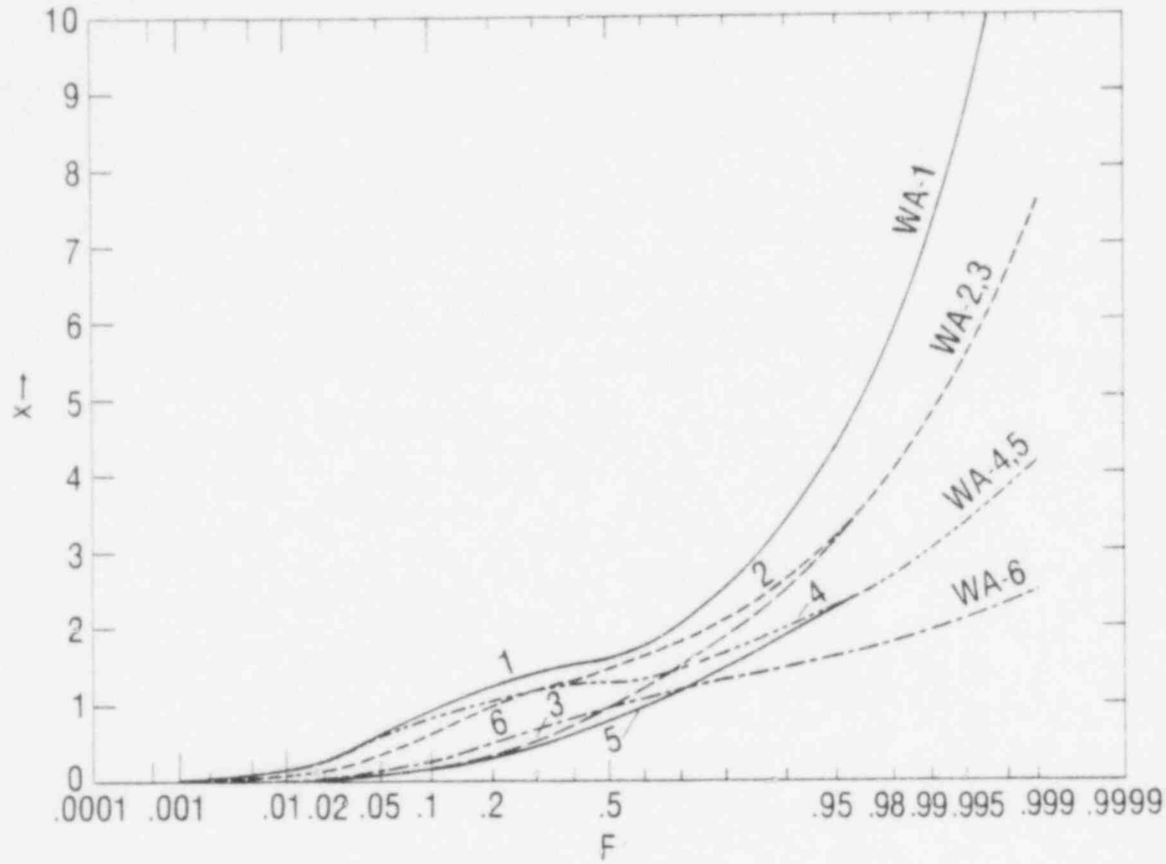


Figure 2 -- Cumulative distribution functions : *WA*-1, . . . *WA*-6



the Log-Normal conditioned quantile estimates by solution (numerical integration) of

$$F = \frac{1}{\sqrt{2\pi}\hat{b}_q} \int_{-\infty}^{\hat{x}_q} \frac{1}{(y - \hat{a}_q)} \exp\left\{-\frac{1}{2}\left[\frac{\ln(y - \hat{a}_q) - \hat{m}_q}{\hat{b}_q}\right]^2\right\} dy \tag{5}$$

and the Gumbel conditioned quantile estimates

$$\hat{x}_q = \hat{m}_q - \hat{a}_q \ln(-\ln F) \tag{6}$$

were determined for specific values of F in the range (0.5, 0.999).

For F specified, the quantile value x is, in practise, the design flood; i.e., the flood magnitude upon which protective measures (structural or nonstructural) are sized. Thus the measure is underdesigned if $(\hat{x} - x) < 0$ and is overdesigned if $(\hat{x} - x) > 0$. The expected over- and underdesign losses, L^+ and L^-, are defined as

$$L^+ = \begin{cases} \alpha^+ k^+ E|\hat{x} - x|^r & ; \hat{x} - x > 0 \\ 0 & ; \hat{x} - x \le 0 \end{cases} \tag{7}$$

$$L^- = \begin{cases} \alpha^- k^- E|\hat{x} - x|^r & ; \hat{x} - x > 0 \\ 0 & ; \hat{x} - x \le 0 \end{cases} \tag{8}$$

where, α^+ and $\alpha^- = 1 - \alpha^+$ denote the probabilities of over- and underdesign, $k^+ \ge 0$ and $k^- \ge 0$, weighting factors reflecting the scale of over- and underdesign costs and $r \ge 0$, a factor defining the analytical shape of the loss functions (see Slack *et al.*: 1975). Hence the expected design loss is given by

$$L = L^+ + L^- \tag{9}$$

If $k^+ = k^- = k$ (i.e., the loss functions are symmetric), the bias, Θ, in the estimates

Table 1 -- Wakeby distributions

Distribution	Parameters					Statistical Characteristics				
	m	a	b	c	d	μ	σ	C_v	γ	λ
$WA-1$	0	1	16.0	4	0.20	1.94	1.34	0.69	4.14	63.74
$WA-2$	0	1	7.5	5	0.12	1.56	0.90	0.58	2.01	14.08
$WA-3$	0	1	1.0	5	0.12	1.18	1.03	0.87	1.91	10.73
$WA-4$	0	1	16.0	10	0.04	1.36	0.51	0.38	1.10	7.69
$WA-5$	0	1	1.0	10	0.04	0.92	0.70	0.76	1.11	4.73
$WA-6$	0	1	2.5	10	0.02	0.92	0.46	0.50	0.00	2.65

Relative to $\gamma = 1.14$ for the Gumbel distribution, three of the six distributions may be regarded as having high skews (*WA*-1, *WA*-2, *WA*-3), two moderate skews (*WA*-4, *WA*-5), and one low skew (*WA*-6). The distributions *WA*-5 and *WA*-6 are less kurtotic and the other Wakeby distributions are more kurtotic than the Gumbel distribution for which $\lambda = 5.4$. In contrast to the distributions *WA*-3, *WA*-5, *WA*-6, the distributions *WA*-1, *WA*-2, and *WA*-4 are more kurtotic than the Log-Normal distribution for comparable values of γ and satisfy the condition of separation.

With respect to each Wakeby distribution, η sequences, each of length $n = 31$ were generated in the manner described by Landwehr *et al.* [1978]. For the q-th sequence conditioned on a particular Wakeby distribution the Wakeby, Log-Normal, and Gumbel parameter estimates, $(\hat{m}_q, \hat{a}_q, \hat{b}_q, \hat{c}_q, \hat{d}_q)$, $(\hat{m}_q, \hat{a}_q, \hat{b}_q)$, and $(\hat{m}_q, \hat{a}_q)$, respectively, were obtained and hence the Wakeby conditioned quantile estimates

$$\hat{x}_q = \hat{m}_q + \hat{a}_q\left[1 - (1 - F)^{\hat{b}_q}\right] - \hat{c}_q\left[1 - (1 - F)^{-\hat{d}_q}\right] \qquad (4)$$

696 108

of x is given by

$$\Theta = \frac{(L^+ - L^-)}{k} \tag{10}$$

where $r = 1$ (i.e., linear loss functions), and the mean square error, Φ, of the estimates is given by

$$\Phi \equiv \frac{L}{k} = \frac{(L^+ + L^-)}{k} \tag{11}$$

where $r = 2$ (i.e., quadratic loss functions). Thus the statistical measures of goodness of estimation, Θ and Φ, are directly related to the economic measures of goodness of design, L^+ and L^-, if the loss functions are symmetric. The use of Φ as a criterion upon which to choose among alternative estimates of x implies that i) the economic loss functions are symmetric and quadratic, and ii) one is indifferent to an over- or an underdesign loss.

Given the η sequences, where η was at least 20,000, conditioned upon a particular Wakeby distribution, the estimates of x, for specific values in the range (0.5, 0.999), were identified as being greater or less than x The probabilities of over- and underdesign were approximated by

$$\tilde{\alpha}^+ = \frac{\eta^+}{\eta} \tag{12}$$

$$\tilde{\alpha}^- = \frac{\eta^-}{\eta} \tag{13}$$

where η^+ denotes the number of estimates that were greater than x, and η^-, the number less than x. Given the set of estimates $\hat{x} > x$, the value $E|\hat{x} - x|^r$ for $r = 1,2$ was approximated by

$$\tilde{E}|\hat{x} - x|^r = \sum_{q=1}^{\eta^+} |\hat{x}_q - x|^r / \eta^+ \tag{14}$$

Similarly, the value $E\,|\,\hat{x} - x\,|^r$ was approximated given the set of estimates $\hat{x} \leq x$.

The expected over- and underdesign losses were taken to be

$$\tilde{L}^+ = \tilde{\alpha}^+ k^+ \tilde{E}\,|\,\hat{x} - x\,|^r \,;\; \hat{x} - x > 0 \tag{15}$$

$$\tilde{L}^- = \tilde{\alpha}^- k^- \tilde{E}\,|\,\hat{x} - x\,|^r \,;\; \hat{x} - x > 0 \tag{16}$$

whereby

$$\tilde{L} = \tilde{L}^+ + \tilde{L}^- \tag{17}$$

For $k^+ = k^- = 1$ (i.e., symmetric, unscaled loss functions), the bias, Θ, and the mean square error, Φ, where approximated by

$$\tilde{\Theta} = \tilde{L}^+ - \tilde{L}^- \tag{18}$$

$$\tilde{\Phi} = \tilde{L} \tag{19}$$

The Log-Normal, (LN), conditioned quantile estimates were obtained by the method of (conventional) moments, MoM, (see e.g., Johnson and Kotz: 1970). The method of maximum likelihood, MxL, a method which in principle is more efficient than MoM when the population is distributed as Log-Normal was not considered. On the basis of some exploratory work, it was noted that for a non-neglible percentage of sequences, the iterative solutions in the course of estimating m showed little if any tendency towards convergence; a drawback to the use of MxL as noted in the statistical literature (see Aitcheson and Brown: 1957 and Johnson and Kotz: 1970).

Three methods for obtaining the Gumbel conditioned quantile estimates were considered: i) MoM, ii) MxL, and iii) probability weighted moments, PWM. These methods were considered previously by Landwehr *et al.* [1979*c*] in the case of sampling from Gumbel populations. The probability weighted moment, $M_{(k)}$, is defined as

$$M_{(k)} = E[X(1 - F)^k] \qquad (20)$$

where k is a real number (see Greenwood *et al.*: 1979). In the case where k is a non-negative integer, an unbiased estimate of $M_{(k)}$, here denoted as $M^*_{(k)}$, is given by

$$M^*_{(k)} = \sum_{j=1}^{n-k} x_j \binom{n-j}{k} / \left(n \binom{n-1}{k}\right) \quad ; \quad k = 0, 1, \ldots \qquad (21)$$

where $x_1 \leq x_2 \leq \ldots \leq x_{n-k}$ (see Landwehr *et al.*: 1979c).

The Wakeby, (WA), conditioned quantile estimates were obtained with the algorithm given by Landwehr *et al.* [1979*b*], as $M^{**}_{(k)}$ with path order [$\forall\ m$; $m = 0$], where $M^{**}_{(k)}$, a biased estimate of $M_{(k)}$, is given by

$$M^{**}_{(k)} = \sum_{j=1}^{n} x_j [(n - j + \alpha)/n]^k / n \qquad (22)$$

where $\alpha = 0.35$ and $x_1 \leq x_2 \leq \ldots \leq x_n$.

Experimental Results: Skewness

For a given population, samples of size n reflect, on the average, skewness less than that characterizing the population. Moment estimates of γ, denoted as $\hat{\gamma}$, are algebraically bounded (see Kirby: 1974) and biased downward (see Wallis *et al.*: 1974 and Landwehr *et al.*: 1978). Based on the η sequences (i.e., samples) of length $n = 31$, the means $\tilde{\mu}(\hat{\gamma})$, and standard deviations, $\tilde{\sigma}(\hat{\gamma})$, of the moment estimates of γ were determined and are given in Table 2.

Table 2 -- Values of $\tilde{\mu}(\hat{\gamma})$ and $\tilde{\sigma}(\hat{\gamma})$ conditioned on WA

Distribution	γ	$\tilde{\mu}(\hat{\gamma})$	$\tilde{\sigma}(\hat{\gamma})$
$WA-1$	4.14	1.69	0.93
$WA-2$	2.01	1.06	0.82
$WA-3$	1.91	1.24	0.66
$WA-4$	1.10	0.66	0.82
$WA-5$	1.11	0.87	0.48
$WA-6$	0.00	-0.06	0.37

On the average, the samples reflect populations with moderate to low skews. The reflection is particularly pronounced in the case of WA-1. In addition, had n been smaller, the downward bias in the estimates of γ would have been even more pronounced. The results for $\hat{\lambda}$ when λ is large are even more striking than for $\hat{\gamma}$, for instance $\tilde{\lambda} = 7.02$ for WA-1 with $n =$ 31 compared to the true value of 63.74. It would appear almost impossible to obtain reasonable moment estimates of γ or λ, at least from samples of normal hydrologic length, and if one admits the hypothesis that floods may be distributed in a Wakeby-like manner with relatively high γ and λ, then $\hat{\gamma}$ and $\hat{\lambda}$ estimated by the method of moments are likely to be severely underestimated, and fitting procedures that use $\hat{\gamma}$ or $\hat{\lambda}$ are likely to give results that do not distinguish between different values of γ or λ. The high skew, high kurtosis hypothesis for flood sequences can be used to explain the apparent lack of success shown by those research studies that have attempted to develop either physically based regional maps of $\hat{\gamma}$, or regression equations for $\hat{\gamma}$ in real space. Note, this is a new and separate difficulty from those that have have already been shown to exist for the estimate of skew in log space (see, Landwehr *et al.*: 1978).

Experimental Results: Quantiles

For each of the six Wakeby populations, the estimates of the upper quantiles, for which $0.5 \leq F \leq 0.999$, as given by the five estimating techniques, were assessed according to bias, Θ, mean square errors, Φ, and expected underdesign loss, L^-, where $k^- = 1$. Tables 3 througth 8 present the approximate values $\tilde{\Theta}$, $\tilde{\Phi}$, and $\tilde{L}^-$, as well as $\tilde{\alpha}^-$, for *WA*-1 through *WA*-6. Table 9 presents an aggregate assessment of the results. From the values given, the values of $\tilde{L}^+$ where $k^+ = 1$ and the values of α^+ may be determined. Also, the values of $\tilde{L}^+$, $\tilde{L}^-$, and $\tilde{L}$ may be determined for any arbitrary values of k^+ and k^-.

696 113

Bias -- Θ

For the upper quantiles, the WA- PWM^{**} estimates generally display the smallest bias. The $LN - MoM$ techniques underestimates the quantiles for all six populations. The G-techniques underestimate the quantiles for the populations with high values of skewness. For populations of moderate skew (i.e., $\gamma \sim 1.14$, the skew of the Gumbel distribution) the biases may be positive or negative. For low values of skewness the G- techniques overestimate the quantiles.

Mean square error -- Φ

The mean square error associated with the estimates provided by the $WA - PWM^{**}$ technique were larger or at least as large as those given by the other techniques over all populations. For samples from low skew distributions, i.e., un-floodlike distributions (e.g., WA-6), the $LN - MoM$ estimates had the smallest Φ values. However, for WA-1 through WA-5, the estimates given by the three G- techniques gave varying values of Φ, but in all cases the values were smaller than those given by the other techniques. It is noted that for the high and low skew populations, $\Phi \cong \Theta^2$.

Also it is noted that among the G- the MxL estimates tended to yield the larger values of Φ relative to the high and low skew populations. Thus amongst the G-techniques the MxL may lead to minimal values of Φ when the population is distributed with near Gumbel skew, but it is not likely to do so if the population is characterized by γ substantially different from γ_G.

Expected underdesign loss -- L^-

As noted above, the mean square error, Φ, is proportional to the expected design loss if the loss functions are symmetric. If such loss functions attain, and if one is indifferent to over-or underdesign, then Φ is a convenient and meaningful measure upon which to base the choice of quantile estimate. However, if underdesign losses are of greater concern, then a more meaningful measure is the expected underdesign loss, L^-.

In the case of the high γ and high λ distributions (i.e., WA-1, WA-2), the $WA - PWM^{**}$ estimates gave the smaller $\tilde{L}^-$ values, inspite of having larger $\tilde{\Phi}$ values. For the other less flood-like distributions, the estimates based on one of the G- techniques gave the smaller $\tilde{L}^-$ values. To reiterate, the use of only a mean square error criterion could lead to some potentially dangerous underdesign, and should only be advocated if one is indifferent to over- or underdesign losses.

Aggregate assessment

If the six Wakeby distributions are considered to attain at different sites in a region, the aggregate (i.e. regional) performance of the quantile estimating techniques may be assessed by the cumulative mean square error, $\sum_{i=1}^{6} \Phi^2(i)$. With respect to the techniques from smallest to largest cumulative mean square errors is: $G - PWM^* \leq G - MoM \leq G - MxL \leq LN - MoM \leq WA - PWM^{**}$. It is noted, however, that the magnitudes of the differences are not large, except with respect to $WA - PWM^{**}$. The assessment masks bias effects, hence the over- or under-design aspects of flood protection measures, but it does provide some appraisal of the total regional error in estimating the quantiles for all sites in a region.

A similar statistic, cumulative squared biases, $\sum_{i=1}^{6} \Theta^2(i)$, yields a ranking for the five techniques of $WA-1 \leq LN-MoM \leq G-MoM \leq G-PWM \leq G-MxL$, an almost perfect reversal of the results for cumulative mean square error. Overall, the WA biases are far smaller than those for any of the other techniques.

Anticipating caveats

It might be said that *If you fit Wakeby's to Wakeby's the results should be good* so let us add that this was not the the purpose of the study. Rather the interest was upon the hypothesis that annual flood sequences are distributed with high skew and high kurtosis and what effect, if any, such an hypothesis would have upon the quantile estimates of interest in flood frequency analysis. However, fitting the Wakeby distribution with the algorithm identified by Landwehr *et al.* [1979*b*] as M^{**} and path [$\forall\ m;\ m = 0$], performs fairly creditably with other high skew but more conventionally kurtotic worlds. For instance, it is possible to generate Log Normal "floods" (see eq. 3), that have μ, σ, and γ identical to those of *WA*-1 "floods" (see Table 1), and to repeat the analysis. The results, given in Table 10, are quite similar to those given in Table 3. Thus whether the distribution is *WA*-1 or *LN*-1, the *WA*- PWM^{**} technique yields estimates that are statistically similar.

There are many other distributions and fitting procedures that might have been selected for inclusion in this study. It appears improbable that any of the more conventional choices would have resulted in at-site extreme quantile estimates that were both less biased and more precise. Without some probability of success in finding a method for which L^- and $\tilde{\Phi}$ would both be minimal the incentive to test other methods becomes somewhat marginal.

However, the possibility exists that if there were sufficient data available in a region to show a separation effect, that there would also be sufficient data to allow for a statistical regionalization, in which the quantile estimates for each site could be made on the basis of the data for the site in question in combination with the data from all other sites in the region. It is believed that such an approach may result in more precise estimates for each site, while showing minimal overall bias, (see Landwehr *et al.*: 1979d).

Conclusions

The biases, Θ, mean square errors, Φ, and expected underdesign losses, L^-, were determined for five quantile estimating techniques relative to the upper quantiles, where $0.5 \le F \le 0.999$, for six specific Wakeby distributions. Although the Wakeby populations had widely varying values of skewness, the samples from the populations reflected populations of more moderate to lower skewness. From an assessment of Θ, Φ, and L^-, the following conclusions were drawn.

1. Over all populations considered, the smallest bias quantile estimates were obtained with the $WA - PWM^{**}$ techniques.
2. If the population skew is much different from the Gumbel skew of about 1.14, either larger or smaller, then the Gumbel techniques give precise (small variability) but inaccurate estimates of the quantile values as shown by $\Phi \cong \Theta^2$ conditioned upon WA-1, WA-2, WA-3, WA-4, and WA-6.
3. No one of the three Gumbel techniques performed consistently better than the other with respect to Θ, Φ, and L^- for all populations. Thus if the population is not known to be Gumbel, one can be rather indifferent to the choice of Gumbel technique, and it does not follow that MxL is preferred.
4. The Gumbel and Log-Normal techniques consistently underestimate the quantiles of high skew populations. To the extent that floods are highly skewed, contrary to the view provided by estimated values of skewness, the use of these techniques lead, on the average, to the underdesign of flood protection measures.
5. If indeed flood distributions are highly skewed, then among the techniques investigated for estimating the upper quantiles, the expected underdesign losses are the smaller when the $WA - PWM^{**}$ technique is used, even though the mean square errors tend to be larger.
6. When considered on an aggregate basis, the Gumbel techniques lead to the smaller values of cumulated mean square errors. However, the value masks the biases which differ considerably among the five techniques.

Table 3

Bias, $\tilde{\Theta}$, and mean square error, $\tilde{\Phi}$, probability of underdesign, $\tilde{\alpha}^-$, expected linear under design loss, $\tilde{L}_1^-$, expected quadratic underdesign loss, $\tilde{L}_2^-$, of quantile estimates given by five fitting procedures conditioned on distribution *WA*-1.

F	.500	.900	.950	.980	.990	.995	.999
Quantile	1.59	3.34	4.28	5.75	7.05	8.54	12.92
Method of Estimation							
				$\tilde{\Theta}$			
$WA-PWM^{**}$	0.02	−0.05	−0.11	−0.16	−0.14	−0.00	1.26
$G-PWM^{*}$	0.16	0.07	−0.24	−0.89	−1.58	−2.46	−5.48
$G-MxL$	0.14	−0.12	−0.50	−1.23	−1.99	−2.93	−6.05
$G-MoM$	0.14	0.21	−0.50	−0.62	−1.25	−2.08	−4.91
$LN-MoM$	0.09	0.12	−0.07	−0.48	−0.93	−1.53	−3.60
				$\tilde{\Phi}$			
$WA-PWM^{**}$	0.03	0.31	0.76	2.43	5.90	14.4	110.7
$G-PWM^{*}$	0.07	0.38	0.67	1.82	3.88	7.9	32.5
$G-MxL$	0.05	0.22	0.56	2.02	4.62	9.5	38.1
$G-MoM$	0.06	0.74	1.23	2.53	4.54	8.3	31.0
$LN-MoM$	0.04	0.53	1.21	3.17	6.10	11.0	36.7

Table 3, Continued

$\tilde{\alpha}^-$

$WA - PWM^{**}$	0.47	0.58	0.60	0.61	0.61	0.60	0.59
$G - PWM^{*}$	0.21	0.51	0.68	0.84	0.91	0.95	0.99
$G - MxL$	0.22	0.65	0.83	0.95	0.98	0.99	1.00
$G - MoM$	0.23	0.47	0.61	0.76	0.83	0.89	0.96
$LN - MoM$	0.33	0.50	0.61	0.71	0.76	0.79	0.85

$\tilde{L}_1^-$

$WA - PWM^{**}$	0.05	0.25	0.40	0.69	0.99	1.38	2.69
$G - PWM^{*}$	0.02	0.20	0.45	1.02	1.66	2.52	5.45
$G - MxL$	0.02	0.25	0.58	1.26	2.00	2.91	6.05
$G - MoM$	0.02	0.19	0.42	0.93	1.52	2.30	5.05
$LN - MoM$	0.04	0.20	0.43	0.91	1.43	2.10	4.35

$\tilde{L}_2^-$

$WA - PWM^{**}$	0.01	0.28	0.38	1.09	2.19	4.23	15.9
$G - PWM^{*}$	0.00	0.11	0.41	[illegible]	3.70	7.74	32.4
$G - MxL$	0.00	0.14	0.52	1.20	4.62	9.35	38.1
$G - MoM$	0.00	0.11	0.39	1.49	3.43	7.11	29.8
$LN - MoM$	0.01	0.12	0.43	1.56	3.21	6.86	26.7

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

696 119

Table 4

Bias, $\tilde{\Theta}$, and mean square error, $\tilde{\Phi}$, probability of underdesign, $\tilde{\alpha}^-$, expected linear under design loss, $\tilde{L}_1^-$, expected quadratic underdesign loss, $\tilde{L}_2^-$, of quantile estimates given by five fitting procedures conditioned on distribution *WA*-2.

F	.500	.900	.950	.980	.990	.995	.999
Quantile	1.43	2.59	3.16	4.00	4.69	5.44	7.45
Method of Estimation							
			$\tilde{\Theta}$				
$WA-PWM^{**}$	0.00	−0.03	−0.06	−0.07	−0.01	0.13	1.04
$G-PWM^{*}$	0.00	0.07	−0.03	−0.25	−0.48	−0.78	−1.73
$G-MxL$	0.00	0.08	−0.02	−0.24	−0.48	−0.77	−1.73
$G-MoM$	0.00	0.08	−0.01	−0.22	−0.45	−0.74	−1.68
$LN-MoM$	0.01	0.06	−0.05	−0.27	−0.50	−0.78	−1.64
			$\tilde{\Phi}$				
$WA-PWM^{**}$	0.02	0.13	0.28	0.77	1.69	3.80	25.0
$G-PWM^{*}$	0.02	0.13	0.19	0.38	0.67	1.17	3.9
$G-MxL$	0.02	0.08	0.12	0.25	0.48	0.92	3.5
$G-MoM$	0.02	0.18	0.29	0.55	0.88	1.46	4.4
$LN-MoM$	0.02	0.15	0.32	0.83	1.56	2.76	8.2

Table 4, Continued

$\tilde{\alpha}^-$

$WA-PWM^{**}$	0.51	0.57	0.59	0.59	0.57	0.56	0.53
$G-PWM^{*}$	0.52	0.46	0.57	0.71	0.79	0.86	0.95
$G-MxL$	0.51	0.42	0.55	0.73	0.84	0.91	0.98
$G-MoM$	0.52	0.47	0.57	0.69	0.77	0.83	0.91
$LN-MoM$	0.49	0.48	0.59	0.69	0.74	0.77	0.82

$\tilde{L}_1^-$

$WA-PWM^{**}$	0.05	0.16	0.24	0.38	0.50	0.65	1.13
$G-PWM^{*}$	0.06	0.10	0.19	0.38	0.58	0.86	1.77
$G-MxL$	0.06	0.08	0.15	0.33	0.53	0.80	1.73
$G-MoM$	0.06	0.11	0.21	0.40	0.61	0.88	1.77
$LN-MoM$	0.05	0.12	0.24	0.50	0.76	1.08	2.03

$\tilde{L}_2^-$

$WA-PWM^{**}$	0.01	0.07	0.15	0.35	0.62	1.06	3.20
$G-PWM^{*}$	0.01	0.03	0.09	0.28	0.57	1.10	3.91
$G-MxL$	0.01	0.02	0.06	0.20	0.45	0.90	3.50
$G-MoM$	0.01	0.04	0.10	0.32	0.65	1.20	4.12
$LN-MoM$	0.01	0.04	0.14	0.49	1.03	1.91	6.16

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

Table 5

Bias, $\tilde{\Theta}$, and mean square error, $\tilde{\Phi}$, probability of underdesign, $\tilde{\alpha}^-$, expected linear underdesign loss, $\tilde{L}_1^-$, expected quadratic underdesign loss, $\tilde{L}_2^-$, of quantile estimates given by five fitting procedures conditioned on distribution *WA*-3.

F	.500	.900	.950	.980	.990	.995	.999
Quantile	0.93	2.49	3.11	3.98	4.68	5.44	7.45
Method of Estimation							
				$\tilde{\Theta}$			
WA − *PWM***	−0.00	−0.01	−0.02	−0.07	−0.12	−0.14	0.10
G − *PWM**	0.09	−0.03	−0.10	−0.26	−0.43	−0.65	−1.44
G − *MxL*	0.07	−0.21	−0.34	−0.58	−0.81	−1.09	−2.02
G − *MoM*	0.09	−0.03	−0.10	−0.24	−0.41	−0.64	−1.42
LN − *MoM*	0.08	−0.05	−0.11	−0.24	−0.38	−0.56	−1.16
				$\tilde{\Phi}$			
WA − *PWM***	0.03	0.18	0.35	0.90	1.88	3.88	21.4
G − *PWM**	0.03	0.17	0.27	0.48	0.72	1.12	3.2
G − *MxL*	0.03	0.17	0.30	0.62	1.02	1.66	4.80
G − *MoM*	0.03	0.21	0.35	0.62	0.94	1.42	2.7
LN − *MoM*	0.04	0.18	0.37	0.63	1.44	2.37	6.6

Table 5, Continued

$\tilde{\alpha}^-$

$WA - PWM^{**}$	0.54	0.53	0.55	0.59	0.60	0.62	0.63
$G - PWM^{*}$	0.31	0.56	0.61	0.68	0.74	0.80	0.91
$G - MxL$	0.34	0.74	0.80	0.86	0.90	0.94	0.98
$G - MoM$	0.31	0.57	0.61	0.68	0.73	0.78	0.88
$LN - MoM$	0.34	0.58	0.63	0.67	0.70	0.73	0.78

$\tilde{L}_1^-$

$WA - PWM^{**}$	0.08	0.17	0.24	0.41	0.59	0.82	1.55
$G - PWM^{*}$	0.03	0.17	0.26	0.40	0.57	0.77	1.51
$G - MxL$	0.03	0.27	0.40	0.62	0.84	1.12	2.02
$G - MoM$	0.03	0.19	0.28	0.44	0.60	0.81	1.55
$LN - MoM$	0.03	0.19	0.29	0.48	0.66	0.89	1.63

$\tilde{L}_2^-$

$WA - PWM^{**}$	0.02	0.08	0.16	0.41	0.81	1.46	4.93
$G - PWM^{*}$	0.00	0.08	0.16	0.34	0.59	1.00	3.14
$G - MxL$	0.01	0.14	0.27	0.65	1.00	1.64	4.80
$G - MoM$	0.00	0.09	0.18	0.39	0.67	1.12	3.40
$LN - MoM$	0.01	0.09	0.19	0.46	0.84	1.44	4.30

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

Table 6

Bias, $\tilde{\Theta}$, and mean square error, $\tilde{\Phi}$, probability of underdesign, $\tilde{\alpha}^-$, expected linear under design loss, $\tilde{L}_1$, expected quadratic underdesign loss, $\tilde{L}_2$, of quantile estimates given by five fitting procedures conditioned on distribution *WA*-4.

F	.500	.900	.950	.980	.990	.995	.999
Quantile	1.28	1.96	2.27	2.69	3.02	3.36	4.18
Method of Estimation							
				$\tilde{\Theta}$			
$WA-PWM^{**}$	0.01	−0.02	−0.01	0.03	0.10	0.21	0.75
$G-PWM^{*}$	0.00	0.03	0.00	−0.07	−0.13	−0.20	−0.41
$G-MxL$	0.00	0.15	0.17	0.16	0.15	0.12	0.02
$G-MoM$	0.00	0.03	0.00	−0.06	−0.12	−0.19	−0.40
$LN-MoM$	0.03	0.02	−0.05	−0.17	−0.28	−0.40	−0.71
				$\tilde{\Phi}$			
$WA-PWM^{**}$	0.00	0.04	0.07	0.40	0.40	0.86	4.97
$G-PWM^{*}$	0.01	0.04	0.05	0.10	0.14	0.20	0.44
$G-MxL$	0.01	0.06	0.10	0.15	0.20	0.26	0.44
$G-MoM$	0.01	0.04	0.06	0.12	0.17	0.23	0.49
$LN-MoM$	0.01	0.04	0.08	0.20	0.37	0.62	1.61

696 124

Table 6, Continued

$\tilde{\alpha}^-$

$WA - PWM^{**}$	0.48	0.57	0.55	0.52	0.49	0.47	0.44
$G - PWM^{*}$	0.53	0.46	0.53	0.62	0.67	0.71	0.78
$G - MxL$	0.52	0.21	0.26	0.32	0.36	0.40	0.480
$G - MoM$	0.53	0.47	0.54	0.62	0.67	0.71	0.78
$LN - MoM$	0.37	0.48	0.61	0.70	0.74	0.77	0.80

$\tilde{L}_1^-$

$WA - PWM^{**}$	0.02	0.08	0.12	0.16	0.19	0.24	0.38
$G - PWM^{*}$	0.04	0.06	0.09	0.15	0.21	0.29	0.48
$G - MxL$	0.04	0.02	0.04	0.07	0.10	0.14	0.25
$G - MoM$	0.04	0.06	0.10	0.16	0.23	0.30	0.49
$LN - MoM$	0.02	0.07	0.14	0.27	0.39	0.53	0.90

$\tilde{L}_2^-$

$WA - PWM^{**}$	0.00	0.02	0.04	0.07	0.11	0.18	0.45
$G - PWM^{*}$	0.00	0.01	0.02	0.06	0.10	0.16	0.39
$G - MxL$	0.00	0.00	0.01	0.02	0.04	0.07	0.19
$G - MoM$	0.00	0.01	0.02	0.06	0.11	0.17	0.42
$LN - MoM$	0.00	0.01	0.05	0.14	0.27	0.48	1.26

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

Table 7

Bias, $\tilde{\Theta}$, and mean square error, $\tilde{\Phi}$, probability of underdesign, $\tilde{\alpha}^-$, expected linear under design loss, $\tilde{L}_1$, expected quadratic underdesign loss, $\tilde{L}_2$, of quantile estimates given by five fitting procedures conditioned on distribution *WA*-5.

F	.500	.900	.950	.980	.990	.995	.999
Quantile	0.78	1.86	2.22	2.67	3.01	3.36	4.18
Method of Estimation							
				$\tilde{\Theta}$			
$WA-PWM^{**}$	−0.01	0.00	0.00	−0.01	−0.02	−0.01	0.17
$G-PWM^{*}$	0.02	−0.03	0.00	0.06	0.10	0.14	0.20
$G-MxL$	0.02	−0.11	−0.10	−0.08	−0.07	−0.05	−0.06
$G-MoM$	0.02	−0.07	−0.05	−0.01	0.02	0.05	0.07
$LN-MoM$	0.05	−0.07	−0.08	−0.09	−0.10	−0.13	−0.20
				$\tilde{\Phi}$			
$WA-PWM^{**}$	0.02	0.07	0.12	0.24	0.45	0.85	4.16
$G-PWM^{*}$	0.01	0.06	0.08	0.14	0.19	0.25	0.41
$G-MxL$	0.01	0.07	0.09	0.12	0.16	0.19	0.26
$G-MoM$	0.01	0.06	0.09	0.14	0.19	0.25	0.41
$LN-MoM$	0.02	0.06	0.10	0.19	0.30	0.45	1.02

Table 7, Continued

$\tilde{\alpha}^-$

$WA-PWM^{**}$	0.55	0.52	0.52	0.55	0.56	0.57	0.59
$G-PWM^{*}$	0.45	0.57	0.51	0.45	0.41	0.40	0.38
$G-MxL$	0.46	0.69	0.65	0.60	0.57	0.56	0.55
$G-MoM$	0.43	0.63	0.58	0.53	0.50	0.49	0.48
$LN-MoM$	0.37	0.62	0.62	0.62	0.63	0.63	0.66

$\tilde{L}_1^-$

$WA-PWM^{**}$	0.06	0.10	0.14	0.20	0.27	0.36	0.64
$G-PWM^{*}$	0.04	0.11	0.12	0.12	0.12	0.13	0.15
$G-MxL$	0.04	0.16	0.17	0.18	0.19	0.21	0.26
$G-MoM$	0.03	0.13	0.15	0.15	0.16	0.17	0.22
$LN-MoM$	0.03	0.13	0.17	0.22	0.27	0.32	0.49

$\tilde{L}_2^-$

$WA-PWM^{**}$	0.01	0.03	0.05	0.11	0.19	0.32	0.92
$G-PWM^{*}$	0.00	0.03	0.04	0.05	0.05	0.06	0.10
$G-MxL$	0.00	0.05	0.07	0.08	0.09	0.11	0.17
$G-MoM$	0.00	0.04	0.07	0.08	0.09	0.09	0.15
$LN-MoM$	0.00	0.04	0.07	0.11	0.16	0.23	0.51

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

Table 8

Bias, $\tilde{\Theta}$, and mean square error, $\tilde{\Phi}$, probability of underdesign, $\tilde{\alpha}^-$, expected linear under design loss, $\tilde{L}_1$, expected quadratic underdesign loss, $\tilde{L}_2$, of quantile estimates given by five fitting procedures conditioned on distribution *WA*-6.

F	.500	.900	.950	.980	.990	.995	.999
Quantile	0.96	1.47	1.62	1.81	1.96	2.12	2.48
Method of Estimation							
				$\tilde{\Theta}$			
$WA-PWM^{**}$	−0.03	0.01	0.01	0.00	0.00	0.03	0.31
$G-PWM^{*}$	−0.12	0.08	0.21	0.36	0.47	0.58	0.83
$G-MxL$	−0.11	0.18	0.34	0.54	0.69	0.83	1.15
$G-MoM$	−0.12	0.04	0.14	0.27	0.37	0.46	0.66
$LN-MoM$	−0.04	−0.02	0.03	0.02	−0.01	−0.05	−0.18
				$\tilde{\Phi}$			
$WA-PWM^{**}$	0.01	0.01	0.02	0.05	0.10	0.19	1.32
$G-PWM^{*}$	0.02	0.02	0.06	0.16	0.26	0.38	0.77
$G-MxL$	0.02	0.05	0.14	0.32	0.52	0.74	1.42
$G-MoM$	0.02	0.02	0.04	0.10	0.17	0.25	0.50
$LN-MoM$	0.01	0.02	0.02	0.03	0.05	0.07	0.16

Table 8, Continued

$\tilde{a}^-$

$WA-PWM^{**}$	0.58	0.46	0.49	0.54	0.54	0.53	0.50
$G-PWM^{*}$	0.93	0.22	0.05	0.01	0.00	0.00	0.00
$G-MxL$	0.91	0.08	0.01	0.00	0.00	0.00	0.00
$G-MoM$	0.92	0.37	0.13	0.03	0.02	0.01	0.00
$LN-MoM$	0.67	0.44	0.43	0.51	0.58	0.65	0.76

$\tilde{L}_1^-$

$WA-PWM^{**}$	0.06	0.04	0.05	0.09	0.12	0.15	0.26
$G-PWM^{*}$	0.13	0.01	0.00	0.00	0.00	0.00	0.00
$G-MxL$	0.12	0.00	0.00	0.00	0.00	0.00	0.00
$G-MoM$	0.12	0.03	0.01	0.00	0.00	0.00	0.00
$LN-MoM$	0.06	0.03	0.03	0.06	0.09	0.13	0.25

$\tilde{L}_2^-$

$WA-PWM^{**}$	0.01	0.00	0.01	0.02	0.04	0.06	0.18
$G-PWM^{*}$	0.02	0.00	0.00	0.00	0.00	0.00	0.00
$G-MxL$	0.02	0.00	0.00	0.00	0.00	0.00	0.00
$G-MoM$	0.02	0.01	0.00	0.00	0.00	0.00	0.00
$LN-MoM$	0.01	0.00	0.00	0.01	0.02	0.04	0.11

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

Table 9

Aggregate outcome: cumulative squared biases, and cumulative mean square errors for the median and right tail quantiles, considering the six populations to occur in one region.

F	.500	.900	.950	.980	.990	.995	.999
Method of Estimation							
			$\sum_{i=1}^{6} \tilde{\Theta}^2$				
$WA - PWM^{**}$	0.00	0.00	0.01	0.04	0.04	0.1	3.4
$G - PWM^{*}$	0.05	0.02	0.12	1.06	3.17	7.5	36.0
$G - MxL$	0.04	0.13	0.52	2.22	5.34	11.1	45.0
$G - MoM$	0.03	0.06	0.26	0.49	1.96	5.3	29.2
$LN - MoM$	0.02	0.03	0.03	0.40	1.35	3.5	17.6
			$\sum_{i=1}^{6} \tilde{\Phi}^2$				
$WA - PWM^{**}$	0.11	0.74	1.59	4.80	10.43	23.9	167.
$G - PWM^{*}$	0.17	0.79	1.32	3.06	5.86	11.0	41.
$G - MxL$	0.14	0.65	1.30	3.50	7.02	13.3	49.
$G - MoM$	0.15	1.25	2.07	4.04	6.92	11.9	41.
$LN - MoM$	0.14	0.98	2.10	5.24	9.80	17.3	54.

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

Table 10

Bias, $\tilde{\Theta}$, and mean square error, $\tilde{\Phi}$, probability of underdesign, $\tilde{\alpha}^-$, expected linear under design loss, $\tilde{L}_1$, expected quadratic underdesign loss, $\tilde{L}_2$, of quantile estimates given by five fitting procedures conditioned on distribution $LN-1$.

F	.500	.900	.950	.980	.990	.995	.999
Quantile	1.55	3.34	4.32	5.85	7.21	8.75	13.18
Method of Estimation							
				$\tilde{\Theta}$			
$WA-PWM^{**}$	0.02	−0.06	−0.15	−0.28	−0.32	−0.23	1.08
$G-PWM^{*}$	0.21	0.01	−0.35	−1.09	−1.85	−2.81	−5.87
$G-MxL$	0.18	−0.31	−0.78	−1.66	−2.53	−3.60	−6.91
$G-MoM$	0.19	0.18	−0.11	−0.76	−1.45	−2.34	−5.24
$LN-MoM$	0.12	0.08	−0.13	−0.56	−1.03	−1.63	−3.56
				$\tilde{\Phi}$			
$WA-PWM^{**}$	0.03	0.40	0.90	2.69	6.3	14.8	110.
$G-PWM^{*}$	0.09	0.45	0.86	2.37	5.1	10.0	38.
$G-MxL$	0.07	0.37	1.06	3.46	7.3	14.1	50.
$G-MoM$	0.08	0.79	1.35	2.86	5.3	9.7	35.
$LN-MoM$	0.05	0.58	1.27	3.261	6.17	11.0	36.

Table 10, Continued

$\tilde{\alpha}^-$

$WA-PWM^{**}$	0.52	0.58	0.60	0.63	0.63	0.62	0.60
$G-PWM^{*}$	0.15	0.54	0.70	0.85	0.92	0.96	0.99
$G-MxL$	0.18	0.74	0.88	0.96	0.99	1.00	1.00
$G-MoM$	0.17	0.47	0.62	0.77	0.84	0.89	0.96
$LN-MoM$	0.31	0.51	0.61	0.72	0.76	0.80	0.84

$\tilde{L}_1^-$

$WA-PWM^{**}$	0.06	0.28	0.45	0.79	1.13	1.57	2.96
$G-PWM^{*}$	0.01	0.26	0.55	1.20	1.93	2.85	5.89
$G-MxL$	0.01	0.41	0.84	1.69	2.55	3.60	6.91
$G-MoM$	0.01	0.23	0.49	1.07	1.70	2.55	5.36
$LN-MoM$	0.03	0.25	0.61	0.99	1.52	2.18	4.34

$\tilde{L}_2^-$

$WA-PWM^{**}$	0.01	0.20	0.48	1.38	2.77	5.3	18.7
$G-PWM^{*}$	0.00	0.18	0.60	2.18	4.89	9.9	37.9
$G-MxL$	0.00	0.30	1.01	3.45	7.35	14.1	49.6
$G-MoM$	0.00	0.16	0.54	1.95	4.32	8.8	33.8
$LN-MoM$	0.00	0.17	0.55	1.82	3.91	7.60	27.0

* indicates $\hat{M}^{*}(k)$ used to estimate PWM

** indicates $\hat{M}^{**}(k)$ used to estimate PWM

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