Houston Lighting & Power Company

Electric Tower
P.O. Box 1700
Houston, Texas 77001

August 8, 1979
ST-HL-AE-367
SFN: V-0100

Mr. Dominic B. Vassallo
Assistant Director
Division of Project Management
United States Nuclear Regulatory Commission
1717 H. Street
Washington, D.C. 20555

Dear Mr. Vassallo:

<div align="center">

South Texas Project
Units 1 & 2
Docket Nos. STN 50-498, 50-499
Transmittal of Technical Publications
Related to Soil Structure Interaction

</div>

The attached technical publications discussing analytical methods related to soil-structure interaction were requested by Mr. E. Licitra of your office in a telephone conversation between Houston Lighting and Power, Mr. Raja Gupta and Mr. Licitra of the Nuclear Regulatory Commission. The articles provide a detailed discussion of the analytical methods proposed to be used in performing soil-structure interaction studies as required by NRC question SEB 130.12.

Mr. Gupta was provided a copy of the attachments by Federal Express courier on August 7, 1979. Should additional information be required, please contact Mr. L. R. Jacobi at (713) 676-7953.

Very truly yours,

E. A. Turner
Vice President
Power Plant Construction
& Technical Services

LRJ:bf
Attachments (4)
CC: Without Attachments
    Director, NRC Office of Inspection & Enforcement
    M. D. Schwarz (Baker & Botts)
    R. Gordon Gooch (Baker & Botts)
    J. R. Newman (Lowenstein, Newman, Reis, Axelrad & Toll)
    D. G. Barker
    A. J. Granger
    R. A. Frazar

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I/I

649   239   7908140 668   A

To	Multiply by
millimeters (mm)	25.40
centimeters (cm)	2.540
meters (m)	0.0254
meters (m)	0.305
kilometers (km)	1.61
meters (m)	0.91
square centimeters (cm^2)	6.45
square meters (m^2)	0.093
square meters (m^2)	0.836
square meters (m^2)	4047.
square kilometers (km^2)	2.59
cubic centimeters (cm^3)	16.4
cubic meters (m^3)	0.028
cubic meters (m^3)	0.765
kilograms (kg)	0.453
kilograms (kg)	907.2
newtons (N)	4.45
newtons (N)	9.81
newtons per square meter (N/m^2)	47.9
kilonewtons per square meter (kN/m^2)	6.9
cubic meters (m^3)	0.0038
liter (dm^3)	3.8
cubic meters (m^3)	1233.
cubic meters/minute (m^3/min)	0.0038
pascals (Pa)	1.00

JOURNAL OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION

Vertical Vibration of Embedded Footings

By Milos Novak[1] and Youpele O. Beredugo[2]

INTRODUCTION

Most of the existing theoretical solutions treat the footing as a rigid body
attached to the surface of an elastic half-space. However, real footings are
usually partially embedded, and experiments indicate that embedment can
considerably affect the dynamic response of footings (7,14).

No rigorous analytical solution of embedded footings is available because
of the obvious mathematical difficulties. The most promising way of approaching
this problem seems to be the finite element analyses as used by Lysmer and
Kuhlemeyer (13), and by Kaldjian (9) for static stiffness. Nevertheless, there
is a need for alternative approximate solutions that would be able to predict
the motion in more degrees-of-freedom and to yield the stiffness and damping
characteristics of embedded footings readily applicable in dynamic analyses of
various structures.

An approximate analytical approach was formulated by Baranov (1), who
assumed that the soil underlying the footing base is an elastic half-space and
that the overlying soil is an independent elastic layer composed of a series
of infinitesimally thin independent elastic layers. The compatibility condition
between the elastic half-space and the overlying elastic layer was satisfied only
at the body and very far from it. Nevertheless, the solution seems to yield
reasonable results in closed forms, and is very versatile and easily applicable
to any vibration mode (3,16). For these reasons, this solution is further extended
in this paper and compared with finite element solutions and with experiments
in order to verify its applicability. Embedment into a stratum is also investigated.

Special attention is devoted to the vertical motion because this is the fundamental vibration mode for checking the theory. A brief consideration of all vibration modes is given in Ref. 16. Coupled horizontal and rocking motion is analyzed in detail in Ref. 3.

EQUATION OF MOTION

Fig. 1 shows a model of the embedded footing-soil system, and the forces acting on the footing. The basic differential equation of motion is

$$m\ddot{w}(t) = P(t) - R_z(t) - N_z(t) \quad \dots \dots \dots \dots \dots \dots \dots (1)$$

in which: m = mass of footing; w = vertical displacement of footing; $P(t)$ = time dependent vertical excitation force; $R_z(t)$ = dynamic vertical reaction at base of footing; and $N_z(t)$ = dynamic vertical reaction along the side surface of footing.

In order to solve Eq. 1, the following assumptions are made:

1. The footing is a rigid cylindrical body with radius r_o.
2. Linear elasticity is assumed.
   The dynamic reaction, $R_z(t)$, is independent of the depth of embedment.
   There is a perfect bond between the sides of the footing and the soil.
5. The excitation force, $P(t)$, is harmonic and acts along the vertical axis.

Under these assumptions, the dynamic reaction at the base can be expressed using elastic half-space solutions. The relation between footing base displacement $w(t)$ and elastic half-space reaction $R_z(t)$ can be written as

$$R_z(t) = Gr_0(C_1 + iC_2)w(t) \quad \dots \dots \dots \dots \dots \dots \dots (2)$$

in which: $\quad C_1 = \dfrac{-f_1}{f_1^2 + f_2^2}, \; C_2 = \dfrac{f_2}{f_1^2 + f_2^2} \quad \dots \dots \dots \dots \dots (3)$

Herein, $f_{1,2}$ = functions of dimensionless frequency $a_0 = \omega r_0 \sqrt{\rho/G}$, Poisson's ratio and stress distribution in the base, G = shear modulus of half space, and r_0 = radius of footing. (Functions $f_{1,2}$ were introduced by Reissner and can be taken from later solutions, e.g., Refs. 4, 12, 18.)

The dynamic reaction, $N_z(t)$, acting on the vertical sides of the footing is a complex function of the embedment depth, l, the dimensionless frequency a_0, the shear modulus, G_s, and density ρ_s, of the adjacent soil layer and also of the quality of the contact between the soil and the footing. If s = the dynamic reaction per unit depth of embedment, then:

$$N_z(t) = \int_0^l s(z,t)dz \quad \dots \dots \dots \dots \dots \dots \dots (4)$$

If the approximate assumption is accepted that s is independent of z, then $s = s(t)$ and Baranov's solution (1), of which the basic assumptions are outlined in the Introduction, can be used. According to Ref. 1, the unit reaction:

$$s = G_s(S_1 + iS_2)w(t) \quad \dots \dots \dots \dots \dots \dots \dots (5)$$

$$s(t) = G_s(S_1 + iS_2)\omega(t)$$
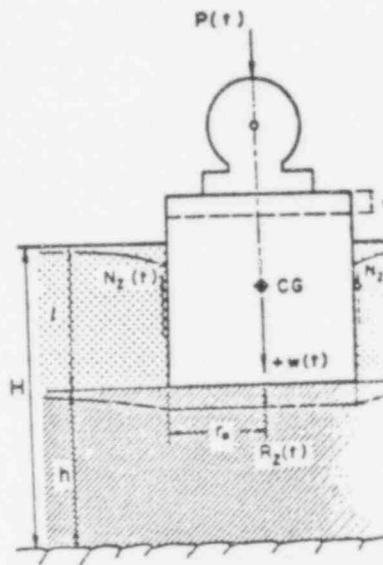
Fig. 1.—Embedded Footing Excited
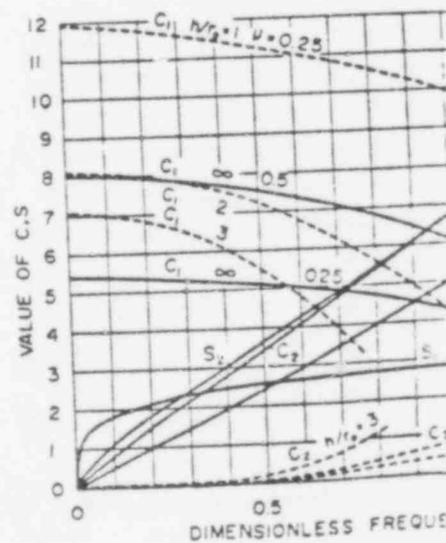
Fig. 2.—Stiffness Parameters C_1, S_1 and Damping Pa Stratum (Dashed Lines) and Side Layer

to the vertical motion because this is the fundamen-
ing the theory. A brief consideration of all vibration
coupled horizontal and rocking motion is analyzed

the embedded footing-soil system, and the forces
and differential equation of motion is

$$\dots \dots \dots \dots \tag{1}$$

 using: w = vertical displacement of footing; $P(t)$
excitation force; $R_z(t)$ = dynamic vertical reaction
$\quad$ = dynamic vertical reaction along the side surface

following assumptions are made:

cylindrical body with radius r_o.
med.
$R_z(t)$, is independent of the depth of embedment.
between the sides of the footing and the soil.
(t), is harmonic and acts along the vertical axis.

the dynamic reaction at the base can be expressed
on. The relation between footing base displacement
and $R_z(t)$ can be written as

$$\dots \dots \dots \dots \dots \dots \tag{2}$$

$$\quad = \frac{f_2}{f_1^2 + f_2^2} \dots \dots \dots \dots \dots \tag{3}$$

dimensionless frequency $a_0 = \omega r_0 \sqrt{\rho / G}$, Poisson's
in the base, G = shear modulus of half space,
(Functions $f_{1,2}$ were introduced by Reissner and
sons, e.g., Refs. 4, 12, 18.)
(t), acting on the vertical sides of the footing is
embedment depth, l, the dimensionless frequency
and density ρ_s, of the adjacent soil layer and also
between the soil and the footing. If s = the dynamic
embedment, then:

$$\dots \dots \dots \dots \tag{4}$$

ion is accepted that s is independent of z, then
on (1), of which the basic assumptions are outlined
ed. According to Ref. 1, the unit reaction:

$$\dots \dots \dots \dots \tag{5}$$

Fig. 1.—Embedded Footing Excited Vertically
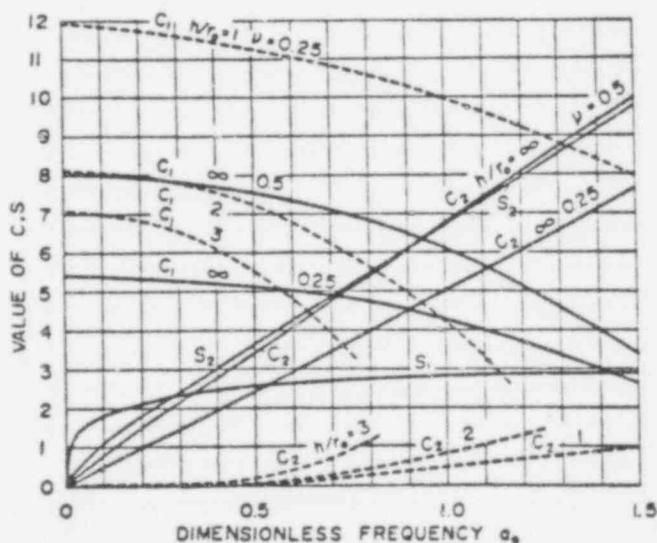
Fig. 2.—Stiffness Parameters C_1, S_1 and Damping Parameters C_2, S_2 for Half-Space, Stratum (Dashed Lines) and Side Layer

in which $S_1 = 2\pi a_0 \dfrac{J_1(a_0)J_0(a_0) + Y_1(a_0)Y_0(a_0)}{J_0^2(a_0) + Y_0^2(a_0)}$ . . . . (6)

$$S_2 = \dfrac{4}{J_0^2(a_0) + Y_0^2(a_0)}$$ . . . . (7)

Herein $J_0(a_0)$, $J_1(a_0)$ = Bessel functions of the first kind of order zero and one, respectively, and $Y_0(a_0)$, $Y_1(a_0)$ = Bessel functions of the second kind of order zero and one. Then the total side reaction is, from Eqs. 4 and 5:

$$N_z(t) = \int_0^l G_s(S_1 + iS_2)w(t)\,dz = G_s l(S_1 + iS_2)w(t)$$ . . . . (8)

Substitution of Eqs. 2 and 8 into Eq. 1 yields the equation of vertical vibrations:

$$m\ddot{w}(t) + Gr_0\left[C_1 + iC_2 + \dfrac{G_s}{G}\dfrac{l}{r_0}(S_1 + iS_2)\right]w(t) = P(t)$$ . . . . (9)

With complex excitation:

$$P(t) = P_0 \exp(i\omega t) = P_0(\cos \omega t + i \sin \omega t)$$ . . . . (10)

the steady-state response is

$$w(t) = w \exp(i\omega t)$$ . . . . (11)

in which P_0 = real force amplitude and w = complex response amplitude. Note the frequency dependent stiffness (spring) constant:

$$k = Gr_0\left(C_1 + \dfrac{G_s}{G}\dfrac{l}{r_0}S_1\right)$$ . . . . (12)

and the frequency dependent damping constant:

$$c = \dfrac{Gr_0}{\omega}\left(C_2 + \dfrac{G_s}{G}\dfrac{l}{r_0}S_2\right)$$ . . . . (13)

Then the real part of the vibration is

$$w(t) = w_0 \cos(\omega t + \phi)$$ . . . . (14)

in which real amplitude

$$w_0 = \dfrac{P_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \dfrac{P_0}{k}\dfrac{1}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2\right]^2 + 4D^2\left(\dfrac{\omega}{\omega_0}\right)^2}}$$ (15)

phase shift $\phi = -a\tan\dfrac{c\omega}{k - m\omega^2}$ . . . . (16)

damping ratio $D = \dfrac{c}{2m\omega_0}$ . . . . (17)

and natural undamped frequency of an embedded footing
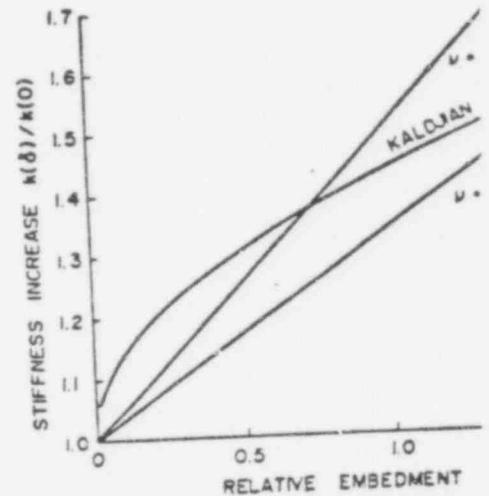
Fig. 3.—Comparison of Stiffness Increase with
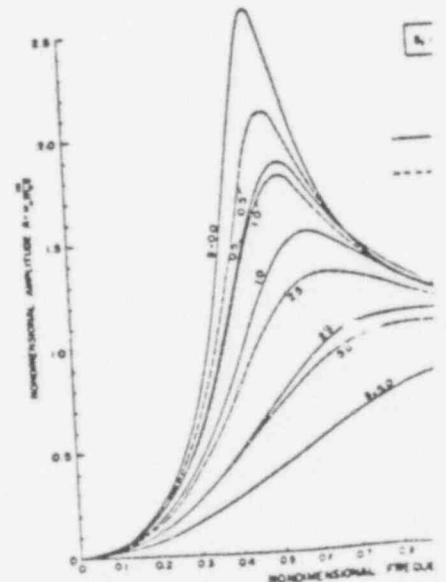
Fig. 4.—Theoretical Response Curves of Footings E
Lines) and Side Fill (Dashed Lines) (Rigid Base Str

$$\frac{s_1 J_0(a_0) + Y_1(a_0) Y_0(a_0)}{J_0^2(a_0) + Y_0^2(a_0)} \quad - \quad \dots \dots (6)$$

$$\dots \dots \dots \dots (7)$$

...sel functions of the first kind of order zero and
..., $Y_1(a_0)$ = Bessel functions of the second kind
n the total side reaction is, from Eqs. 4 and 5:

$$\dots (t)\, dz = G_s l(S_1 + iS_2) w(t) \quad \dots \dots \dots (8)$$

into Eq. 1 yields the equation of vertical vibrations:

$$\frac{G_s}{G}\frac{l}{r_0}(S_1 + iS_2)\Bigg] w(t) = P(t) \quad \dots \dots (9)$$

$$\cos \omega t + i \sin \omega t) \quad \dots \dots \dots (10)$$

$$\dots \dots \dots \dots (11)$$

amplitude and w = complex response amplitude.
ent stiffness (spring) constant:

$$\dots \dots \dots (12)$$

nt damping constant:

$$\dots \dots \dots (13)$$

bration is

$$\dots \dots \dots (14)$$

$$\frac{}{\cdot)^2} = \frac{P_0}{k}\frac{1}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2\right]^2 + 4D^2\left(\dfrac{\omega}{\omega_0}\right)^2}} \quad (15)$$

$$\frac{c\,\omega}{- m\omega^2} \quad \dots \dots (16)$$

$$\dots \dots \dots (17)$$

...ency of an embedded footing
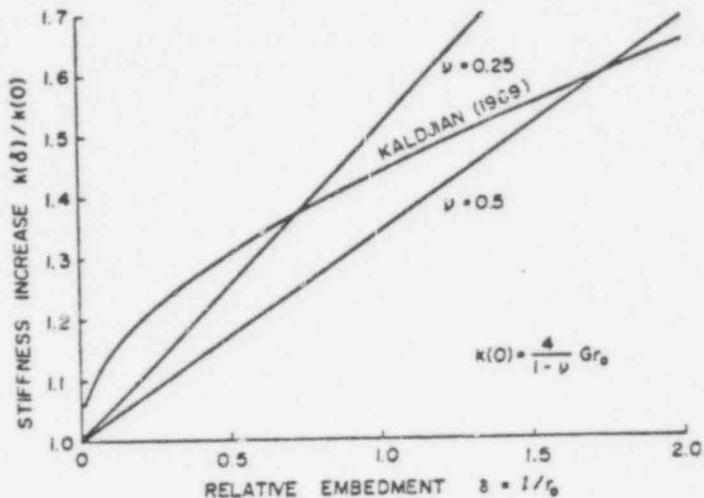
Fig. 3.—Comparison of Stiffness Increase with Finite Element Solution
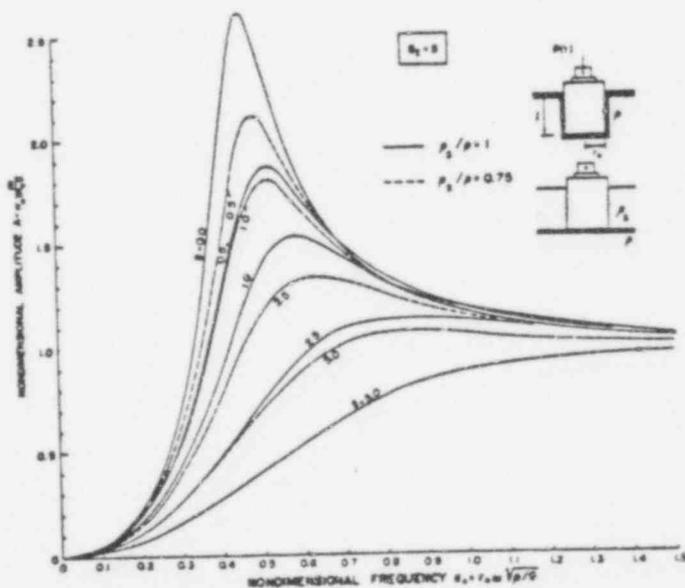
Fig. 4.—Theoretical Response Curves of Footings Embedded in Undisturbed Soil (Full Lines, and Side Fill (Dashed Lines) (Rigid Base Stress Distribution, $B_z = 5$)

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{Gr_0}{m}\left(C_1 + \frac{G_s}{G}\frac{l}{r_0}S_1\right)} \quad \dots\dots\dots\dots(18)$$

(The natural undamped frequency must be determined from Eq. 18 by trial and error approach because it appears in both C_1 and S_1.)
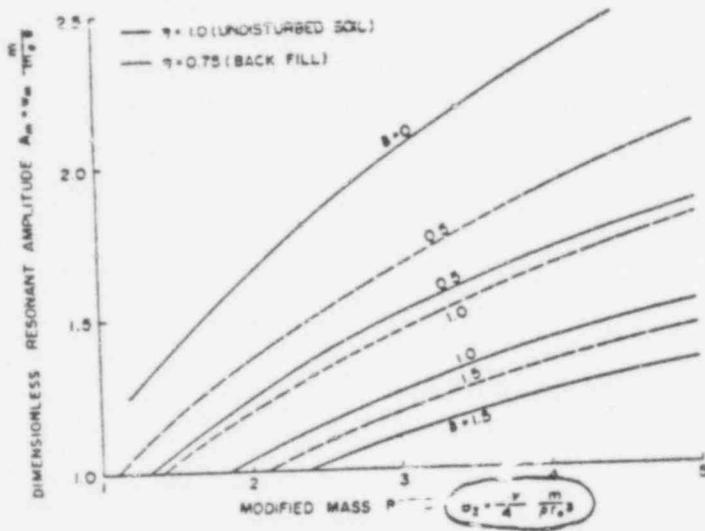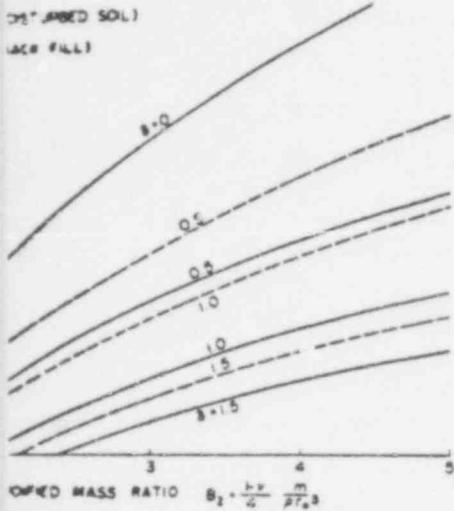
Fig. 5.—Dimensionless Resonant Amplitudes Versus Mass Ratio for Various Embedments $\delta = l/r_0$ (Undisturbed Soil and Side Fill with $\eta = \rho_s/\rho = 0.75$)
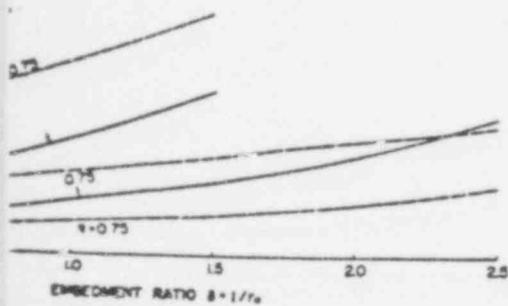
Fig. 6.—Dimensionless Resonant Frequencies Versus Relative Embedment for Various Mass Ratios $B_z(\eta = \rho_s/\rho)$

In most practical problems, the excitation force, $P(t)$, is caused by rotation of an unbalanced mass, m_e, and in Eq. 15 the exciting force amplitude is:

$$P_u = m_e \epsilon \omega^2 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(19)$$

in which e = eccentricity of rotating mass; it is so introduce the dimensionless amplitude, $A = w_a m/($

EXAMPLES OF NUMERICAL RESULTS

From Eq. 15 the response curves of embedded with various assumptions concerning both the b. pertinent stiffness parameters C_1 and S_1, and dar are shown in Fig. 2 and given in a polynomial

Fig. 7.—Resonant Amplitude Ratio Versus Embedme Mass Ratio; Undisturbed Soil $\eta = 1$, Side Fill $\eta = 0.7$

Fig. 8.—Resonant Frequency Ratio Versus Embedm Side Fill)

$$\frac{G_s}{G}\frac{l}{r_o} S_1) \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (18)$$

quency must be determined from Eq. 18 by trial
appears in both C_1 and S_1.)

plitudes Versus Mass Ratio for Various Embed-
and Side Fill with $\eta = \rho_s/\rho = 0.75$)

Frequencies Versus Relative Embedment for Various

the excitation force, $P(t)$, is caused by rotation
nd in Eq. 15 the exciting force amplitude is:

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (19)$$

in which e = eccentricity of rotating mass; it is sometimes convenient to also
introduce the dimensionless amplitude, $A = w_0 m/(m_e e)$.

EXAMPLES OF NUMERICAL RESULTS

From Eq. 15 the response curves of embedded footings can be computed
with various assumptions concerning both the base and side reactions. The
pertinent stiffness parameters C_1 and S_1, and damping parameters C_2 and S_2
are shown in Fig. 2 and given in a polynomial form in Tables 1 and 2 to
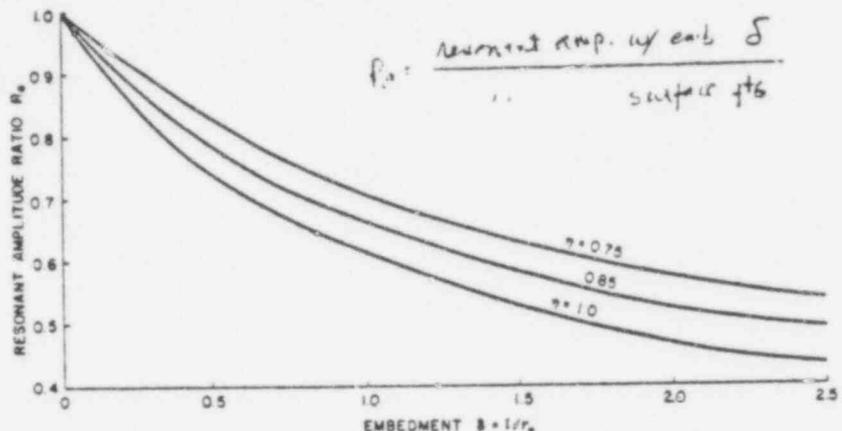
Fig. 7.—Resonant Amplitude Ratio Versus Embedment (Approximately Valid for any
Mass Ratio; Undisturbed Soil $\eta = 1$, Side Fill $\eta = 0.75, 0.85$)
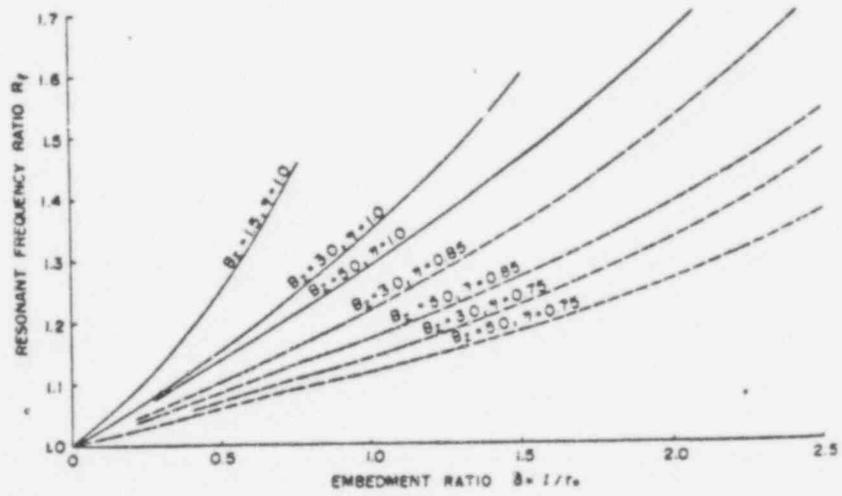
Fig. 8.—Resonant Frequency Ratio Versus Embedment (——— Undisturbed Soil, ———
Side Fill)

facilitate the computation. Curves concerning strata were calculated with $\nu = 0.25$ from Warbarton (20) and are shown in Fig. 2 in dashed lines. Curves
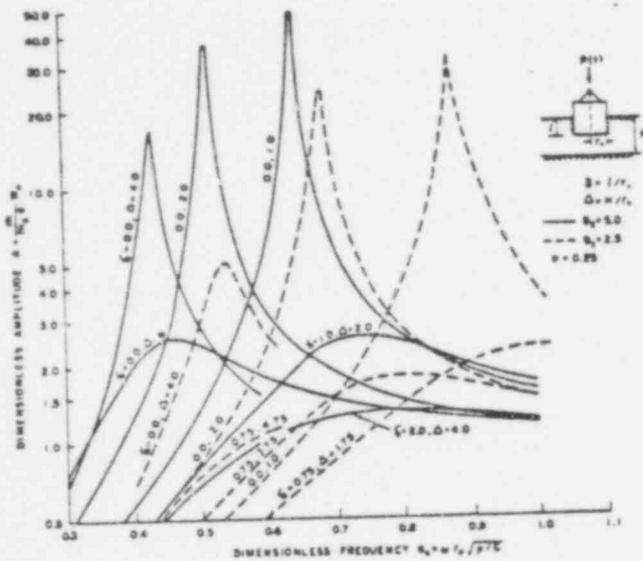
Fig. 9.—Theoretical Response Curves for Vertical Vibration of Footings Embedded in Elastic Stratum (Undisturbed Soil, Various Embedments, and Various Thicknesses of Elastic Stratum)
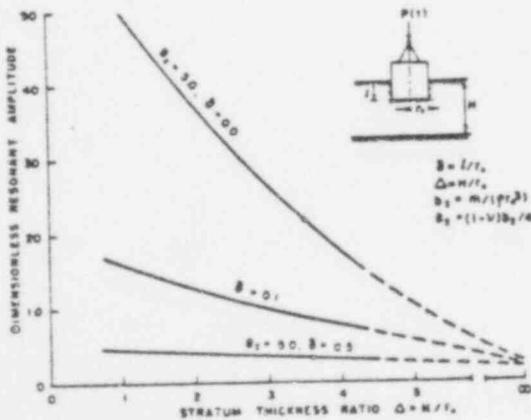
Fig. 10.—Variation of Theoretical Resonant Amplitudes of Vertical Vibration of Footing Embedded in Elastic Stratum with Thickness of Stratum (Undisturbed Soil, Various Embedments)

concerning half-space were computed from Bycroft (4). It will be noticed that $S_1 = 0$ for $a_0 \rightarrow 0$ which results from the assumption that the base overlying

layer is independent of the underlying half-space a:

In Fig. 3 the static stiffness increase found by k stiffness increases obtained from Eq. 12 with C for a rather representative frequency $a_0 = 0.8$. embedment, δ, $k(0)$ = stiffness for surface footi element method and thus, Fig. 3 yields an idea abou of the approximate analytical approach. It may b tend to overestimate the rigidity increase with s shown later herein.

Embedment in Half-Space.—In Fig. 4, the the given for a rigid base stress distribution. The curve a situation where the footing is embedded in an are in reasonable agreement with the finite elem Kuhlemeyer (13). The dashed lines show a differ is surrounded by a side fill whose density $\rho_s =$ was assumed that $G_s/G = (\rho_s/\rho)^3$.]

The curves shown in Fig. 4 were computed f the effect of embedment very distinctly. The vari with embedment for other mass ratios can be seen amplitude ratio R_a = resonant amplitude with emb of surface footing, was found practically indepen 7 can be used for any footing embedded in a half- amplitude reduction.

The natural frequency (ω_0) variations appear ratio too. However, the resonant frequency (freq ratio R_f = resonant frequency with embedment δ footing, highly depends on mass ratio (Fig. 8).

Embedment in Stratum.—With equal ease, the in a stratum can be analyzed using the proper f the stiffness and damping parameters $C_{1,2}$. Sid the same as before.

Examples of theoretical response curves are sh of relative embedment δ and two values of m of embedment, as well as a strong variability of the involved, are obvious. The variations of resona and stratum thickness are plotted in Fig. 10. embedment considerably reduces the dependanc graphs shown do not apply for other mass rat trends to be expected. Any particular situation c: as long as the base reactions are available.

It can be seen that the omission of embe overestimation of amplitudes in the case of a omission of layering can lead to substantial und

SIMPLIFIED DESIGN ANALYSIS

Calculation of amplitudes and resonant freque fied if stiffness parameters C_1 and S_1 are taken as f

...concerning strata were calculated with ν = ...and are shown in Fig. 2 in dashed lines. Curves
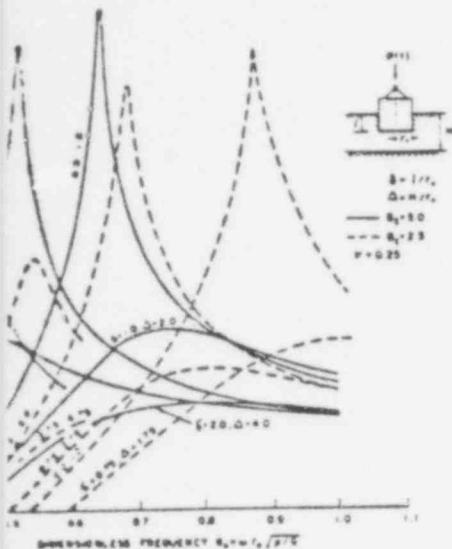
Curves for Vertical Vibration of Footings Embedded
..d Soil, Various Embedments, and Various Thicknesses
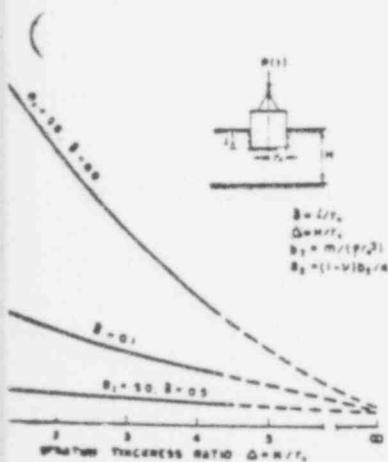
...al Resonant Amplitudes of Vertical Vibration of Footing
...th Thickness of Stratum (Undisturbed Soil, Various

computed from Bycroft (4). It will be noticed that
...ults from the assumption that the base overlying

layer is independent of the underlying half-space and has an infinite span.

In Fig. 3 the static stiffness increase found by Kaldjian (9) is compared with stiffness increases obtained from Eq. 12 with $C_1 = 4/1\nu$ and S_1 computed for a rather representative frequency $a_0 = 0.8$. [$k(\delta)$ = stiffness with the embedment, δ, $k(0)$ = stiffness for surface footing.] Kaldjian used the finite element method and thus, Fig. 3 yields an idea about the accuracy and limitations of the approximate analytical approach. It may be noted that all the theories tend to overestimate the rigidity increase with small embedments as will be shown later herein.

Embedment in Half-Space.—In Fig. 4, the theoretical response curves are given for a rigid base stress distribution. The curves shown in full lines describe a situation where the footing is embedded in an undisturbed soil. The curves are in reasonable agreement with the finite element solution by Lysmer and Kuhlemeyer (13). The dashed lines show a different case in which the footing is surrounded by a side fill whose density $\rho_s = 0.75 \rho$. [In this example it was assumed that $G_s/G = (\rho_s/\rho)^3$.]

The curves shown in Fig. 4 were computed for a high mass ratio to show the effect of embedment very distinctly. The variations of resonant amplitudes with embedment for other mass ratios can be seen in Figs. 5, 6, and 7. Resonant amplitude ratio R_a = resonant amplitude with embedment δ/resonant amplitude of surface footing, was found practically independent of mass ratio. Thus Fig. 7 can be used for any footing embedded in a half-space to estimate the resonant amplitude reduction.

The natural frequency (ω_0) variations appear highly independent of mass ratio too. However, the resonant frequency (frequency at maximum amplitude) ratio R_f = resonant frequency with embedment δ/resonant frequency of surface footing, highly depends on mass ratio (Fig. 8).

Embedment in Stratum.—With equal ease, the response of footings embedded in a stratum can be analyzed using the proper functions f_1 and f_2 to compute the stiffness and damping parameters $C_{1,2}$. Side reaction parameters $S_{1,2}$ are the same as before.

Examples of theoretical response curves are shown in Fig. 9 for several values of relative embedment δ and two values of mass ratio. The essential effect of embedment, as well as a strong variability of the response with all the parameters involved, are obvious. The variations of resonant amplitudes with embedment and stratum thickness are plotted in Fig. 10. With higher embedments, the embedment considerably reduces the dependance on the stratum thickness. The graphs shown do not apply for other mass ratios; however, they indicate the trends to be expected. Any particular situation can be analyzed without difficulty as long as the base reactions are available.

It can be seen that the omission of embedment can result in unrealistic overestimation of amplitudes in the case of a stratum. And, contrarily, the omission of layering can lead to substantial underestimation of amplitudes.

SIMPLIFIED DESIGN ANALYSIS

Calculation of amplitudes and resonant frequencies can be considerably simplified if stiffness parameters C_1 and S_1 are taken as frequency independent (constant),

1300 — DECEMBER 1972 — SM 12 — SM 12 — EMBEDDED FOOTINGS

Table 1.—Stiffness and Damping Parameters for Half Space and Side Layers

ν (1)	Values (2)	Constant parameters (3)	Validity range (4)
	(a) Half Space		
0.0	$C_1 = 4.00 - 0.08356 a_0 + 0.6346 a_0^2 - 2.600 a_0^3 + 1.801 a_0^4 - 0.3646 a_0^5$ $C_2 = 3.438 a_0 + 0.5742 a_0^2 - 1.154 a_0^3 + 0.7433 a_0^4$	$\bar{C}_1 = 3.90$ $\bar{C}_2 = 3.50$	$0 \leq a_0 \leq 1.5$
0.25	$C_1 = 5.37 + 0.364 a_0 - 1.41 a_0^2$ $C_2 = 5.06 a_0$	$\bar{C}_1 = 5.20$ $\bar{C}_2 = 5.00$	
0.5	$C_1 = 8.00 + 2.180 a_0 - 12.63 a_0^2 + 20.73 a_0^3 - 16.47 a_0^4 + 4.458 a_0^5$ $C_2 = 7.414 a_0 - 2.986 a_0^2 + 4.324 a_0^3 - 1.782 a_0^4$	$\bar{C}_1 = 7.50$ $\bar{C}_2 = 6.80$	
	(b) Side Layer	$\bar{S}_1 = 2.70$ $\bar{S}_2 = 6.70$	$0 \leq a_0 \leq 2.00$
any value	$S_1 = 0.2153 a_0 + 2.760 a_0/a_0 + 0.06084$ $S_2 = 6.059 a_0 + 0.7022 a_0/a_0 + 0.01616$		

Compute for the circular foundation

Table 2.—Stiffness and Damping Parameters for Stratum

h/r_0 (1)	Stratum ν = 0.25 (2)	Constant parameters (3)	Validity range (4)
1.0	$C_1 = 12.23 - 1.178 a_0 - 0.3056 a_0^2 - 1.177 a_0^3 + 0.4160 a_0^4$ $C_2 = 0.2395 a_0^2 + 0.5646 a_0^3 + 0.0227 a_0^4 - 0.3403 a_0^5 + 0.203 a_0^6$	$\bar{C}_1 = 10.0$ $\bar{C}_2 = 0.30$	$0 \leq a_0 \leq 1.50$
2.0	$C_1 = 8.13 + 0.8516 a_0 - 3.664 a_0^2 - 8.289 a_0^3 + 11.18 a_0^4 - 3.978 a_0^5$ $C_2 = 0.004044 a_0 - 0.7386 a_0^2 + 13.27 a_0^3 - 39.61 a_0^4 + 49.8 a_0^5 - 26.95 a_0^6 + 5.069 a_0^7$	$\bar{C}_1 = 7.00$ $\bar{C}_2 = 0.45$ $\bar{C}_1 = 5.5$	$0 \leq a_0 \leq 1.25$ $0 \leq a_0 \leq 0.81$

649 249

h/r_0		Constant parameters	Validity range
0.0	$C_1 \approx 4.00 - 0.08356\,a_0 + 0.6346\,a_0^2 - 2.600\,a_0^3 + 01\,a_0^4 - 0.3646\,a_0^5$	$\bar{C}_1 = 3.90$	$0 \le a_0 \le 1.5$
0.25	$C_2 = 3.438\,a_0 + 0.5742\,a_0^2 - 1.154\,a_0^3 + 0.7433\,a_0^4$	$\bar{C}_2 = 3.50$	
	$C_1 = 5.37 + 0.364\,a_0 - 1.41\,a_0^2$	$\bar{C}_1 = 5.20$	
	$C_2 = 5.06\,a_0$	$\bar{C}_2 = 5.00$	
0.5	$C_1 = 8.00 + 2.180\,a_0 - 12.63\,a_0^2 + 20.73\,a_0^3 - 16.47\,a_0^4 + 4.458\,a_0^5$	$\bar{C}_1 = 7.50$	
	$C_2 = 7.414\,a_0 - 2.986\,a_0^2 + 4.324\,a_0^3 - 1.782\,a_0^4$	$\bar{C}_2 = 6.80$	
	(b) Side Layer		
any value	$S_1 = 0.2153\,a_0 + 2.760\,a_0/a_0 + 0.06084$	$\bar{S}_1 = 2.70$	$0 \le a_0 \le 2.00$
	$S_2 = 6.059\,a_0 + 0.7022\,a_0/a_0 + 0.01616$	$\bar{S}_2 = 6.70$	

Table 2.—Stiffness and Damping Parameters for Stratum

h/r_0 (1)	Stratum $\nu = 0.25$ (2)	Constant parameters (3)	Validity range (4)
1.0	$C_1 = 12.23 - 1.178\,a_0 - 0.3056\,a_0^2 - 1.177\,a_0^3 + 0.4160\,a_0^4$	$\bar{C}_1 = 10.0$	$0 \le a_0 \le 1.50$
	$C_2 = 0.2395\,a_0^2 + 0.5646\,a_0^3 + 0.0227\,a_0^4 - 0.3403\,a_0^5 + 0.203\,a_0^6$	$\bar{C}_2 = 0.30$	
2.0	$C_1 = 8.13 + 0.8516\,a_0 - 3.664\,a_0^2 - 8.289\,a_0^3 + 11.18\,a_0^4 - 3.978\,a_0^5$	$\bar{C}_1 = 7.00$	$0 \le a_0 \le 1.25$
	$C_2 = 0.004044\,a_0 - 0.7386\,a_0^2 + 13.27\,a_0^3 - 39.61\,a_0^4 + 49.8\,a_0^5 - 26.95\,a_0^6 + 5.069\,a_0^7$	$\bar{C}_2 = 0.45$	
3.0	$C_1 = 7.04 + 0.4659\,a_0 - 6.989\,a_0^2$	$\bar{C}_1 = 5.5$	$0 \le a_0 \le 0.81$
	$C_2 = 0.7361\,a_0^2 - 1.462\,a_0^3 + 3.573\,a_0^4$	$\bar{C}_2 = 0.65$	
4.0	$C_1 = 6.579 - 0.2422\,a_0 - 0.3889\,a_0^2 - 29.69\,a_0^3 + 7.711\,a_0^4 + 76.44\,a_0^5 - 77.42\,a_0^6$	$\bar{C}_1 = 4.30$	$0 \le a_0 \le 0.62$
	$C_2 = 0.02804\,a_0 + 3.02\,a_0^2 + 7.458\,a_0^3 - 184.2\,a_0^4 + 655.7\,a_0^5 - 804.9\,a_0^6 + 314.2\,a_0^7$	$\bar{C}_2 = 1.00$	

and damping parameters C_2 and S_2 as proportional to dimensionless frequency, a_0 (Fig. 2). These assumptions seem well justified for the embedment parameters S_1 and S_2 if $a_0 \geqslant 0.1$, as they do for C_2 for a half space; C_2 for a stratum is less linear but very small, thus adding little to the total damping. The constancy of C_1 may be generally questioned (see Ref. 12); nevertheless, it seems acceptable in the shown frequency range. Altogether, the simplifying assumptions are consistent with the generally accepted practice in design analysis of surface footings (17).

Therefore, assume that $C_1 = \bar{C}_1$ and $S_1 = \bar{S}_1$ and that:

$$C_2 = \bar{C}_2 a_0, \quad S_2 = \bar{S}_2 a_0 \quad \dots \dots (20)$$

in which $\bar{C}_{1,2}$ and $\bar{S}_{1,2}$ = constants whose values can be readily established
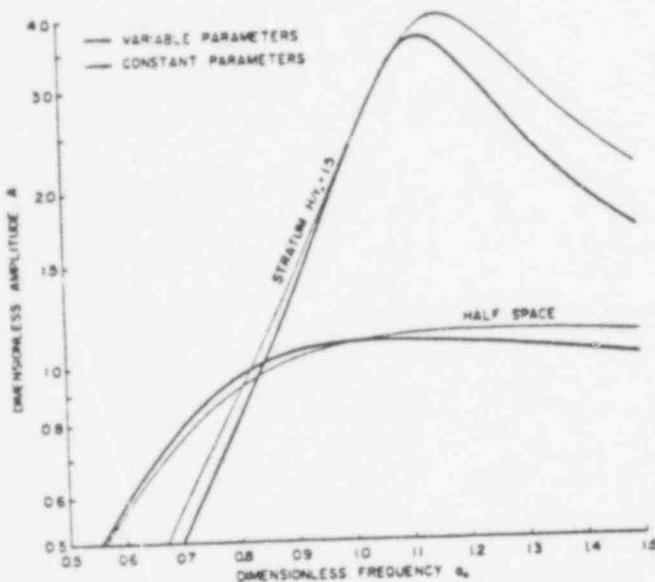
Fig. 11.—Comparison of Response Curves Computed with Variable and Constant Parameters ($l/r_0 = 0.5$, $b_0 = 8.1$, $\rho/\rho_s = 0.75$ and $G_s/G = 0.5$)

from Fig. 2, Tables 1 and 2 or any other suitable formulae for $f_{1,2}$. Several suitable values of $\bar{C}_{1,2}$ and $\bar{S}_{1,2}$ are given in Tables 1 and 2.

With constant stiffness parameters, $\bar{C}_1$ and $\bar{S}_1$ substituted in Eq. 12 in place of C_1 and S_1, the frequency independent stiffness constant is obtained, and with it, the natural undamped frequency ω_0 directly follows from Eq. 18.

Substitution of Eqs. 20 into Eq. 13 yields the frequency independent damping constant for embedded footings

$$c = r_0^3 \sqrt{\rho G} \left(\bar{C}_2 + \bar{S}_2 \frac{1}{r_0} \sqrt{\frac{\rho_s}{\rho} \frac{G_s}{G}} \right) \quad \dots \dots (21)$$

or the damping ratio

$$D = \frac{1}{2\sqrt{b_0}} \frac{\left(\bar{C}_2 + \bar{S}_2 \frac{1}{r_0} \sqrt{\frac{\rho_s}{\rho} \frac{G_s}{G}} \right)}{\sqrt{\bar{C}_1 + \frac{G_s}{G} \frac{1}{r_0} \bar{S}_1}} \quad \dots \dots$$

in which mass ratio $b_0 = m/\rho r_0^3$.

Then, the vertical amplitude is obtained from E interest.

The amplitude at natural frequency ω_0 (slightly amplitude) is simply:

$$w_0(\omega_0) = \frac{P_0}{k} \frac{1}{2D} \quad \dots \dots$$

or with frequency variable excitation (Eq. 19):

$$w_0(\omega_0) = \frac{m_e e}{m} \frac{1}{2D} \quad \dots \dots$$

as in any one degree-of-freedom system (Voigt mo

Values of G can be determined by tests, or f 17.

In Fig. 11, example of response curves comp constant parameters are shown for embutment in The differences decrease with increasing embedm appear to yield an accuracy sufficient for practica

COMPARISON WITH EXPERIMENTS

In order to assess the practical applicability solution, the theoretical results have been compar laboratory experiments (5), and with a series of University of Western Ontario by the second wri

In the latter tests, two concrete blocks were × 27 in. × 48 in. (0.686 m × 0.686 m × 1.2 × 38 in. × 48 in. (0.483 m × 0.966 m × 1.2 concrete block was cast directly in the foundatior consisted of a mechanical oscillator LAZAN ar coupled with a Kopp Variator.

Different embedments were obtained by care the footing sides to the appropriate depth. The m at the test site are: (1) De ity of undisturbed sor (2) Poisson's ratio = 0.36; (3) shear modulus kg/cm²); (4) equivalent radius of footing = 1 of footing = 15.48; and (6) Lysmer's modified m

The experiments are described in more deta the response curves in rocking modes are also g

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$$\left. \frac{\rho_s}{\rho} \frac{G_s}{G} \right) \dots \dots \dots \dots \dots (21)$$

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or the damping ratio

$$D = \frac{1}{2\sqrt{b_0}} \frac{\left(\bar{C}_2 + \bar{S}_2 \frac{l}{r_0} \sqrt{\frac{\rho_s}{\rho} \frac{G_s}{G}} \right)}{\sqrt{\bar{C}_1 + \frac{G_s}{G} \frac{l}{r_0} \bar{S}_1}} \dots \dots (22)$$

in which mass ratio $b_0 = m/\rho r_0^3$.

Then, the vertical amplitude is obtained from Eq. 15 for any frequency of interest.

The amplitude at natural frequency ω_0 (slightly smaller than the maximum amplitude) is simply:

$$w_0(\omega_0) = \frac{P_0}{k} \frac{1}{2D} \dots \dots \dots \dots (23)$$

or with frequency variable excitation (Eq. 19):

$$w_0(\omega_0) = \frac{m_e e}{m} \frac{1}{2D} \dots \dots \dots \dots (24)$$

as in any one degree-of-freedom system (Voigt model).

Values of G can be determined by tests, or found in literature, e.g., Ref. 17.

In Fig. 11, examples of response curves computed with both variable and constant parameters are shown for embedment in a stratum and in a half space. The differences decrease with increasing embedment. The constant parameters appear to yield an accuracy sufficient for practical purposes.

COMPARISON WITH EXPERIMENTS

In order to assess the practical applicability of the approximate analytical solution, the theoretical results have been compared with earlier field tests (14), laboratory experiments (5), and with a series of field tests carried out at the University of Western Ontario by the second writer.

In the latter tests, two concrete blocks were used with dimensions 27 in. × 27 in. × 48 in. (0.686 m × 0.686 m × 1.22 m) (square base) and 19 in. × 38 in. × 48 in. (0.483 m × 0.966 m × 1.22 m) (rectangular base). Each concrete block was cast directly in the foundation pit. The excitation equipment consisted of a mechanical oscillator LAZAN and a 220 v. three-phase motor coupled with a Kopp Variator.

Different embedments were obtained by carefully removing the soil around the footing sides to the appropriate depth. The major soil and footing properties at the test site are: (1) Density of undisturbed soil = 103.0 lb/ft³ (1,650 kg/m³); (2) Poisson's ratio = 0.38; (3) shear modulus = 6.6 lb/ft² × 10⁵ lb/ft² (320 kg/cm²); (4) equivalent radius of footing = 1.26 ft (0.344 m); (5) mass ratio of footing = 15.48; and (6) Lysmer's modified mass factor = 2.40.

The experiments are described in more detail in Refs. 2 and 16, in which the response curves in rocking modes are also given.

The comparison of the theory with the experi
nonlinearity which makes the nondimensional reso
cies dependent upon the intensity of excitation.
can be analyzed using a procedure given in Re
inevitable scatter in the comparison of experiment.
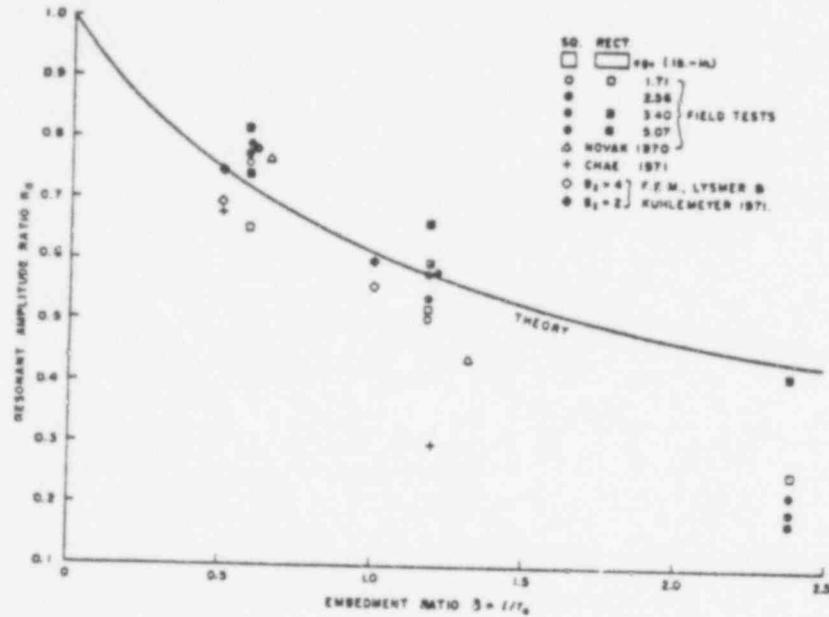
Fig. 12.—Comparison of Theoretical and Experimental Resonant Amplitude Ratio Variations with Embedment in Undisturbed Soil (Theoretical Curve Approximately Applies for all Mass Ratios)

Fig. 14.—Comparison of Theoretical Resonant Ampl of Footings Embedded in Back Fill Having Density R
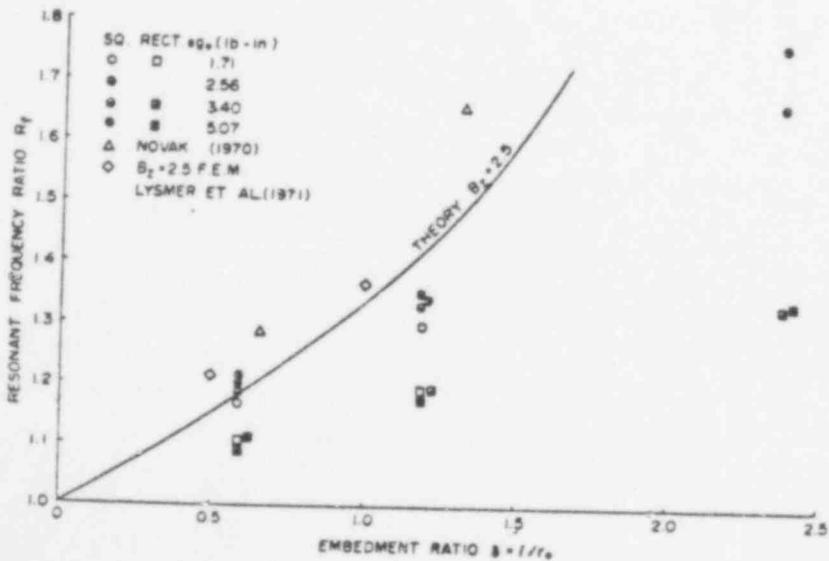
Fig. 13.—Comparison of Theoretical and Experimental Resonant Frequency Ratio Variations with Embedment in Undisturbed Soil

Fig. 15.—Comparison of Theoretical Resonant Fre Footing Embedded in Back Fill Having Density Rati

A second difficulty is due to the fact that the
of the surface footing are consistently found t
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The comparison of the theory with the experiments is complicated due to nonlinearity which makes the nondimensional resonant amplitudes and frequencies dependent upon the intensity of excitation. (Nonlinear response curves can be analyzed using a procedure given in Ref. 15.) Nonlinearity causes an inevitable scatter in the comparison of experimental results with a linear theory.
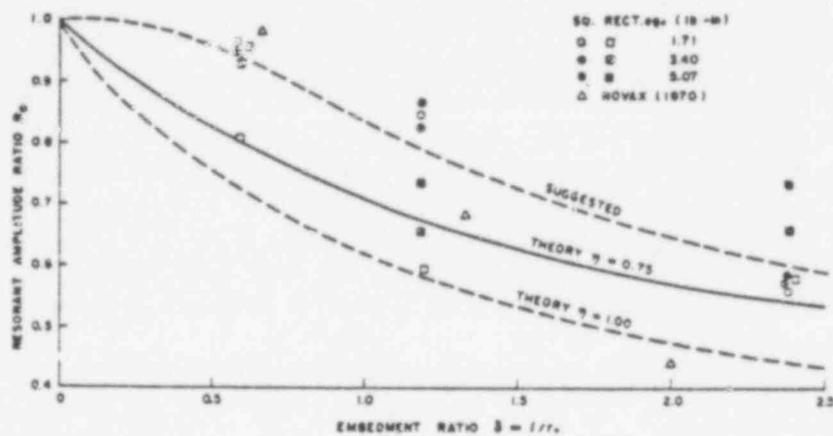
Fig. 14.—Comparison of Theoretical Resonant Amplitude Reduction with Field Tests of Footings Embedded in Back Fill Having Density Ratio $\eta = \rho_s/\rho = 0.765$

oretical and Experimental Resonant Frequency Ratio
Undisturbed Soil
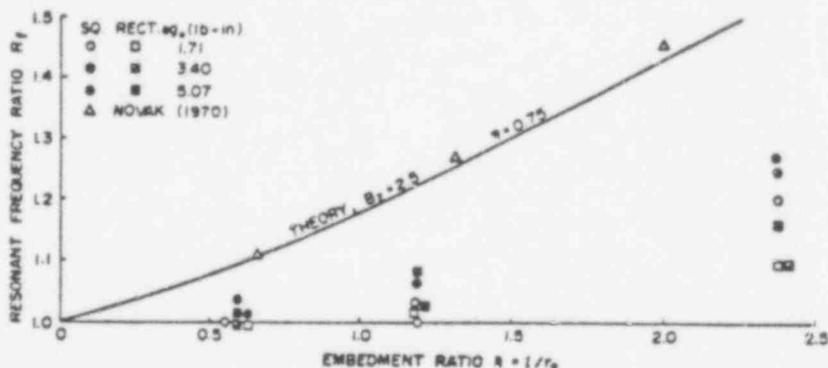
Fig. 15.—Comparison of Theoretical Resonant Frequency Ratio with Field Tests of Footing Embedded in Back Fill Having Density Ratio $\eta = \rho_s/\rho = 0.765$

A second difficulty is due to the fact that the measured resonant amplitudes of the surface footing are consistently found to be two times to three times greater than the theoretical values. [This was also found in previous experiments (14).] This difference may be caused by the reflection of elastic waves from layer interfaces and fissures, variations in the stress distribution in the footing base, and partly by the shape of the footing base.

However, as far as the effect of embedment is concerned, the differences

in absolute values of amplitudes can become less substantial if the relative variations are compared. These relative variations in resonant amplitudes and frequencies are plotted in Figs. 12 to 15.

In Fig. 12, the relative variations in resonant (maximum) amplitudes are shown ve us relative embedment depth for undisturbed soil. Experiments with both square and rectangular bases are presented. Lysmer and Kuhlemeyer's results of their finite element solution are plotted also. They indicate a good agreement with the approximate analytical solution. The field experiments are also in good agreement with the theory except for the highest embedment at which most experiments exhibit greater amplitude reduction. Also, the strong effect of nonlinearity is visible. Usually greater strength of the upper cohesive soil layer may contribute to a greater amplitude reduction at full embedment. Chae's laboratory experiments feature a much stronger amplitude reduction than both theory and field tests, probably due to the conditions of laboratory tests.

In Fig. 13 resonant frequency ratio is plotted against embedment. In most experiments the frequency increase was considerably smaller than the theory predicts, particularly with the rectangular base. The reason for this may have been that the bond between the soil and the footing was not perfect, as the theory assumes.

Figs. 14 and 15 present an analogous comparison for a footing surrounded by a backfill. According to Fig. 14, the theory tends to overestimate the effect of embedment, especially for small embedments. This seems to indicate that there is a lack of bond between the footing and the backfill. Perfect force transmission is assumed in the theory, which can hardly be satisfied with shallow embedments where the horizontal soil pressure and the frictional bond are very low. These conclusions are further substantiated by Fig. 15. In this figure previous experiments (14) agree with the theory very well. The backfill (loess loam) was compacted very heavily and possibly a better bond between the fill and the footing was achieved. Because the bond depends also on the roughness of the footing sides, this seems to be a further factor affecting the quality of the force transmission between footing and the backfill.

From Figs. 14 and 15, the conclusion can be drawn that for backfill the effects of embedment upon both the resonant frequencies and amplitudes can be considerably reduced. This reduction depends on the soil used, i.e., on its compaction and roughness of the footing sides. Therefore, with footings cast into forms, an even smaller effect of embedment can be expected than that observed in the experiments described. The reduced effect of the backfill can be accounted for in the theory by considering $\rho_s < \rho$ and $G_s < G$.

SUMMARY AND CONCLUSIONS

The effect of embedment upon vertical forced vibration of a rigid footing was investigated both theoretically and experimentally in the field. The conclusions can be summarized as follows:

1. The approximate analytical solution compares favorably with the finite element solution and can easily be applied to the analysis of footings and structures supported by embedded foundations.

2. The real effect of embedment depends on e difference between undisturbed soil and bacl important in the case of a stratum.

3. The theory and experiments agree qualit; in resonant amplitudes and an increase in resor embedment depth and increasing density of the

4. The experimental resonant amplitudes ar theoretical values computed with a rigid base of two to three in the described case.

5. There is a reasonable agreement between t relative variations of the resonant amplitudes a with the embedment in the case of the undistu increase in resonant frequencies is much smalle of resonant amplitudes are in better agreement.

6. The resonant (maximum) amplitude reduc of the mass, while the corresponding frequency

ACKNOWLEDGMENT

APPENDIX I.—STIFFNESS AND DAMPING P/

Stiffness and damping parameters presented i from Eqs. 3, 6, and 7 by curve fitting to facilitate is sufficient. Functions $f_{1,2}$ were taken from reactions and from Warburton (Ref. 20) for st reproduction. (The sign convention of these w this paper.) Some of the parameters are also ; stratum reactions can be found in Refs. 10 ar can be adjusted according to the frequency ran;

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asily be applied to the analysis of footings and structures
oundations.

2. The real effect of embedment depends on embedment depth, with a marked difference between undisturbed soil and backfill, and becomes particularly important in the case of a stratum.

3. The theory and experiments agree qualitatively that there is a decrease in resonant amplitudes and an increase in resonant frequencies with increasing embedment depth and increasing density of the backfill.

4. The experimental resonant amplitudes are consistently higher than the theoretical values computed with a rigid base stress distribution by a factor of two to three in the described case.

5. There is a reasonable agreement between the theoretical and experimental relative variations of the resonant amplitudes and of the resonant frequencies with the embedment in the case of the undisturbed soil. With the backfill the increase in resonant frequencies is much smaller than predicted. The variations of resonant amplitudes are in better agreement.

6. The resonant (maximum) amplitude reduction is essentially independent of the mass, while the corresponding frequency ratio is mass dependent.

ACKNOWLEDGMENT

APPENDIX I.—STIFFNESS AND DAMPING PARAMETERS

Stiffness and damping parameters presented in Tables 1 and 2 were obtained from Eqs. 3, 6, and 7 by curve fitting to facilitate the calculations. Their accuracy is sufficient. Functions $f_{1,2}$ were taken from Bycroft (Ref. 4) for half-space reactions and from Warburton (Ref. 20) for stratum reactions by mechanical reproduction. (The sign convention of these writers have been maintained in this paper.) Some of the parameters are also plotted in Fig. 2. More data on stratum reactions can be found in Refs. 10 and 11. The constant parameters can be adjusted according to the frequency range of interest.

APPENDIX II.—REFERENCES

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APPENDIX III.—NOTATION

The following symbols are used in this paper:

$A = w_0 m/(m_e e)$ = nondimensional displacement amplitude;

A_m = nondimensional resonant (maximum) amplitude of displacement of embedded footing;

a_m = nondimensional frequency at resonant (maximum) amplitude of embedded footing;

$a_0 = r_0 \omega \sqrt{\rho/G}$ = nondimensional frequency;

$B_1 = 1 - \nu/4 b_0$ = Lysmer's modified mass ratio;

$b_0 = m/\rho r_0^3$ = dimensionless mass ratio;

C_1 = half-space (stratum) stiffness;

$\bar{C}_1$ = frequency independent half meter;

C_2 = half-space (stratum) damping;

$\bar{C}_2$ = constant related to C_2;

c = damping constant for embed

D = damping ratio;

e = eccentricity of unbalanced r

f_1, f_2 = components of Reissner's di vibration;

G = shear modulus of soil beneat

G_s = shear modulus of backfill (si

g_0 = weight of unbalanced rotatin

H = total thickness of stratum;

h = thickness of stratum under f

$i = \sqrt{-1}$ = imaginary unit;

k = stiffness (spring) constant fo

$k(\delta)$ = stiffness with embedment δ;

l = depth of embedment of footi

m = mass of footing;

m_e = unbalanced rotating mass;

$N_z(t)$ = vertical dynamic reaction alo

$P(t)$ = excitation force;

P_0 = excitation force amplitude;

R_a = resonant (maximum) amplitu with embedment δ/resonant

R_f = resonant frequency ratio = r ment δ/resonant frequency (

$R_z(t)$ = vertical dynamic reaction at

r_0 = radius of cylindrical footing: or rectangular footing (see R

S_1, S_2 = frequency dependent stiffnes to embedment;

$\bar{S}_1$ = frequency independent (cons embedment;

$\bar{S}_2$ = constant related to S_2;

$s(t)$ = vertical side dynamic reactio

t = time;

$w(t)$ = vertical displacement of rigid

w = complex amplitude of vertica

w_m = real resonant (maximum) am

w_0 = real amplitude of vibration;

z = vertical coordinate;

$\Delta = H/r_0$ = relative thickness of stratum;

$\delta = l/r_0$ = embedment ratio;

$\eta = \rho_s/\rho$ = density ratio;

ν = Poisson's ratio;

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far.*, 1957, pp. 55-58.

ION

are used in this paper:

imensional displacement amplitude;
imensional resonant (maximum) amplitude of displace-
of embedded footing;
limensional frequency at resonant (maximum) amplitude
nbedded footing;
limensional frequency;
ner's modified mass ratio;

$b_0 = m/\rho r_0^3$ = dimensionless mass ratio;

C_1 = half-space (stratum) stiffness parameter;

$\bar{C}_1$ = frequency independent half-space (stratum) stiffness para-
meter;

C_2 = half-space (stratum) damping parameter;

$\bar{C}_2$ = constant related to C_2;

c = damping constant for embedded footing;

D = damping ratio;

e = eccentricity of unbalanced rotating masses;

f_1, f_2 = components of Reissner's displacement function for vertical
vibration;

G = shear modulus of soil beneath footing;

G_s = shear modulus of backfill (side layer);

g_0 = weight of unbalanced rotating mass m_s;

H = total thickness of stratum;

h = thickness of stratum under footing base;

$i = \sqrt{-1}$ = imaginary unit;

k = stiffness (spring) constant for embedded footing;

$k(\delta)$ = stiffness with embedment δ;

l = depth of embedment of footing;

m = mass of footing;

m_s = unbalanced rotating mass;

$N_z(t)$ = vertical dynamic reaction along embedded footing sides;

$P(t)$ = excitation force;

P_0 = excitation force amplitude;

R_a = resonant (maximum) amplitude ratio = resonant amplitude
with embedment δ/resonant amplitude of surface footing;

R_f = resonant frequency ratio = resonant frequency with embed-
ment δ/resonant frequency of surface footing;

$R_z(t)$ = vertical dynamic reaction at base of footing;

r_0 = radius of cylindrical footing; also equivalent radius of square
or rectangular footing (see Ref. 17, p. 347);

S_1, S_2 = frequency dependent stiffness and damping parameters due
to embedment;

$\bar{S}_1$ = frequency independent (constant) stiffness parameter due to
embedment;

$\bar{S}_2$ = constant related to S_2;

$s(t)$ = vertical side dynamic reaction per unit depth of embedment;

t = time;

$w(t)$ = vertical displacement of rigid footing at time t;

w = complex amplitude of vertical displacement;

w_m = real resonant (maximum) amplitude;

w_0 = real amplitude of vibration;

z = vertical coordinate;

$\Delta = H/r_0$ = relative thickness of stratum;

$\delta = l/r_0$ = embedment ratio;

$\eta = \rho_s/\rho$ = density ratio;

ν = Poisson's ratio;

ρ = mass density of elastic half-space (soil);

ρ_s = mass density of side layer (backfill);

ϕ = phase angle;

ω = circular frequency of excitation;

ω_m = resonant circular frequency (at maximum amplitudes); and

ω_0 = natural circular frequency.

Bearing Capacity Theory from Expe

By A. Verghese Chummar

INTRODUCTION

The plastic equilibrium theories on bearing capaci
on failure surfaces either assumed or derived fro
The failure surfaces observed in the experiments (
soils by Selig and McKee (10), Verghese (12) and
the shape of the failure surfaces adopted in the
given in Fig. 1 between some of the observed ar
reveals a marked discrepancy. The existing theorie
bearing capacity fairly accurately until it was esta
long footings, the plane strain conditions prevail an
ϕ under plane strain is about 1.1 times the value (
Fig. 2 gives the experimental values of the bearin
against plane-strain values of ϕ and the N_γ values gi
theories. The comparison clearly indicates that t
accurately the bearing capacity of long footings. A
herein to determine the actual shape of the failu
bearing capacity theory based on that surface. I
data, the present study is restricted to the case
cohesionless soils.

FAILURE SURFACE

The shape of the failure surface is determined from
by different research workers by deducing a patt
the observed ones.

DYNAMIC ANALYSIS OF EMBEDDED STRUCTURES

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SUMMARY

The paper presents simplified rules to account for embedment and soil layering in the soil-structure interaction problem, to be used in dynamic analyses. The relationship between the spring method, and a direct solution (in which both soil and structure are modeled with finite elements and linear members) is first presented. It is shown that for consistency of the results with the two solution methods the spring method should be performed in the following three steps:

1. Determination of the motion of the massless foundation (having the same shape as the actual one) when subjected to the same input motion as the direct solution. For an embedded foundation it will yield, in general, both translations and rotations.
2. Determination of the frequency dependent subgrade stiffness for the relevant degrees of freedom. This step yields the so-called "soil springs".
3. Computations of the response of the real structure supported on frequency dependent soil springs and subjected at the base of these springs to the motion computed in step 1.

The first two steps require, in general, finite element methods, which would make the procedure not attractive. It is shown in the paper, however, that excellent approximations can be obtained, on the basis of 1-dimensional wave propagation theory for the solution of step 1, and correction factors modifying for embedment the corresponding springs of a surface footing on a layered stratum, for the solution of step 2. Use of these rules not only provides remarkable agreement with the results obtained from a full finite element analysis, but results in substantial savings of computer execution and storage requirements. This frees the engineer to perform extensive studies, varying the input properties over a wide range to account for uncertainties, in particular with respect to the soil properties.

1. Introduction

Significant efforts have been directed in rece... years to formulate engineering solutions to the problems of vibration of foundations and seismic response of buildings. The problem is of ...cial interest in the seismic analysis and design of massive buildings such as nuclear containment structures.

Building foundations, and nuclear reactors in particular, are usually buried to some extent beneath the surface of the ground. This embedment has in many cases considerable affect on the dynamic response of the structure, both in terms of relative frequency contents and amplitude of the resulting motions. Because of the complicated boundary conditions that must be satisfied in a theoretical formulation, rigorous analytical solutions for embedded foundations are nonexistent at present. Hence, numerical (finite element), experimental and approximate analytical techniques are currently being used to provide a solution to the problem at hand for these complicated geometries. An awareness of the effects associated with embedment, coupled with the availability of numerical solutions and the lack of rigorous solutions, has been the basis in recent times for discrediting the spring method as a tool for analyses, particularly in the nuclear power industry. The detraction of the spring method has been argued by some researchers on the basis of comparisons between the classical half space method, and more involved finite element solutions. Many of these comparisons are not meaningful, since they were based on inappropriate values for both the "spring constant" and the "support motion." In fact, the spring method and finite element solutions can be shown to be mathematically equivalent; if they are classified as different, it is because of inconsistencies in their implementation.

It is the purpose of this paper to show the relationship between a more general spring method, and the solutions provided by direct finite element procedures, and to present practical rules for use in dynamic analysis.

2. The basic superposition theorem

Referring to Fig. 1, assume that the general equations of motion of a structure-foundation system are given by the matrix equation

$$M\ddot{U} + C\dot{Y} + KY = 0 \qquad (1)$$

where M, C, K, are the system mass, damping and stiffness matrices; U and Y are the absolute and relative displacements with respect to some general ground reference system. The solution of this equation is equivalent to the solution of the two matrix equations

$$M_1 \ddot{U}_1 + C\dot{Y}_1 + KY_1 = 0 \qquad (2)$$

$$M \ddot{Y}_2 + C\dot{Y}_2 + KY_2 = -M_2\ddot{U}_1 \qquad (3)$$

where $U_1 = Y_1 + U_g$, $U = U_1 + Y_2$, and $M = M_1 + M_2$. M_1 excludes the mass of the structure, while M_2 excludes the mass of the soil. U_g is some generalized ground motion vector. The equivalence of 2) with 1) is demonstrated by simple addition. In Eq. 2, the response of the massless structure is found first, and will be referred to as the kinematic interaction. The results of this step are then used in Eq. 3) which shall define the inertial interaction, and which is solved by application of fictitious inertia forces applied to the structure alone.

In the solution of the second step, it is irrelevant whether the soil is modelled with finite elements, or equivalently, with a (far-coupled) matrix of stiffness functions modelling the subgrade, and defined at the soil-structure interface. These stiffness functions can be

symmetry may be introduced into the model of the structure in step 3 (which means that a tor-sional "spring" and "dashpot" must be evaluated in step 2) and the effect of changes in the mass and stiffness of the structure be evaluated without having to rerun an entire analysis.

3.   Approximate Solution for Circular, Embedded Foundations

The solutions presented in the following sections have been obtained with a three-dimensional axisymmetric finite element formulation. A fundamental feature of the program used is the exact representation of the model boundary which separates the finite element region from the semi-infinite continuum (the free field). This consistent energy transmitting boundary was developed for the plane strain case by Waas and Lysmer (11), (7), and was extended to the three-dimensional case by the first author (2), (3), (5). In essence, it can be regarded as a virtual extension of the finite element mesh to infinity, and can be placed without loss of accuracy immediately next to the foundation.

In the following section, it will be assumed that the motion which the ground experiences before any structure (or hypothetical massless foundation) has been built, can be described by means of one-dimensional wave propagation theory. The control motion, i.e., the specified earthquake record, will be assumed to take place at the free surface in the "free field." Motions at other points in the free field can be obtained by the so-called "deconvolution" process, which makes use of one-dimensional wave-propagation theory.

3.1   Approximations to the Kinematic Interaction

The third author investigated the kinematic interaction problem in a parametric study (8), using a suite of embedment ratios E/R and stratum ratios H/R, covering a range of values typically found in nuclear reactor design, and proposed rules to approximate the kinematic interaction.

Referring to Fig. 3, a unit harmonic displacement was specified at the free surface, and deconvolved to bedrock. Using the finite element program, frequency dependent transfer functions u, ϕ for the displacement and rotation of the massless foundation, relative to the motion at the free surface, were determined. Similarly, the frequency dependent transfer functions for the displacement in the free field at the elevation of the foundation, and for the pseudorotation of the free field

$$\phi_B (\Omega) = \frac{u_A - u_B}{E} \approx \frac{1 - u_B}{E}$$

were computed. Typical results are found in Fig. 4. The transfer functions u and u_B, ϕ and ϕ_B were then compared for a range of embedment and stratum depth values, and rules to approximate these functions were suggested (8). A simplified version of these rules is given below.

Let $F(\Omega)$ be the Fourier transform of the acceleration at the free surface in the free field (in most cases, the design earthquake). The translation and rotation of the massless, rigid foundation are then given approximately by

$$\ddot{u} = IFT \begin{cases} F(\Omega) \cdot \left[\cos\left(\frac{\pi}{2} \cdot \frac{f}{f_n}\right) \right] & \text{if } f < 0.7\, f_n \\ F(\Omega) \cdot [0.453] & \text{if } f > 0.7\, f_n \end{cases}$$

sp 3 (which means that a tor-
he effect of changes in the
to rerun an entire analysis.

obtained with a three-dimen-
sature of the program used is
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nergy transmitting boundary

(7), and was extended to the
sence, it can be regarded as
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n which the ground experiences
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otion, i.e., the specified
face in the "free field."
so-called "deconvolution"
ory.

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R, covering a range of values
approximate the kinematic

fied at the free surface, and
ency dependent transfer
foundation, relative to the
quency dependent transfer
n of the foundation, and for

er functions u and u_g,
depth values, and rules to
rsion of these rules is given

he free surface in the free
nd rotation of the massless,

f_n

f_n

$$\ddot{\phi} = IFT \begin{cases} F(\Omega) \cdot \left[0.257 \, (1-\cos \frac{\pi}{2} \cdot f/f_n)/R \right] & \text{if } f \leq f_n \\ F(\Omega) \cdot \left[0.257/R \right] & \text{if } f > f_n \end{cases}$$

(ϕ is positive clockwise)

where IFT stands for Inverse Fourier Transformation; f_n is the fundamental shear beam frequency of the embedment region (for uniform soil properties in the embedment region, this value is given by $f_n = C_s/4E$ with C_s being the shear wave velocity, and E the depth of embedment. The expression in square brackets describes an approximation to the transfer functions for the translation and rotation of the massless foundation. For surface footings, E = 0, $f_n = \infty$, $\cos 0 = 1$; it follows that no kinematic interaction takes place for this case.

The procedure described yields satisfactory results for a wide range of embedment ratios, see for instance Fig. 5. It should be noted, however, that the rotational component is sensitive to the lateral soil conditions, and particularly to the flexibility of the lateral walls. For flexible sidewalls, the actual rotation is significantly smaller, and in the extreme case of no sidewalls, the rotation even changes sign! Nevertheless, the contribution of the rotational component to the response of the structure is in most cases not very significant. For nuclear containment structures, the effect of the rotational component on the structural response is of the order of 15-20 percent.

3.2 Approximations for the Stiffness Functions

As with the kinematic interaction, the values of the subgrade stiffness functions (impedance functions) depend only on the geometric configuration of the foundation and on the properties of the founding soil. These functions can be evaluated using analytical, experimental, or numerical methods.

The results presented in this paper are based on parametric studies performed by the fourth writer to determine approximate expressions for stiffness functions of circular, embedded foundations (1). For each particular geometry, the static values were evaluated with two or three meshes (a fine, a standard, and a coarse mesh), and the results were corrected for mesh size error in a manner similar to that described in Ref. 4. The study was limited to the coupled horizontal translation and rotation (rocking-swaying) of the rigid, circular, embedded foundation in a homogeneous stratum. For this particular case, the force displacement relationship can be written

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{Bmatrix} K_{xx} & K_{x\phi} \\ K_{\phi x} & K_{\phi\phi} \end{Bmatrix} \begin{Bmatrix} u \\ \phi \end{Bmatrix}$$

where F = the horizontal force; M = the rocking moment; and u, ϕ, are the corresponding displacements (rotations). The elements K_{xx}, $K_{x\phi}$, $K_{\phi\phi}$ of the stiffness matrix depend on the frequency of excitation Ω of the forces (moments). Since these forces and the resulting displacements are generally not in phase with each other, these elements are complex functions of frequency. Each stiffness function is of the form $K^0 (1 + 2i\beta) (k + ia_0 c)$, where K^0 is the static stiffness, β a measure of the internal damping in the soil (of a hysteretic nature), $i = \sqrt{-1}$, and a_0 is the dimensionless frequency $\Omega R/C_s$. (Ω is the circular frequency of the motion and excitation, R the radius of the foundation slab, and C_s a reference shear wave velocity.)

k and c are frequency dependent coefficients normalized with respect to the static stiffness. The coefficient c is related to the energy loss by radiation.

Using the program described earlier, the dynamic stiffness functions were computed for a range of embedment and stratum depth ratios, and written as described above as:

$$K_{xx} = K^o_{xx} (k_{11} + ia_o \cdot c_{11}) (1 + 2i\beta)$$

$$K_{x\phi} = K^o_{x\phi} (k_{12} + ia_o \cdot c_{12}) (1 + 2i\beta)$$

$$K_{\phi\phi} = K^o_{\phi\phi} (k_{22} + ia_o \cdot c_{22}) (1 + 2i\beta)$$

Analysis of the results obtained provided then the following approximations to the static values K^o_{xx}, $K^o_{x\phi}$, $K^o_{\phi\phi}$, and to the dynamic stiffness coefficients k_{11}, k_{12}, c_{11}, c_{12}, c_{22}:

Static Values:

$$K^o_{xx} = \frac{8GR}{2-\nu} (1 + \frac{1}{2} \frac{E}{H}) (1 + \frac{2}{3} \frac{E}{R}) (1 + \frac{5}{4} \frac{E}{H})$$

$$K^o_{x\phi} = K^o_{xx} R (\frac{2}{5} \frac{E}{R} - 0.03) \qquad (\phi \text{ is positive clockwise})$$

$$K^o_{\phi\phi} = \frac{8GR^3}{3(1-\nu)} (1 + \frac{1}{6} \frac{R}{H}) (1 + 2\frac{E}{R}) (1 + 0.71\frac{E}{H})$$

In these formulas, G = the shear modulus of soil underneath the mat; R = the radius of the foundation, E = the depth of embedment, H = the depth to bedrock, and ν = Poisson's ratio. Dynamic Stiffness Coefficients:

k_{11}, k_{22} half space solution (i.e., Ref. 10)

$$c_{11} = \begin{cases} 0.88\frac{a_o}{a_{ol}} & \text{for } a_o < \frac{\pi}{2} \frac{R}{H} = a_{ol} \\ \\ \text{Half space solution for } a_o \geq a_{ol} \end{cases}$$

$$c_{22} = \begin{cases} 0.58\frac{a_o}{a_{o2}} & \text{for } a_o < \frac{\pi}{2} \frac{R}{H} \frac{C_p}{C_s} = a_{o2} \\ \\ \text{Half space solution for } a_o \geq a_{o2} \end{cases}$$

$$k_{12} = k_{11}, \quad c_{12} = c_{11}$$

where β is the internal (hysteretic) damping in the soil; C_p and C_s are the dilatational wave velocities in the subgrade; a_{ol} and a_{o2} are the (nondimensional) fundamental shear beam and dilatation frequencies of the stratum, as defined above.

Except for the stiffness coefficient k_{11}, which displays a somewhat wavy nature, the suggested approximation for the stiffness and damping coefficients provide reasonable substitutes for the true functions. It can be observed that the radiation damping coefficients c_1, c_2 are larger for the embedded case than for the surface footing; therefore, the suggested procedure should give conservative results for an embedded structure.

spect to the static stiffness.

functions were computed for a
cribed above as:

(8)

(8)

(8)

pproximations to the static
nts k_{11}, k_{12}, c_{11}, c_{12}, c_{22}:

$\frac{5}{4} \frac{F}{R}$)

a positive clockwise)

$1 + 0.71 \frac{F}{R}$)

e mat; R = the radius of the
ck, and ν = Poisson's ratio.

Ref. 10)

$\flat \, a_{o1}$

$a2$

$\flat \, a_{o2}$

nd C_s are the dilatational wave
1) fundamental shear beam and

a somewhat wavy nature, the
ents provide reasonable sub-
radiation damping coefficients
ooting; therefore, the suggested
ucture.

4. Soil-Structure Interaction Problem

Once the input motion and the base impedances are known, the last step is reduced to a simple dynamic analysis of a multidegree of freedom system. The "stiffness" and "damping" terms can be added directly to the corresponding terms for the structure in a solution in the frequency domain. A time domain solution, and a modal solution in particular (9), offer a simpler physical interpretation of the results. They require, however, frequency independent stiffness and damping coefficients, and for the latter, in addition the existence of normal modes.

A number of comp   tive studies indicate that it is more important to reproduce correctly the static stiffnesses han their complete frequency variation. It is also worth noticing that the increase in stiffness due to embedment is very sensitive to the properties of the lateral soil, which may be disturbed. Considering, in addition, the uncertainties in the soil properties and its nonlinear behavior, it is clear that engineering judgment is needed in the selection of the most appropriate model, and that parametric studies, varying the assumed conditions within reasonable limits, are advisable. This, of course, is equally true whether the analysis is carried out in a single step (one-pass me hod) or in three steps as suggested here.

The spring method has the advantage of being less time-con uming when approximations are used for the kinematic interaction problem. It allows, therefore, more parametric studies, and the accuracy of each step is subject to better control. Of particular importance is the possibility in this method to make use of symmetry or cylindrical conditions if the foundation meets these requirements even if the structure does not (which is a frequent situation). The coupling between the corresponding terms will come in naturally in the third step. From a practical standpoint, the procedure has an additional advantage when the design motion is specified by a broad band response spectrum not tailored to the soil conditions at the site. If the direct approach is applied to such a case, deamplification of certain frequency components with depth resulting from the use of one single wave pattern (i.e., vertically propagating waves) may lead to unconservative estimates for the motions of the structure. Under such conditions, it may be better to regard the design motion as an "average" motion in the vicinity of the structure, and to use it directly as input to Step 3.

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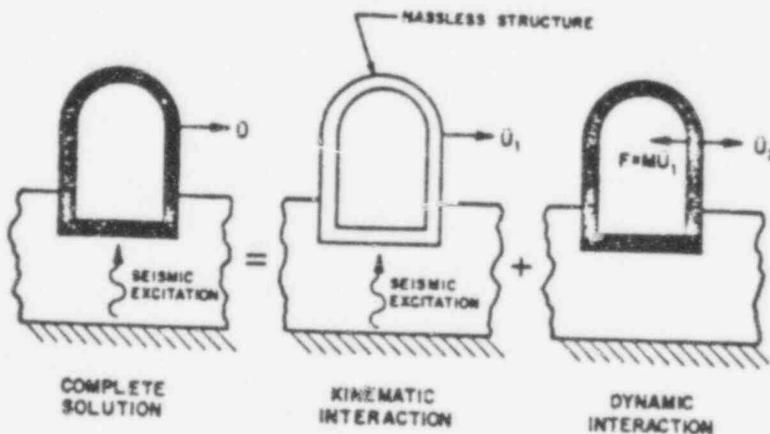
FIGURE 1
SUPERPOSITION THEOREM

ar Foundations Embedded in an
Cambridge, Massachusetts, 1975.

on Layered Media," MIT Research
publication No. 384, January,

is of Footings on Layered Media,"

ular Foundations," Journ. Eng.

t for Layered Strata," paper
., ASCE, 1977.

ar Behavior in Soil-Structure
November, 1976.

Structures," Journ. Eng. Mech.

Embedded Circular Foundations,"
usetts, 1975.

mis for Structures with
1. 99, March, 1973.

of Footings," Journ. Eng. Soil
971.

mics Problems in Semi-infinite
, Berkeley, 1972.

TURE

$F = MU_1$ $\ddot{U}_2$

DYNAMIC
INTERACTION

KINEMATIC INTERACTION   SUBGRADE IMPEDANCES

TOTAL SOLUTION   =   ①   +   ②   +   ③

FIGURE 2
THE 3-STEP SOLUTION

MASSLESS FOUNDATION   FREE FIELD $U_0 = 1$ (HARMONIC)

BEDROCK
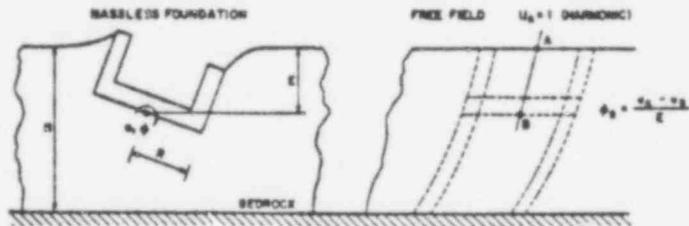
FIGURE 3
KINEMATIC INTERACTION PROBLEM

TRANSLATION

ROTATION

FIGURE 4
MOTION OF MASSLESS FOUNDATION, TRANSFER FUNCTIONS (ABS. VALUE)
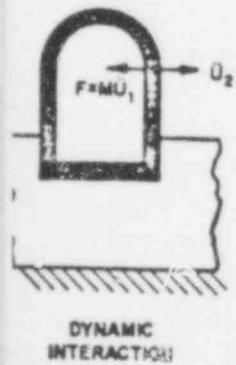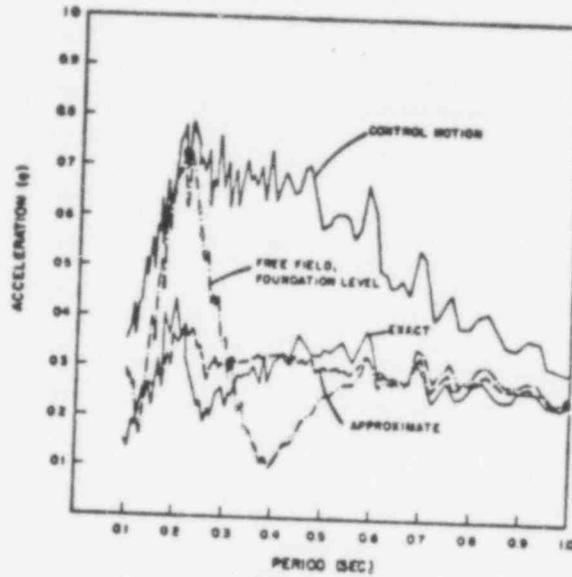
FIG. 5
MOTION OF MASSLESS FOUNDATION, RESPONSE SPECTRA
1% OSCILLATOR DAMPING
H/R = 2, E/R = 1
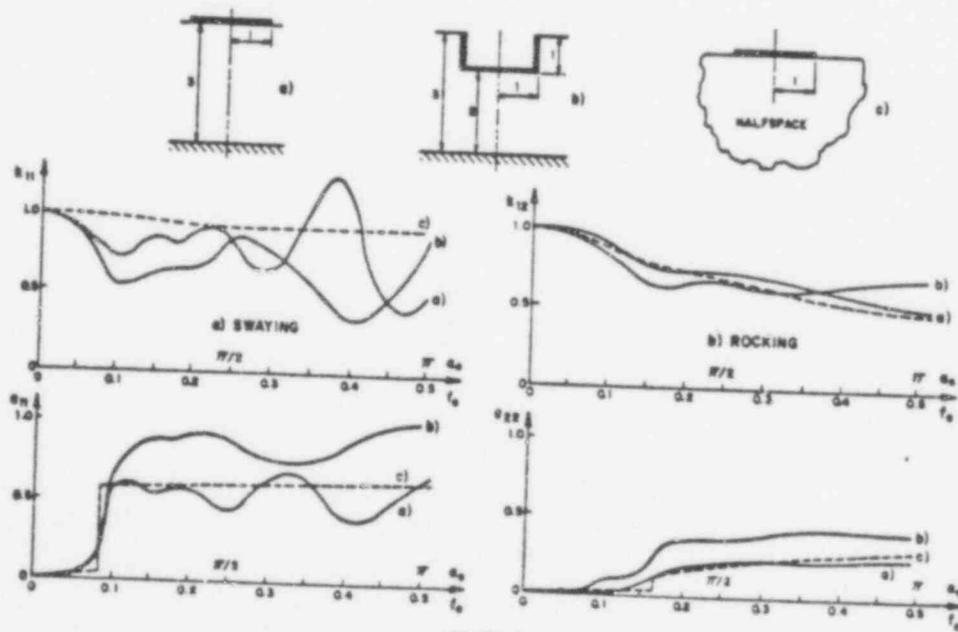
FIGURE 6
DYNAMIC STIFFNESS COEFFICIENTS, $\nu = 1/3$, $\beta = 0.05$

TORSIONAL AND COUPLED VIBRATIONS OF EMBEDDED FOOTINGS

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SUMMARY

An approximate analytical solution is presented for torsional vibrations of footings partially embedded into a semi-infinite medium or a stratum. Simple formulas derived for pure torsional motion make it possible to apply a correction for the effect of embedment to the known solutions of surface footings. The solution completes an approach to the analysis of all modes of footing vibrations, including the coupled modes. The approach to coupled modes is illustrated by the solution of coupled response involving horizontal translation, rocking and torsion. Formulas are presented for stiffness and damping coefficients that can be used in the analysis of embedded footings or structures supported by such footings.

Field experiments were conducted with concrete footings featuring circular, square and rectangular bases and variable embedment depths. The experimental results were compared with theoretical predictions of pure torsional vibrations.

INTRODUCTION

This paper complements an approach to the analysis of embedded footing vibration which related to pure vertical, horizontal and rocking modes[13, 14] and coupled horizontal and rocking modes.[4] Pure torsional response is treated herein as well as response in a coupled mode involving torsional, horizontal and rocking components.

There is experimental evidence that the dynamic response of footings can be considerably affected by their partial embedment beneath the surface of the soil.[12, 13, 22] However, a rigorous analytical solution of this effect is very difficult and finite element solutions appear the most promising approach to this problem.[11, 20]

The approximate analytical solution presented in this paper offers advantages of simplicity, the possibility to solve any number of degrees of freedom, and great versatility, thereby making it possible to consider layering and to introduce the dynamic soil reactions into the analysis of structures resting on embedded footings.

The solution assumes a rigid cylindrical footing, linear isotropic elasticity and a perfect bond between the footing and the soil. The total dynamic reaction of soil is composed of a reaction acting in the footing base and on the footing vertical sides. The base reaction is derived from an elastic half-space which is assumed to model the soil under the footing base. The side reaction is derived as a reaction of an independent layer overlying the half-space. As well, this overlying layer is considered to be composed of a series of infinitesimally thin independent layers which facilitates the analysis of more complicated vibration modes involving non-uniform displacements of footing sides. Such assumptions were first adopted by Baranov;[2] however, he did not consider the torsional modes analyzed herein.

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The assumptions are approximate, primarily because the compatibility condition between the soil under the base and the overlying layer is satisfied only at the footing and very far from it. Nevertheless, the same theoretical approach to embedment yielded a reasonable agreement with the finite element solution for vertical vibration[14] and considerably improved the agreement between the theory and experiments for coupled horizontal and rocking vibration.[6] Thus, it appears useful to obtain some information on the torsional response of embedded footings too because large differences were observed between the strict elastic theory and experiments with surface footings.[21] Finally, the approach presented offers a way of applying an approximate correction for embedment to the already numerous solutions for surface footings.[1, 6, 7, 9, 10, 16, 21]

SIDE REACTIONS OF THE LAYER

Firstly, dynamic reactions will be determined acting on the footing sides if the cylindrical body performs harmonic torsional oscillations with frequency ω around its vertical axis.

With the omission of body forces and with the notation according to Love, the equations of motion of elastic media in cylindrical co-ordinates are

$$
\left.
\begin{array}{l}
(\lambda + 2G)\dfrac{\partial \Delta}{\partial r} - \dfrac{2G}{r}\dfrac{\partial \omega_z}{\partial \theta} + 2G\dfrac{\partial \omega_\theta}{\partial z} = \rho\dfrac{\partial^2 u}{\partial t^2} \\[2mm]
(\lambda + 2G)\dfrac{\partial \Delta}{r\partial \theta} - 2G\dfrac{\partial \omega_r}{\partial z} + 2G\dfrac{\partial \omega_z}{\partial r} = \rho\dfrac{\partial^2 v}{\partial t^2} \\[2mm]
(\lambda + 2G)\dfrac{\partial \Delta}{\partial z} - \dfrac{2G}{r}\dfrac{\partial}{\partial r}(r\omega_\theta) + \dfrac{2G}{r}\dfrac{\partial \omega_r}{\partial \theta} = \rho\dfrac{\partial^2 w}{\partial t^2}
\end{array}
\right\}
\tag{1}
$$

in which relative volume change:

$$
\Delta = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}
$$

and components of rotational vector.

$$
\left.
\begin{array}{l}
\omega_r = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial w}{\partial \theta} - \dfrac{\partial v}{\partial z}\right) \\[2mm]
\omega_\theta = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial r}\right) \\[2mm]
\omega_z = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial(rv)}{\partial r} - \dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\right)
\end{array}
\right\}
\tag{3}
$$

With pure torsional vibration of the layer around the vertical axis, both vertical and radial components of the motion vanish, i.e.

$$
w(r, \theta, t) = 0, \quad u(r, \theta, t) = 0
\tag{4}
$$

and the equations of motion, equations (1), reduce to

$$
(\lambda + 2G)\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + G\frac{1}{r}\frac{\partial v}{\partial r} - G\frac{v}{r^2} + G\frac{\partial^2 v}{\partial r^2} = \rho\frac{\partial^2 v}{\partial t^2}
\tag{5}
$$

With the boundary condition along the circumference of the cylinder:

$$
v(r_0, \theta, t) = r_0\,\zeta_0\,e^{i\omega t}
\tag{6}
$$

the solution to equation (5) can be sought in the form:

$$
v = R(r)\Phi(\theta)\,e^{i\omega t}
\tag{7}
$$

Substitution of equation (7) into equation (5) yields

$$
r^2\frac{R''}{R} + r\frac{R'}{R} + \left(\frac{\rho\omega^2}{G}r^2 - 1\right) = -\frac{\lambda + 2G}{G}\frac{\Phi''}{\Phi}
\tag{8}
$$

This equation can be split into two ordinary differential equations for $R(r)$ and $\Phi(\theta)$

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + (k^2 r^2 - 1) = n^2 \tag{9}$$

$$-\frac{1}{\alpha^2} \frac{\Phi''}{\Phi} = n^2 \tag{10}$$

in which

$$k^2 = \frac{\rho \omega^2}{G}, \quad \alpha^2 = \frac{G}{\lambda + 2G} \tag{11}$$

Solution to equation (10) is

$$\Phi = A_n \cos n\alpha\theta + B_m \sin n\alpha\theta \tag{12}$$

Motion v is independent of angle θ which, with respect to equation (7), requires that in (12) $n = 0$ and thus $\Phi = A_n$.

Equation (9) now takes the form:

$$r^2 R'' + rR' + (\kappa^2 r^2 - 1) R = 0 \tag{13}$$

The solution to this equation is

$$R = C H_1^{(1)}(kr) + D H_1^{(2)}(kr) \tag{14}$$

in which $H_1^{(1)}$, $H_1^{(2)}$ = Hankel functions of the first order, first and second kind, respectively; C, D = constants to be determined.

From the asymptotic form of Hankel functions (see, e.g. Reference 5, p. 616) it can be seen that with excitation coming from the body (outward travelling waves) only the function $H_1^{(2)}$ comes into consideration and thus $C = 0$. From equations (7), (12) and (14) motion

$$v = A H_1^{(2)}(kr) e^{i\omega t} \tag{15}$$

in which A = a constant given by the boundary condition, equation (6). Comparison of equations (6) and (15) yields, with $r = r_0$ and $kr_0 = a_0$, the constant $A = \zeta_0 r_0 / H_1^{(2)}(a_0)$. Thus, motion at a distance r

$$v = \frac{\zeta_0 r_0}{H_1^{(2)}(a_0)} H_1^{(2)}(kr) e^{i\omega t} \tag{16}$$

in which the dimensionless frequency, $a_0 = r_0 \omega \sqrt{(\rho/G)}$.

The torsional stiffness of the layer is obtained from the stresses along the circumference of the cylinder. From general formulas, $\sigma_r = \sigma_\theta = \tau_{rz} = 0$ and circumferential shear

$$\tau_{r\theta} = G \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \bigg|_{r_0} \tag{17}$$

Integration of moment $r_0 \tau_{r\theta}$ along the circumference of the cylinder yields the relation between the total dynamic torsional reaction N_ζ of a layer of thickness l and cylinder motion $\zeta_0 e^{i\omega t}$:

$$N_\zeta(t) = -2\pi G \frac{r_0^2 l}{H_1^{(2)}(a_0)} [a_0 H_1^{(2)}(a_0) - 2 H_2^{(2)}(a_0)] \zeta_0 e^{i\omega t} \tag{18}$$

or split into real and imaginary parts:

$$N_\zeta(t) = G r_0^2 l (S_{\zeta 1} + i S_{\zeta 2}) \zeta_0 e^{i\omega t} \tag{19}$$

in which

$$S_{\zeta 1} = 2\pi \left(2 - a_0 \frac{J_0 J_1 + Y_0 Y_1}{J_1^2 + Y_1^2} \right) \tag{20}$$

and

$$S_{\zeta 2} = \frac{4}{J_1^2 + Y_1^2} \tag{21}$$

Here, $J_0(a_0)$, $J_1(a_0)$ are Bessel functions of the first kind of order zero and one respectively, and $Y_0(a_0)$, $Y_1(a_0)$ are Bessel functions of the second kind of order zero and one.

Function $S_{\zeta 1}$ has a meaning of a frequency dependent stiffness parameter while $S_{\zeta 2}$ represents a frequency dependent damping parameter. These parameters are shown as functions of a_0 in Figure 1. In Appendix I, approximate polynomial expressions are given which can facilitate the computations.
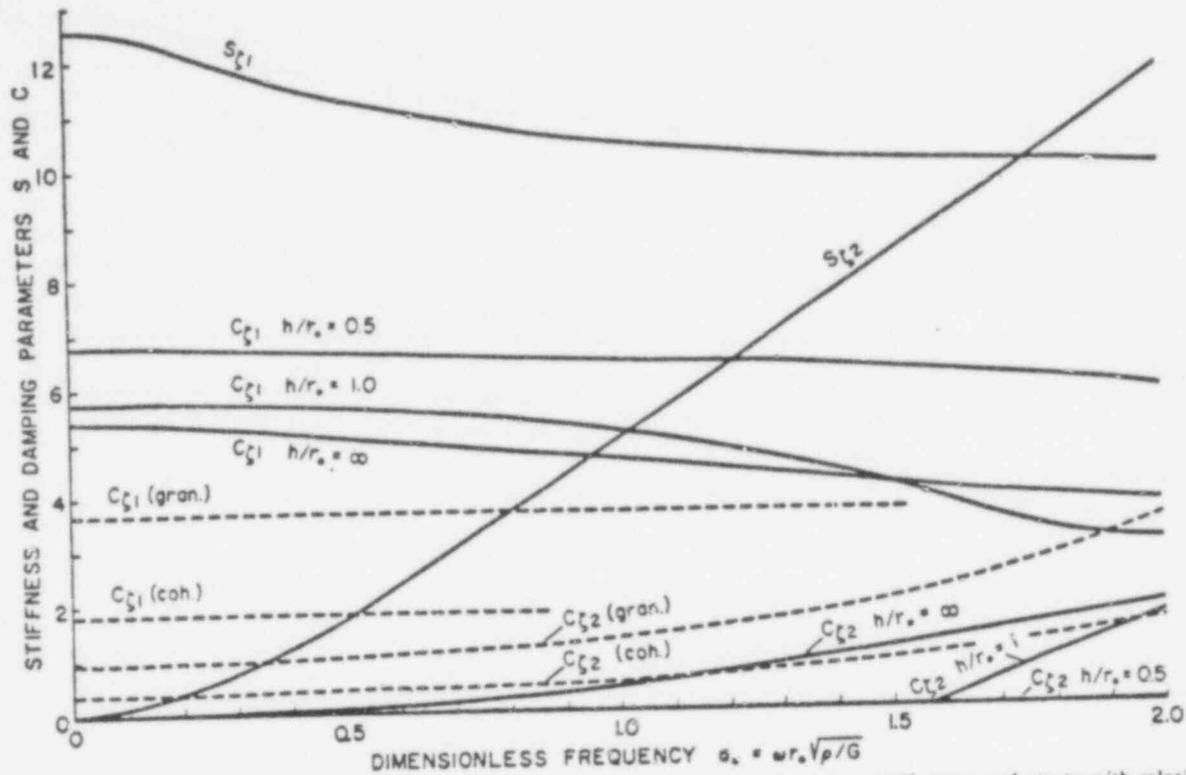
Figure 1. Stiffness parameters $S_{\zeta 1}$, $C_{\zeta 1}$ and damping parameters $S_{\zeta 2}$, $C_{\zeta 2}$ for side layer, half-space and strata with relative thickness $h/r = 0.5$ and 1 ($h/r = \infty$ denotes half space)

PURE TORSIONAL VIBRATIONS OF EMBEDDED FOOTINGS

The equation of torsional oscillations ζ of an embedded cylindrical footing about the vertical axis of symmetry is

$$I_\zeta \ddot{\zeta} = M_\zeta(t) - R_\zeta(t) - N_\zeta(t) \tag{22}$$

in which I_ζ = mass moment of inertia of footing about vertical axis Z, $M_\zeta(t)$ = moment of excitation, $R_\zeta(t)$ = torsional reaction of soil in footing base and $N_\zeta(t)$ = torsional reaction of layer adjacent to footing vertical sides.

The base reaction may be written in terms of displacement functions $f_{\zeta 1}$ and $f_{\zeta 2}$ (taken, e.g. from References 6 or 16):

$$R_\zeta(f) = -Gr_0^3 \frac{1}{f_{\zeta 1} + if_{\zeta 2}} \zeta(t) = Gr_0^3 (C_{\zeta 1} + iC_{\zeta 2}) \zeta(t) \tag{23}$$

in which stiffness parameter $C_{\zeta 1}$ and damping parameter $C_{\zeta 2}$ are

$$C_{\zeta 1} = \frac{-f_{\zeta 1}}{f_{\zeta 1}^2 + f_{\zeta 2}^2}, \quad C_{\zeta 2} = \frac{f_{\zeta 2}}{f_{\zeta 1}^2 + f_{\zeta 2}^2} \tag{24}$$

The side reaction is, according to equation (19),

$$N_\zeta(t) = G_s r_0^3 l(S_{\zeta 1} + iS_{\zeta 2}) \zeta(t) \tag{25}$$

in which G_s = shear modulus of the side layer and $S_{\zeta 1,2}$ are given by equations (20) and (21). Substitution of equations (23) and (25) into equation (22) provides the differential equation of motion:

$$I_\zeta \ddot{\zeta}(t) + Gr_0^3\left[C_{\zeta 1} + \frac{G_s}{G}\frac{I}{r_0}S_{\zeta 1} + i\left(C_{\zeta 2} + \frac{G_s}{G}\frac{I}{r_0}S_{\zeta 2}\right)\right]\zeta(t) = M_\zeta(t) \tag{26}$$

With complex excitation:

$$M_\zeta(t) = M_{\zeta 0}e^{i\omega t} = M_{\zeta 0}(\cos\omega t + i\sin\omega t) \tag{27}$$

in which $M_{\zeta 0}$ = the real moment amplitude and ω = excitation frequency, the particular solution describing the steady-state motion is $\zeta(t) = \zeta_c e^{i\omega t}$ in which ζ_c = complex displacement amplitude.

The real part of the motion is

$$\zeta(t) = \zeta_0\cos(\omega t + \phi) \tag{28}$$

With the notation of the frequency dependent stiffness (spring) constant:

$$k_\zeta = Gr_0^3\left(C_{\zeta 1} + \frac{G_s}{G}\frac{I}{r_0}S_{\zeta 1}\right) \tag{29}$$

and the frequency dependent damping constant:

$$c_\zeta = \frac{Gr_0^3}{\omega}\left(C_{\zeta 2} + \frac{G_s}{G}\frac{I}{r_0}S_{\zeta 2}\right) \tag{30}$$

the real amplitude of torsional vibration

$$\zeta_0 = \frac{M_{\zeta 0}}{\sqrt{[(k_\zeta - I_\zeta\omega^2)^2 + (c_\zeta\omega)^2]}} = \frac{M_{\zeta 0}}{k_\zeta}\frac{1}{\sqrt{\{[1 - (\omega/\omega_0)^2]^2 + 4D_\zeta^2(\omega/\omega_0)^2\}}} \tag{31}$$

and the phase shift:

$$\phi = -a\tan\frac{c_\zeta\omega}{k_\zeta - I_\zeta\omega^2}$$

In equation (31) natural undamped frequency ω_0 and damping ratio D_ζ are

$$\omega_0 = \sqrt{(k_\zeta/I_\zeta)}, \quad D_\zeta = c_\zeta/2I_\zeta\omega_0 \tag{32}$$

The natural undamped frequency of an embedded footing must be determined by a trial and error approach as it appears in k_ζ too. Equations (31) and (32) are formally equal to those of a single degree of freedom system (Voigt model). Often the dimensionless amplitude $A_\zeta = \zeta_0 Gr_0^3/M_{\zeta 0}$ is useful. This dimensionless amplitude depends only on inertia ratio B_ζ and a_0 as can be seen.

In many practical problems the excitation is due to rotation of an unbalanced mass m_e with the eccentricity e_m acting at a distance r_c from the axis of the footing. The excitation moment amplitude is $M_{\zeta 0} = m_e e_m r_c\omega^2$ and a dimensionless rotation amplitude may be introduced, $A_\zeta = \zeta_0 I_\zeta/(m_e e_m r_c)$.

The predicted response depends on functions $C_{\zeta 1,2}$ substituted into the above formulas. Four typical cases can be considered: the footing can be embedded in the half-space or in a stratum and surrounded by undisturbed soil or backfill depending on the type of construction.

Embedment in the half-space

Stiffness parameter $C_{\zeta 1}$ and damping parameter $C_{\zeta 2}$ were computed from equations (24) with displacement functions $f_{\zeta 1,2}$ taken from Bycroft.[6] These parameters are shown in Figure 1 in full lines and are given in Appendix I in a polynomial form to facilitate the computation.

The stiffness increase due to embedment was computed from equation (29) with frequencies $a_0 = 0$ and 1 and is compared with the static finite element solution by Kaldjian[8] in Figure 2. The agreement is quite reasonable.

Examples of theoretical response curves computed from equation (31) with various relative embedments $\delta = I/r_0$ are shown in Figure 3.
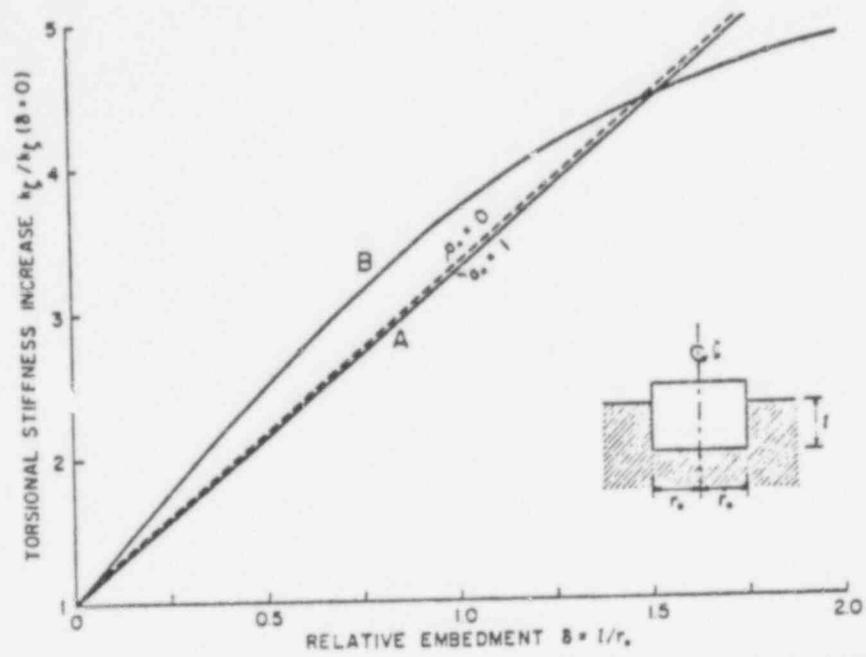
Figure 2. Torsional stiffness increase due to embedment: (A) approximate dynamic theory, (B) Kaldjian's[8] static finite element solution
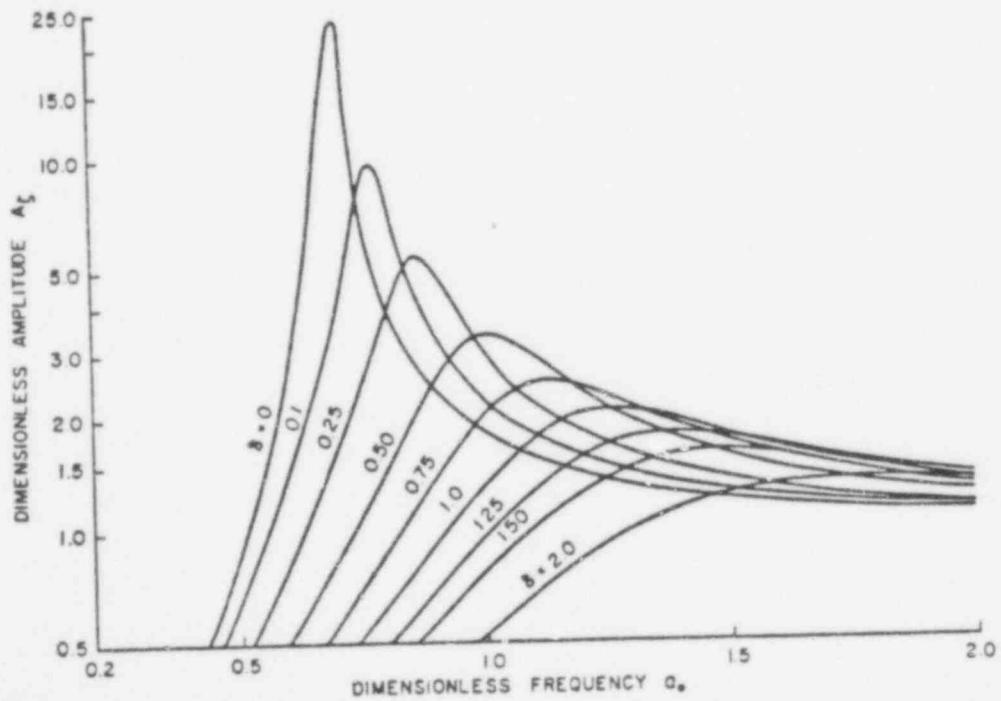
Figure 3. Theoretical response curves of torsional vibrations of footings embedded in elastic half-space (inertia ratio $B_\zeta = 10$, $\rho_s/\rho = 1$, quadratic excitation)

It can be seen that the theory indicates a considerable increase in the resonant frequencies and reduction in the resonant amplitudes due to embedment.

Weissmann[21] approximately considered slip in regions of high shear stress and included hysteretic damping of soil. Functions $C_{\zeta 1,2}$ derived from his solution are shown in Figure 1 in dashed lines for both granular and cohesive soils respectively. These functions considerably deviate from the strictly elastic solution, however, but may sometimes yield results closer to reality.

Embedment in a stratum

Embedment in a stratum can be considered with equal ease. The functions $C_{\zeta 1}$ and $C_{\zeta 2}$ for two strata featuring relative thicknesses $h/r_0 = 0.5$ and 1 are given in Appendix I and are shown in Figure 1. They were computed from Bycroft's functions $f_{1,2}$.[8]

Base reactions for strata of any thickness can be obtained from Awojobi's solution.[1] The corresponding stiffness and damping parameters are also given in Table II (Appendix I).

Examples of response curves obtained from equation (31) with Awojobi's base reactions are shown in Figure 4 for sublayers having relative thicknesses $h/r_0 = 1, 2, 5$ and ∞. In Figures 5 and 6 variations of
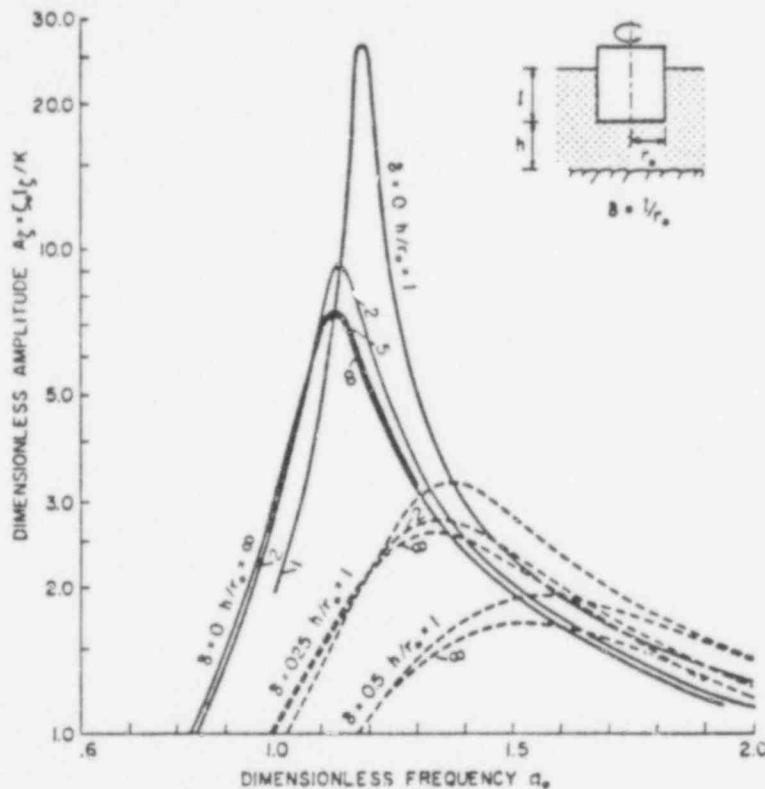
Figure 4. Theoretical response curves of torsional vibrations of footings embedded in strata with $h/r_0 = 1, 2, 5$ and ∞. (Embedment ratios $\delta = 0, 0.25$ and 0.50, $B_\zeta = 3.5$, $\rho_s/\rho = 1$, quadratic excitation)

resonant (maximum) amplitudes and corresponding resonant frequencies with stratum thickness and embedment ratio are shown for inertia ratios $B_\zeta = 2, 4$ and 10. These graphs were also calculated with Awojobi's base reactions. With $\delta = 0$ the solution corresponds to that for surface footings, i.e. to that of Awojobi or Bycroft according to the base reactions used. There is very little difference between Bycroft's and Awojobi's solutions with $h/r_0 = \infty$.

It can be seen from Figures 4–6 that the effect of embedment in a stratum is even more pronounced than with that of embedment in the half-space because of little or no geometric damping generated through the base. Figures 5 and 6 can be used to assess the effects of embedment, inertia ratio and layering.

649 276
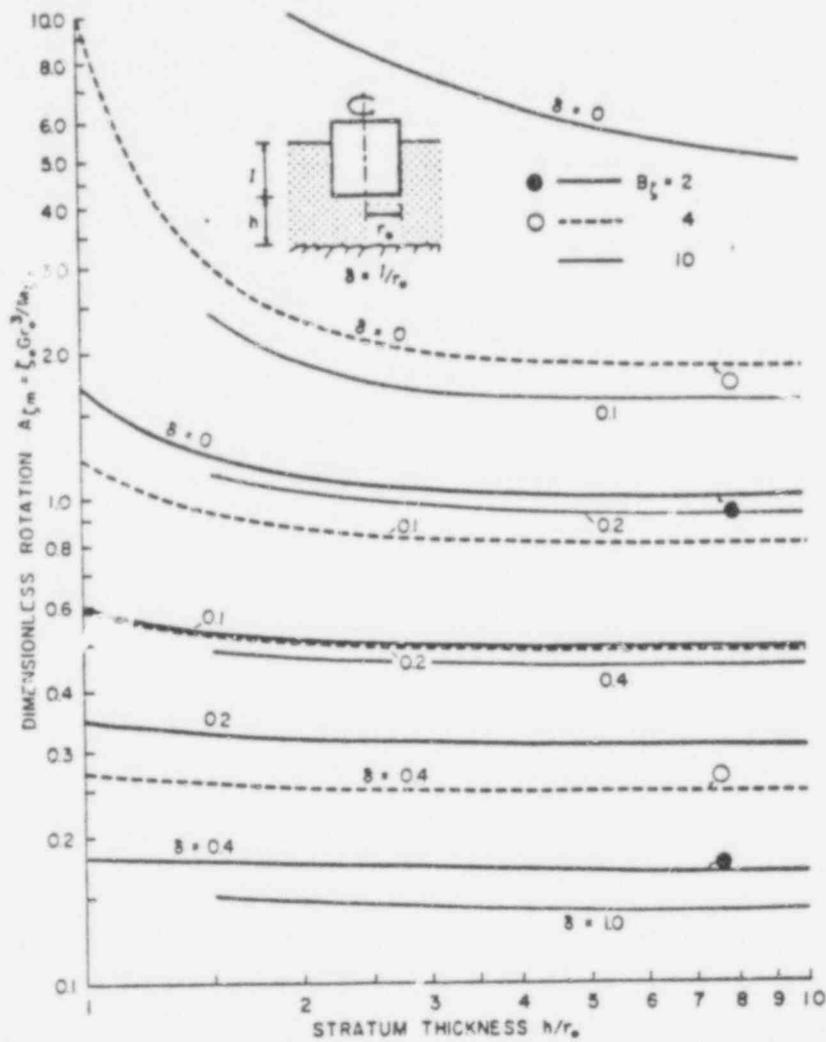
Figure 5. Dimensionless resonant amplitudes of torsional vibrations of footings embedded in strata for several embedment ratios and inertia ratios $B_\zeta = 2$, 4 and 10. ($B_\zeta = I_\zeta / r \rho_0^5$; circles indicate finite element solution, Reference 20)
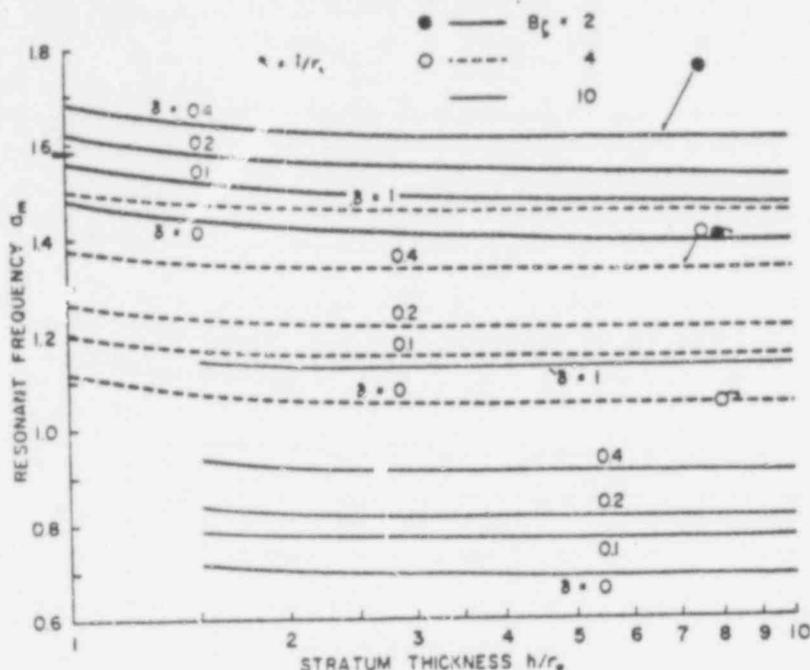
Figure 6. Dimensionless resonant frequencies of torsional vibrations of footings embedded in strata for several embedment ratios δ and inertia ratios $B_\zeta = 2$, 4 and 10. ($B_\zeta = I_\zeta/\rho r_0^5$; circles indicate finite element solution, Reference 20)

Also plotted in Figures 5 and 6 are resonant amplitudes and frequencies obtained for $h/r_0 = 8$ by Wass and Lysmer[20] using the finite element technique. Their results are shown as circles; arrows indicate with which analytical curve they compare. The agreement is quite good. (The finite element results shown were interpolated to match the parameters B_ζ used here. The differences in resonant frequencies with $\delta = 0.4$ are of little practical importance as the corresponding response curves are rather flat.)

SIMPLIFIED DESIGN ANALYSIS

The prediction of the response from the above formula is not difficult; however, the computation can be considerably simplified if stiffness factors $C_{\zeta 1}$ and $S_{\zeta 1}$ are considered as constant (frequency independent) and damping factors $C_{\zeta 2}$ and $S_{\zeta 2}$ as proportional to dimensional frequency a_0. It can be seen from Figure 1 the degree to which these assumptions are acceptable. They seem reasonable and quite adequate for applications with respect to all the physical uncertainties involved.

Thus, stiffness parameters can be considered approximately constant, i.e. $C_{\zeta 1} = \bar{C}_{\zeta 1}$ and $S_{\zeta 1} = \bar{S}_{\zeta 1}$, where the constants denoted by bars can be readily obtained from Figure 1 or Appendix I in which some suitable values are given. With constant stiffness parameters $\bar{C}_{\zeta 1}$ and $\bar{S}_{\zeta 1}$, the frequency independent stiffness constant is obtained from equation (29). Then, the natural frequency ω_0 can be directly calculated from equation (32).

Proportionality of damping factors to frequency assumes:

$$C_{\zeta 2} = \bar{C}_{\zeta 2} a_0 \quad \text{and} \quad S_{\zeta 2} = \bar{S}_{\zeta 2} a_0 \tag{33}$$

in which $\bar{C}_{\zeta 2}$ and $\bar{S}_{\zeta 2}$ are constants whose values can be again established from Figure 1 or Appendix I.

Substitution of equations (33) into equation (30) yields the frequency independent damping constant for torsional vibrations of embedded footings:

$$c_\zeta = r_0^4 \sqrt{(\rho G)} \left[\bar{C}_{\zeta 2} + \bar{S}_{\zeta 2} \frac{l}{r_0} \sqrt{\left/ \left(\frac{\rho_s}{\rho} \frac{G_s}{G} \right) \right.} \right] \tag{34}$$

The frequency independent damping ratio is from equations (32) and (29)

$$D_\zeta = \frac{1}{2\sqrt{B_\zeta}}\left[C_{\zeta 2} + S_{\zeta 2}\frac{l}{r_0}\sqrt{\left(\frac{\rho_s}{\rho}\frac{G_s}{G}\right)}\right]\Big/\sqrt{\left(C_{\zeta 1} + \frac{G_s}{G}\frac{l}{r_0}S_{\zeta 1}\right)} \tag{35}$$

in which inertia ratio $B_\zeta = I_\zeta/\rho r_0^5$. For surface footings, equation (35) yields $D_\zeta = 0.17/\sqrt{B_\zeta}$ and numerical results close to those of Richart and co-workers (Reference 17, p. 226).

With D_ζ obtained, the 'resonant' amplitude at natural frequency ω_0 (slightly smaller than the maximum amplitude) is simply $\zeta_0(\omega_0) = M_{\zeta 0}/(k_\zeta 2D_\zeta)$ or with quadratic excitation $\zeta_0(\omega_0) = m_s e_m r_c/(2D_\zeta I_\zeta)$. The maximum amplitude is $1/\sqrt{(1 - D_\zeta^2)}$ times larger.

An example of comparison of response curves computed with variable parameters and with constant parameters is shown in Figure 7. ($\bar{C}_\zeta = 0.5$ was used for the half-space in this case because it yielded better agreement in the frequency range employed.) The agreement appears satisfactory for design purposes. As even better agreement can be obtained with a mass adjustment.
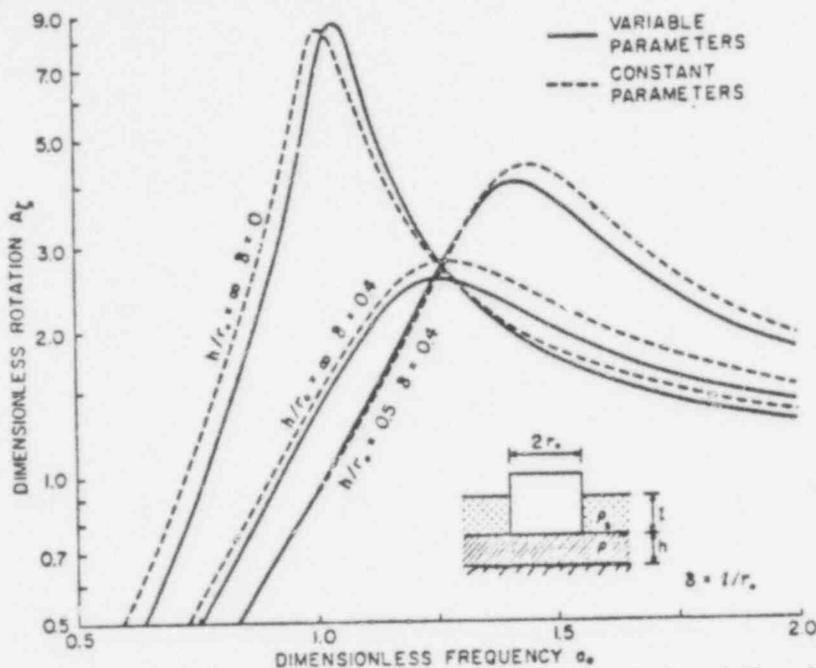
Figure 7. Comparison of response curves of torsional vibrations computed with both variable and constant parameters ($B_\zeta = 4.18$, backfill with $\rho_s/\rho = 0.8$, $G_s/G = 0.5$, quadratic excitation, Bycroft's base reactions)

In practical applications, the effect of embedment may be reduced due to backfill and an imperfect bond between the footing and the soil. The effect of backfill can be accounted for by considering $G_s < G$ and $\rho_s < \rho$ in equations (29) and (35). Possible lack of bond between the footing and the soil can be approximately taken into account by considering Weissmann's base reactions (dashed lines in Figure 1) and intuitively reduced values of $S_{\zeta 1,2}$.

COUPLED VIBRATION INVOLVING TORSIONAL, HORIZONTAL AND ROCKING COMPONENTS

If the centre of gravity of the footing does not lie on the vertical geometric axis of the footing, a coupled motion in three degrees of freedom is produced by a horizontal force Q, by a moment M_ζ in the horizontal plane and by a moment in the vertical plane M_ψ. The components of this motion are the horizontal translation $u(t)$, rotation in the horizontal plane $\zeta(t)$ and rotation in the vertical plane (rocking) $\psi(t)$. This motion can also be solved under the same assumptions as introduced for pure torsional vibration.

Consider the centre of gravity the origin of co-ordinates. With the notation according to Figure 8 and with the omission of product moments of inertia and of coupling between displacements and reactions in
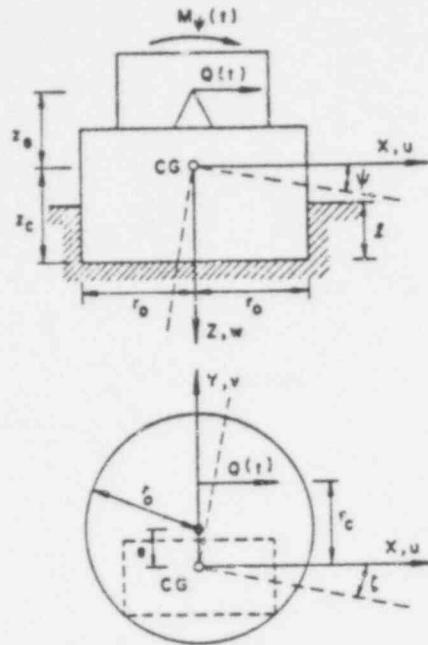
Figure 8. Notation for coupled vibration of footing in three degrees of freedom involving horizontal translation, rocking and torsion

the footing base,[10,19] the equations of the motion are

$$m\ddot{u}(t) + r_0[G(C_{u1} + iC_{u2}) + G_s \delta(S_{u1} + iS_{u2})]u(t)$$

$$+ er_0[G(C_{u1} + iC_{u2}) + G_s \delta(S_{u1} + iS_{u2})]\zeta(t)$$

$$+ r_0^2\left\{-G\frac{z_c}{r_0}(C_{u1} + iC_{u2}) + G_s \delta\left[\left(\frac{1}{2}\delta - \frac{z_c}{r_0}\right)(S_{u1} + iS_{u2})\right]\right\}\psi(t) = Q(t) \tag{36a}$$

$$I_\psi \ddot{\psi}(t) + r_0^2\left[G_s \delta\left(\frac{\delta}{2} - \frac{z_c}{r_0}\right)(S_{u1} + iS_{u2}) - G\frac{z_c}{r_0}(C_{u1} + iC_{u2})\right]u(t)$$

$$+ er_0^2\left[G_s \delta\left(\frac{\delta}{2} - \frac{z_c}{r_0}\right)(S_{u1} + iS_{u2}) - G\frac{z_c}{r_0}(C_{u1} + iC_{u2})\right]\zeta(t)$$

$$+ r_0^3\left\{G(C_{\psi1} + iC_{\psi2}) + G_s \delta\left[S_{\psi1} + iS_{\psi2} + \left(\frac{\delta^2}{3} - \delta\frac{z_c}{r_0} + \frac{z_c^2}{r_0^2}\right)(S_{u1} + iS_{u2})\right]\right.$$

$$\left. + G\frac{z_c^2}{r_0^2}(C_{u1} + iC_{u2})\right\}\psi(t) = M_\psi(t) \tag{36b}$$

$$I_\zeta \ddot{\zeta}(t) + er_0[G(C_{u1} + iC_{u2}) + G_s \delta(S_{u1} + iS_{u2})]u(t)$$

$$+ \{e^2 r_0[G(C_{u1} + iC_{u2}) + G_s \delta(S_{u1} + iS_{u2})]$$

$$+ [G_s r_0^3 \delta(S_{\zeta1} + iS_{\zeta2}) + Gr_0^3(C_{\zeta1} + iC_{\zeta2})]\}\zeta(t)$$

$$+ er_0^2\left[-G\frac{z_c}{r_0}(C_{u1} + iC_{u2}) + G_s \delta\left(\frac{1}{2}\delta - \frac{z_c}{r_0}\right)(S_{u1} + iS_{u2})\right]\psi(t) = M_\zeta(t) \tag{36c}$$

In these equations, C_ζ and S_ζ are the parameters derived above. Parameters C_u, S_u and C_ψ, S_ψ are analogous functions related to horizontal translation u and rocking ψ and can be found in Reference 4.

With harmonic excitation in complex form

$$Q(t) = Q_0 e^{i\omega t}, \quad M_\psi(t) = M_{\psi 0} e^{i\omega t}, \quad M_\zeta(t) = M_{\zeta 0} e^{i\omega t} \tag{37}$$

the response is

$$u(t) = u_c e^{i\omega t}, \quad \psi(t) = \psi_c e^{i\omega t}, \quad \zeta(t) = \zeta_c e^{i\omega t} \tag{38}$$

in which subscript c denotes complex amplitudes. Substitution of equations (37) and (38) into equations (36) yields for the complex amplitudes of coupled motion:

$$\left. \begin{aligned}
[(k_{uu} - m\omega^2) + i\omega c_{uu}] u_c + (k_{u\psi} + i\omega c_{u\psi}) \psi_c + (k_{u\zeta} + i\omega c_{u\zeta}) \zeta_c &= Q_0 \\
(k_{u\psi} + i\omega c_{u\psi}) u_c + [(k_{\psi\psi} - I_\psi \omega^2) + i c_{\psi\psi}] \psi_c + (k_{\psi\zeta} + i\omega c_{\psi\zeta}) \zeta_c &= M_{\psi 0} \\
(k_{u\zeta} + i\omega c_{u\zeta}) u_c + (k_{\psi\zeta} + i\omega c_{\psi\zeta}) \psi_c + [(k_{\zeta\zeta} - I_\zeta \omega^2) + i c_{\zeta\zeta}] \zeta_c &= M_{\zeta 0}
\end{aligned} \right\} \tag{39}$$

in which frequency dependent stiffness constants:

$$\left. \begin{aligned}
k_{uu} &= Gr_0\left(C_{u1} + \frac{G_s}{G}\delta S_{u1}\right) \\[6pt]
k_{u\psi} &= -Gr_0\left[z_c C_{u1} + \frac{G_s}{G}\delta(z_c - \tfrac{1}{2}r_0\delta) S_{u1}\right] \\[6pt]
k_{u\zeta} &= e k_{uu} \\[6pt]
k_{\psi\psi} &= Gr_0^3\left[C_{\psi 1} + \left(\frac{z_c}{r_0}\right)^2 C_{u1} + \frac{G_s}{G}\delta S_{\psi 1} + \frac{G_s}{G}\delta\left(\frac{\delta^2}{3} + \frac{z_0^2}{r_0^2} - \delta\frac{z_c}{r_0}\right) S_{u1}\right] \\[6pt]
k_{\psi u} &= k_{u\psi} \\[6pt]
k_{\psi\zeta} &= -Ger_0\left[z_c C_{u1} + \frac{G_s}{G}\delta(z_c - \tfrac{1}{2}r_0\delta) S_{u1}\right] = e k_{u\psi} \\[6pt]
k_{\zeta\zeta} &= Gr_0^3\left(C_{\zeta 1} + \frac{G_s}{G}\delta S_{\zeta 1} + \frac{e^2}{r_0^2}C_{u1} + \frac{G_s}{G}\delta\frac{e^2}{r_0^2}S_{u1}\right) \\[6pt]
k_{\zeta u} &= e k_{uu} = k_{u\zeta} \\[6pt]
k_{\zeta\psi} &= k_{\psi\zeta}
\end{aligned} \right\} \tag{40}$$

and frequency dependent damping constants:

$$\left. \begin{aligned}
c_{uu} &= \frac{Gr_0}{\omega}\left(C_{u2} + \frac{G_s}{G}\delta S_{u2}\right) \\[6pt]
c_{u\zeta} &= c_{uu}e \\[6pt]
c_{u\psi} &= -\frac{Gr_0}{\omega}\left[z_c C_{u2} + \frac{G_s}{G}\delta(z_c - \tfrac{1}{2}r_0\delta) S_{u2}\right] \\[6pt]
c_{\psi\psi} &= \frac{Gr_0^3}{\omega}\left[C_{\psi 2} + \frac{z_c^2}{r_0^2}C_{u2} + \frac{G_s}{G}\delta S_{\psi 2} + \frac{G_s}{G}\delta\left(\frac{\delta^3}{3} + \frac{z_c^2}{r_0^2} - \delta\frac{z_c}{r_0}\right) S_{u2}\right] \\[6pt]
c_{\psi u} &= c_{u\psi} \\[6pt]
c_{\psi\zeta} &= -\frac{Ger_0}{\omega}\left[z_c C_{u2} + \frac{G_s}{G}\delta(z_c - \tfrac{1}{2}r_0\delta) S_{u2}\right] \\[6pt]
c_{\zeta u} &= c_{u\zeta} \\[6pt]
c_{\zeta\psi} &= c_{\psi\zeta} \\[6pt]
c_{\zeta\zeta} &= \frac{Gr_0^3}{\omega}\left(C_{\zeta 2} + \frac{G_s}{G}\delta S_{\zeta 2} + \frac{e^2}{r_0^2}C_{u2} + \frac{G_s}{G}\delta\frac{e^2}{r_0^2}S_{u2}\right)
\end{aligned} \right\} \tag{41}$$

From equations (39), the complex amplitudes $u_c = u_1 + iu_2$, etc. can be obtained with any excitation frequency. The real and imaginary parts of the complex amplitudes yield the real amplitudes:

$$u_0 = \sqrt{(u_1^2 + u_2^2)}, \quad \psi_0 = \sqrt{(\psi_1^2 + \psi_2^2)}, \quad \zeta_0 = \sqrt{(\zeta_1^2 + \zeta_2^2)} \tag{42}$$

With quadratic excitation, these amplitudes can be made dimensionless according to the formulas:

$$A_x = \frac{m}{m_e e_m} u_0, \quad A_\psi = \frac{I_\psi}{m_e e_m z_e} \psi_0, \quad A_\zeta = \frac{I_\zeta}{m_e e_m r_c} \zeta_0 \tag{43}$$

The natural frequencies can be obtained from equation (39) by setting the right sides and the terms labelled by i equal to zero and solving the eigenvalue problem by trial and error.

The reciprocity relations in equations (40) and (41) hold in this approach but need not hold in general as shown in References 10 and 19.

An example of the response of embedded footings in three degrees of freedom is shown in Figure 9 for eccentricity $e = r_0/3$ and for various embedment depths. Dashed lines denoted $a_{1,2,3}$ indicate the first, second and third resonances of the surface footing ($\delta = 0$).

The first resonant peak shows clearly in all three components of the motion.

The second resonant region is dominated by torsion and can hardly be recognized in the translation component A_x.

The third resonant peak is completely suppressed in all three components.

These relations, of course, depend on the properties of the footing. Nevertheless, the example illustrates possible importance of the torsional component and the effect of embedment on the response in the three resonant regions. It can be seen that the response above the second resonance depends very little on the embedment. With small eccentricities (e) and medium or low mass ratios the third resonance usually appears suppressed.

The analysis of the coupled response can again be considerably simplified if constant parameters C and S are introduced as it was shown in the case of pure torsional vibration. The efficiency of such a simplified approach in two degrees of freedom was demonstrated in Reference 4. With constant parameters C and S introduced into equations (40) and (41), frequency independent stiffness and damping matrices are obtained that can be readily used in the analysis of any structure supported by embedded or surface footings, as discussed in Reference 23.

COMPARISON WITH EXPERIMENTS

To assess the applicability of the theory to circular footings and also to footings featuring square and rectangular bases (cross-sections), field experiments were conducted with three concrete blocks at The University of Western Ontario. The blocks were cast directly into neatly cut excavations. The embedment was changed by removing the soil in three steps. The effect of backfill was investigated by step-wise backfilling of the soil. Pure torsional vibrations were excited by means of a torsional mechanical oscillator with exchangeable eccentric masses.

The soil was a brown, silty clay underlain by a glacial till. The shear modulus and Poisson's ratio of the soil were obtained from wave velocities measured in the field. These measurements were complemented by laboratory investigations. The following properties of the soil were found:

Bulk density of undisturbed soil = 110 lb/ft³ (1760 kg/m³)

Bulk density of backfill = 100 lb/ft³ (1602 kg/m³)

Water content of soil = 15 per cent

Void ratio of undisturbed soil = 0·7

Shear modulus of undisturbed soil = 1·13 × 10⁶ lb/ft² (549 kg/cm²)
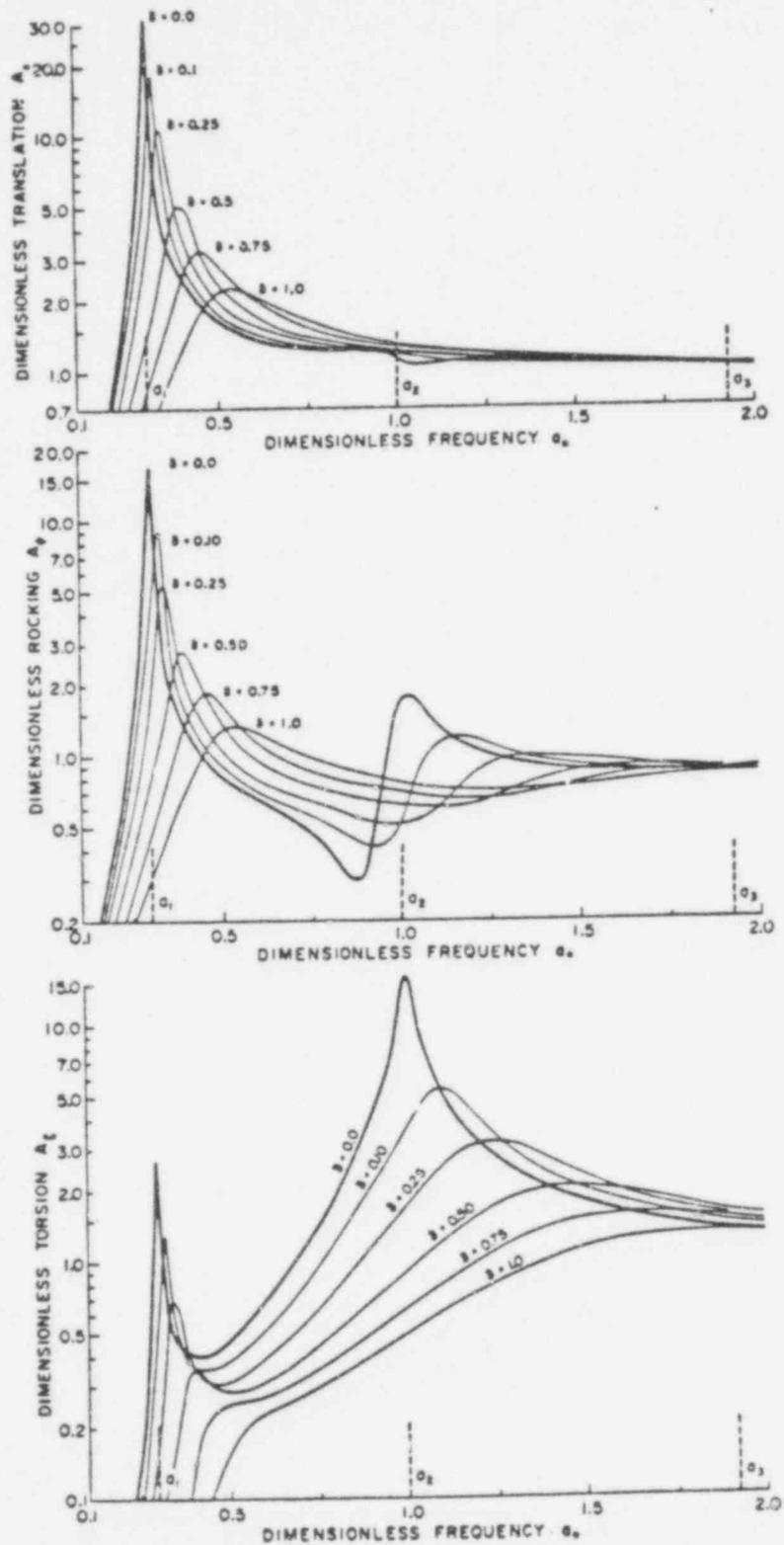
Figure 9. Example of coupled response of embedded footing in three degrees of freedom ($B_\zeta = 4.56, e = r_0/3$, $G_s = G$; translation, rocking and torsional components, quadratic excitation)

The main data on the test footings are given in Table I. The equivalent radii r_0 for square and rectangular footings were computed from the equality of polar moments of inertia of the base.

Table I

Shape of base	Base area ft² (m²)	I_ζ lb ft² s²	r_0 ft (m)	B_ζ
Circular	3·98 (0·369)	54·5 (7·54)	1·125 (0·343)	8·65
Square	5·06 (0·469)	77·6 (10·70)	1·284 (0·391)	6·50
Rectangular	5·01 (0·464)	93·2 (12·88)	1·330 (0·405)	6·55

A typical example of steady torsional response measured at various embedments is shown in Figure 10. The foundation base was circular in this case and the excitation intensity was the same for all embedments.
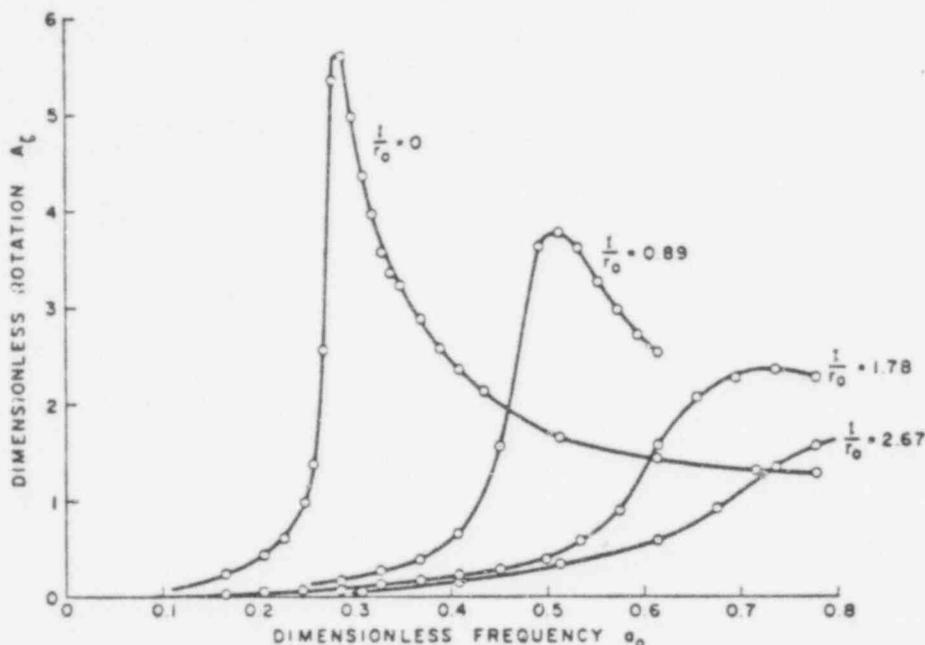
Figure 10. Measured response curves of cylindrical footing at various embedments ($m_e e_m r_\zeta = 0·00326$ lb ft s², undisturbed soil, $B_\zeta = 8·65$)

Comparison of Figure 10 with Figure 3 indicates a good qualitative agreement in the major effects of embedment manifested through drastic reduction in resonant amplitudes and large increase in resonant frequencies. The former effect is primarily due to increased geometric damping while the latter effect confirms an increase in stiffness.

With other foundation shapes the changes in response due to embedment were similar to those depicted in Figure 10. A comparison of response curves observed with various footings is shown in Figure 11 for surface footings and for an embedment of one foot. It can be seen that embedment was most efficient with the rectangular footing. This indicates the efficiency with which torque can be transmitted into the surrounding soil.

The quantitative comparison of theoretical and experimental resonant amplitudes and resonant frequencies is somewhat complicated by non-linearity of the response which renders the dimensionless resonant amplitudes and frequencies dependent on the excitation intensity as shown in Figures 12 and 13. In Figure 13, resonant rotations are shown in full lines, resonant frequencies in dashed lines and embedment is indicated as 0, 1 or 2 ft.

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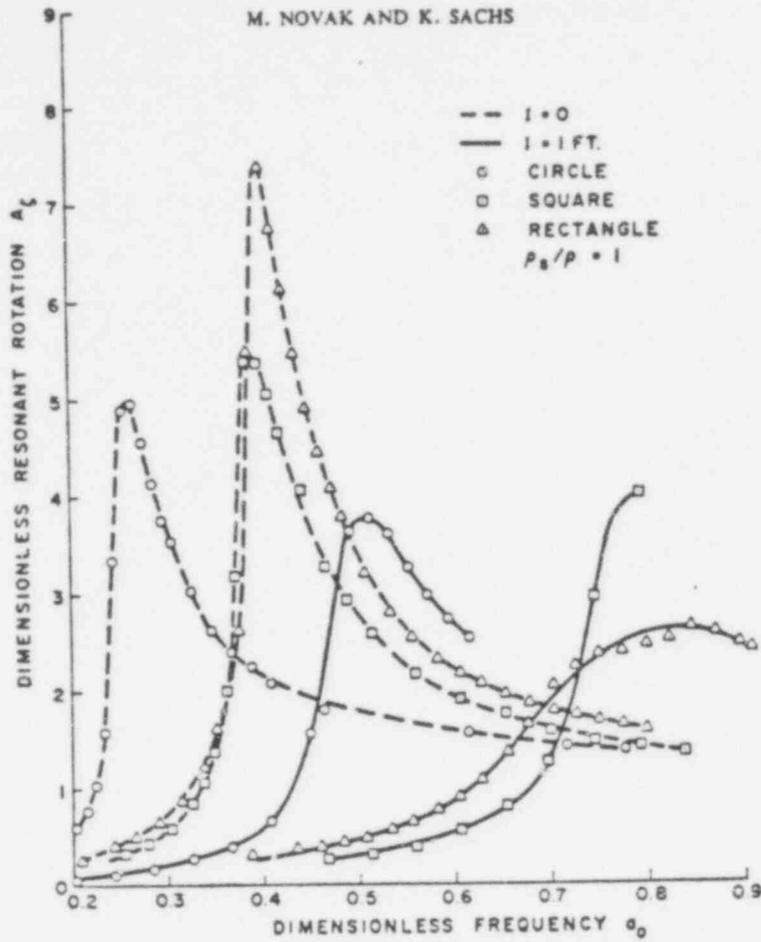
Figure 11. Measured response curves of cylindrical, square and rectangular footings with embedment 0 and 1 ft (undisturbed soil)
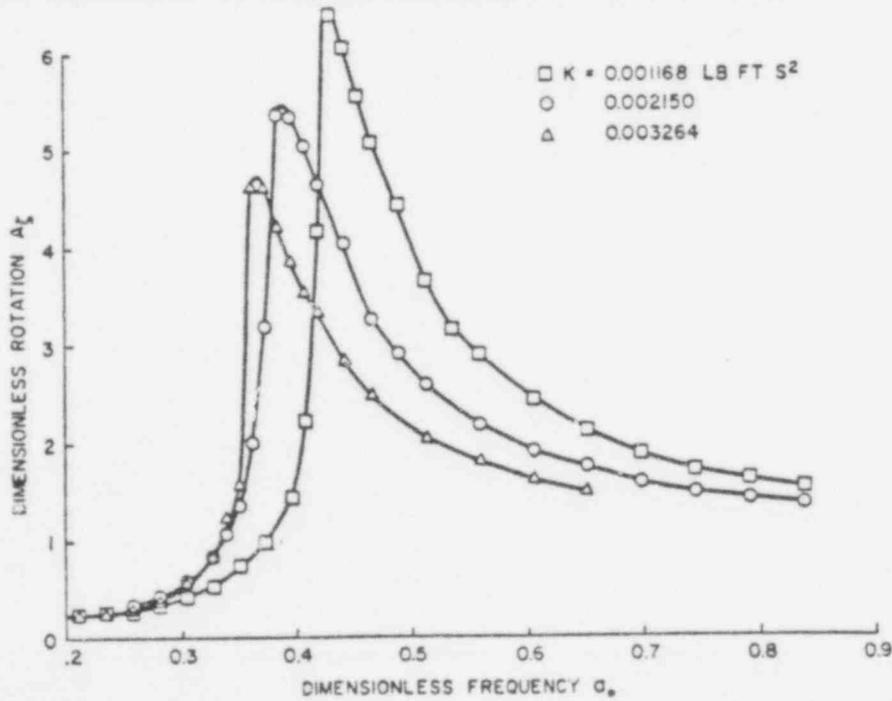
Figure 12. Dimensionless experimental response curves of torsional vibrations measured with three excitation intensities (square footing, embedment = 0, $B_\zeta = 6.5$, quadratic excitation)
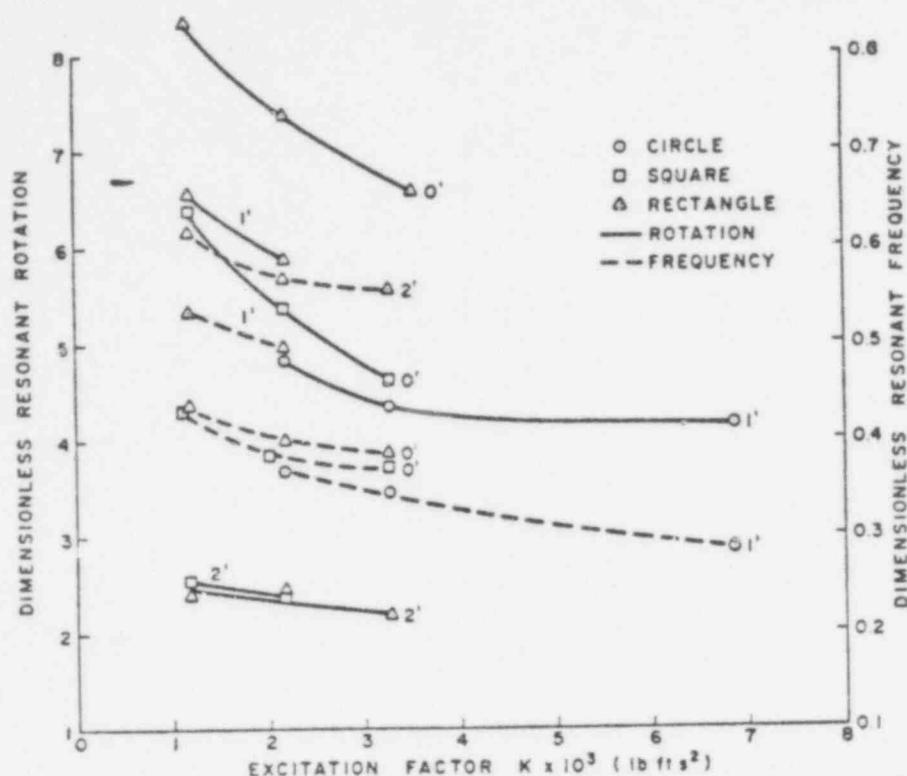
Figure 13. Measured dimensionless resonant rotations $A_{\zeta m}$ and resonant frequencies a_m versus excitation factor $K = m_e e_m r_e$ (zero embedment or backfill 1 or 2 ft)

It appears that there is a trend for both the resonant amplitudes and resonant frequencies to decline with the increasing intensity of excitation. (The measurements were conducted with increasing intensity of excitation.) These changes seem to indicate a damping producing slip between the footing and the soil in regions of high shear stresses as suggested by Whitman[22] for surface footings. The theory presented herein assumes a perfect bond between the footing and the soil; therefore, the measurements conducted with minimum excitation intensities are used, for the most part, in further comparisons shown in Figures 14 and 15.

In Figure 14 the theoretical and experimental resonant amplitudes are compared.

For surface footings, the experimental amplitudes are considerably *smaller* than the theoretical ones, in agreement with observations made by Whitman.[22] For embedded footings the experimental amplitudes are *larger* than the theoretical values; however, the differences are smaller than with surface footings in the case of undisturbed soil and are quite small for the rectangular footing. For backfill the agreement is very poor. It can be seen from Figure 14 that there is a region of small embedments ($l/r_0 \cong 0.1–0.5$) in which the theoretical and experimental amplitudes must coincide.

The comparison of theoretical and experimental resonant frequencies is shown in Figure 15. The experimental resonant frequencies are consistently much lower than the theoretical predictions. The differences do not diminish with increasing embedment and appear largest for the circular footing. (Of course, there is a dependence on the rather inaccurate value of the shear modulus.) Nevertheless, it is clear that the torsional stiffness is much less than the elastic theory predicts.

The experimental frequencies and amplitudes of surface footings reported here are in reasonable agreement with Fry's experiments and with Weissmann's modified theory.[21]

It can be concluded that the quantitative agreement between the strictly elastic theory and the experiments is, in general, poor, particularly as far as resonant frequencies and the effect of backfill are concerned. These differences may be attributed to limited capacity of the soil to transmit the torque from the footing through shear. It is desirable for any theory to allow for a slip in the regions of high shear stresses. The presented
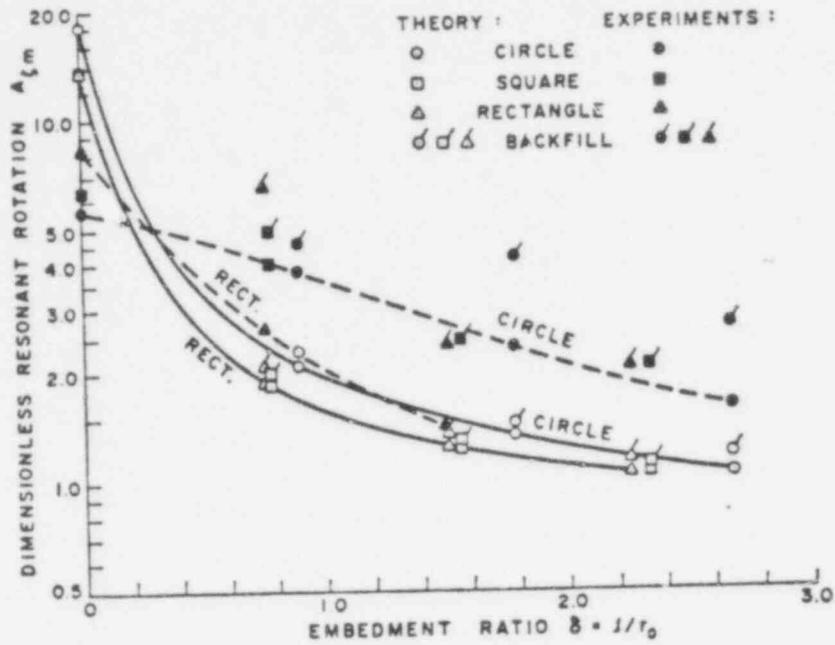
Figure 14. Comparison of theoretical and experimental resonant amplitudes of torsional vibrations (—— theory; · · · · · experiments)
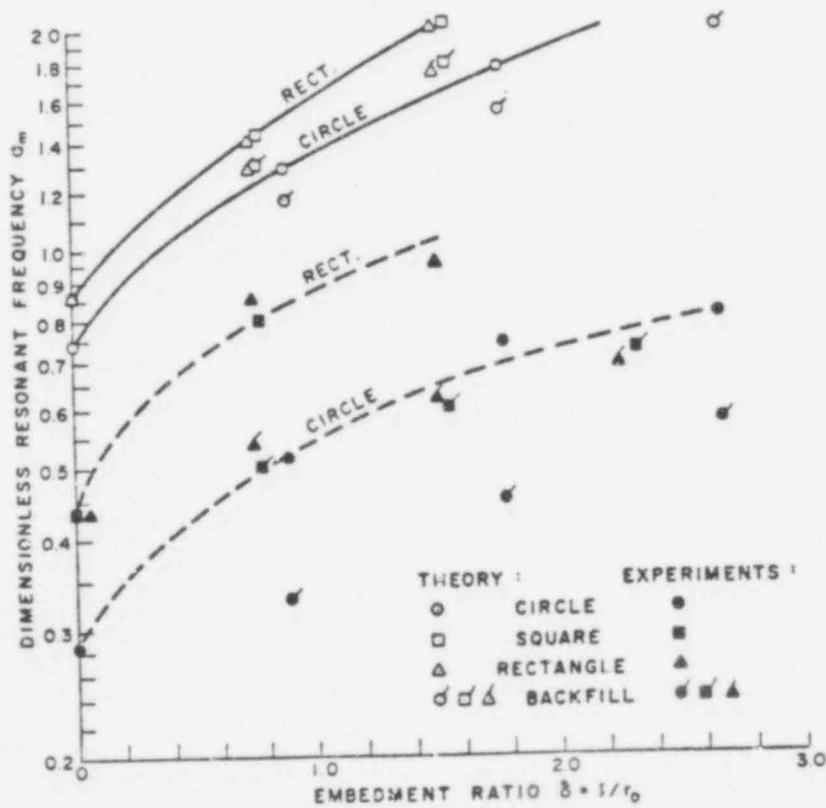
Figure 15. Comparison of theoretical and experimental resonant frequencies of torsional vibration (—— theory; · · · · · experiments)

theory makes it possible to include such modified soil reactions in the footing base. Such a modification would bring the theoretical predictions closer to experimental results.

CONCLUSIONS

Embedment produces a drastic decrease in resonant amplitudes and a marked increase in resonant frequencies of torsional vibrations. These effects are particularly pronounced for footings embedded in shallow strata with relative depth h/r_0 smaller than about 2 where the omission of embedment and internal damping can yield unrealistically large resonant amplitudes. With embedment taken into account, layering appears much less important.

The coupled response involving translation, rocking and torsion is strongly affected by embedment in the regions of the first and second resonances but depends very little on embedment in the region of the third resonance where there is usually no appreciable increase in amplitudes.

The approximate analytical solution agrees quite well with the available finite element solutions and requires very little computing time, and layers with any thickness can be readily considered.

The field experiments show a good qualitative agreement with the theory in the reduction of resonant amplitudes and in the increase in resonant frequencies due to embedment but the quantitative agreement in general is poor. Best agreement is found in resonant amplitudes for small embedments and rectangular footings, worst agreement appears for surface footings, and for circular footings in general.

The real resonant amplitudes are smaller than those theoretically predicted for surface footings and are larger than predicted for embedded footings. Only for small embedments fair agreement can be expected.

The resonant frequencies are considerably lower than the predicted values. Also the effect of backfill is in general much smaller than predicted.

The differences observed can be attributed to the slip in the regions of high shear. The inclusion of this effect appears necessary in order to improve the reliability of the theoretical predictions of the torsional response of footings no matter what kind of approach is used.

APPENDIX I

Stiffness and damping parameters

Stiffness and damping parameters are given in Table II, calculated from papers indicated. From Awojobi's solution,[1] the parameters can be obtained for any stratum thickness with $a_0 > r_0/h$ when the equivalent dimensionless frequency is introduced

$$a_e = \sqrt{[a_0^2 - r_0^2/h^2]} \tag{44}$$

Table II. Stiffness and damping parameters for torsional vibration

Stiffness and damping parameters	Constant parameters	Validity range
Side layer		
$S_{\zeta 1} = 12 \cdot 58 - 1 \cdot 01 a_0 - 5 \cdot 912 a_0^2$	$\bar{S}_{\zeta 1} = 12 \cdot 4$	$0 \leqslant a_0 \leqslant 0 \cdot 2$
$S_{\zeta 1} = 12 \cdot 59 - 1 \cdot 855 a_0 - 3 \cdot 349 a_0^2 + 5 \cdot 335 a_0^3 - 2 \cdot 76 a_0^4 + 0 \cdot 495 a_0^5$	$\bar{S}_{\zeta 1} = 10 \cdot 2$	$0 \cdot 2 \leqslant a_0 \leqslant 2 \cdot 0$
$S_{\zeta 2} = 9 \cdot 04 a_0^2$	$\bar{S}_{\zeta 2} = 2 \cdot 0$	$0 \leqslant a_0 \leqslant 0 \cdot 2$
$S_{\zeta 2} = 7 \cdot 5 a_0 - \dfrac{3 \cdot 726 a_0}{0 \cdot 455 + a_0}$	$\bar{S}_{\zeta 2} = 5 \cdot 4$	$0 \cdot 2 \leqslant a_0 \leqslant 2 \cdot 0$
Half space*		
$C_{\zeta 1} = 5 \cdot 333 + 0 \cdot 032 a_0 - 1 \cdot 358 a_0^2 + 0 \cdot 7434 a_0^3 - 0 \cdot 1414 a_0^4$	$\bar{C}_{\zeta 1} = 4 \cdot 3$	$0 \leqslant a_0 \leqslant 2 \cdot 0$
$C_{\zeta 2} = 0 \cdot 486 a_0^2$	$\bar{C}_{\zeta 2} = 0 \cdot 7$	$0 \leqslant a_0 \leqslant 2 \cdot 0$
Stratum $h/r_0 = 0 \cdot 5$*		
$C_{\zeta 1} = 6 \cdot 78 - 0 \cdot 5377 a_0 + 0 \cdot 4545 a_0^2 - 0 \cdot 2206 a_0^3$	$\bar{C}_{\zeta 1} = 6 \cdot 5$	$0 \leqslant a_0 \leqslant 2 \cdot 5$
$C_{\zeta 2} = 0$	$\bar{C}_{\zeta 2} = 0$	$0 \leqslant a_0 \leqslant 3 \cdot 14$
Stratum $h/r_0 = 1 \cdot 0$*		
$C_{\zeta 1} = 5 \cdot 75 - 0 \cdot 4191 a_0 + 1 \cdot 381 a_0^2 - 2 \cdot 115 a_0^3 + 0 \cdot 5927 a_0^4$	$\bar{C}_{\zeta 1} = 5 \cdot 2$	$0 \leqslant a_0 \leqslant 2 \cdot 00$
$C_{\zeta 2} = 0$	$\bar{C}_{\zeta 2} = 0$	$0 \leqslant a_0 \leqslant 1 \cdot 57$
$C_{\zeta 2} = -6 \cdot 28 + 4 \cdot 0 a_0$	$\bar{C}_{\zeta 2} = 0 \cdot 6$	$1 \cdot 57 \leqslant a_0 \leqslant 2 \cdot 00$
Any stratum thickness†		
$C_{\zeta 1} = 5 \cdot 333 - 1 \cdot 0678 a_e^2 + 0 \cdot 5607 a_e^4 - 0 \cdot 2715 a_e^6 + 0 \cdot 0739 a_e^8 - 0 \cdot 0109 a_e^{10}$	Depends on h/r_0	$\begin{cases} a_0 > r_0/h \\ a_e \leqslant 2 \cdot 0 \end{cases}$
$C_{\zeta 2} = 0 \cdot 755 a_e^3 - 0 \cdot 4228 a_e^5 + 0 \cdot 1631 a_e^7 - 0 \cdot 0302 a_e^9 + 0 \cdot 00204 a_e^{11}$		

* Derived from Bycroft.[6]
† Derived from Awojobi.[1]

APPENDIX II

Notation

A = constant

A_n = function

$A_x = u_0 m/(m_e e_m)$ = dimensionless amplitude of horizontal component of coupled vibration

$A_\psi = \psi_0 I_\psi/(m_e e_m z_e)$ = dimensionless amplitude of rocking component of coupled motion

$A_\zeta = \zeta_0 I_\zeta/K$ = dimensionless amplitude of torsional vibration with quadratic excitation

$A_\zeta = \zeta_0 G r_0^3/M_{\zeta 0}$ = dimensionless amplitude of torsional vibration with constant force amplitude

$A_{\zeta m} = \zeta_m I_\zeta/K$ = dimensionless resonant (maximum) amplitude of torsional vibration

$a_e = [a_0^2 - (r_0/h)^2]^{\frac{1}{2}}$ = equivalent dimensionless frequency for strata

a_m = dimensionless resonant frequency (at maximum amplitude)

$a_0 = \omega r_0 \sqrt{(\rho/G)}$ (or $= \omega r_0 \sqrt{(\rho_s/G_s)}$) = dimensionless frequency of excitation

B_m = function

$B_\zeta = I_\zeta/(\rho r_0^5)$ = inertia ratio for torsional vibration

C = integration constant

C_{u1}, C_{u2} = elastic half space stiffness and damping parameters for horizontal translation

$C_{\psi 1}, C_{\psi 2}$ = elastic half space stiffness and damping parameters for rocking

$C_{\zeta 1}, C_{\zeta 2}$ = elastic half space (stratum) stiffness and damping parameters for torsional vibration

$\bar{C}_{\zeta 1}, \bar{C}_{\zeta 2}$ = frequency independent elastic half space (stratum) stiffness and damping parameters for torsional vibration

c_{ij} = constant of damping force acting in direction i due to displacement (velocity) in direction j

c_ζ = equivalent damping constant for torsional vibration

D = integration constant

D_ζ = damping ratio for torsional vibration

e = eccentricity of centre of gravity

e_m = eccentricity of rotating mass

$f_{\zeta1}, f_{\zeta2}$ = components of Reissner's displacement function for torsional vibration

G = shear modulus of elastic medium (soil beneath footing)

G_s = shear modulus of side layer (backfill)

H = height of footing

$H_1^{(1)}, H_1^{(2)}$ = Hankel functions of the first order, first or second kind respectively

h = thickness of elastic stratum

I_ψ = mass moment of inertia of footing about horizontal axis

I_ζ = mass moment of inertia of footing about vertical axis

$i = \sqrt{(-1)}$

J_0, J_1 = Bessel functions of first kind of order 0 and 1 respectively

$K = m_e e_m r_e$ = excitation factor with quadratic excitation

k = constant

k_{ij} = constant of elastic restoring force acting in direction i due to displacement in direction j

k_ζ = stiffness (spring) constant for torsional vibration

l = depth of embedment of footing

M_ψ = excitation moment about horizontal axis

M_ζ = excitation moment about vertical axis

$M_{\psi 0}$ = amplitude of moment M_ψ

$M_{\zeta 0}$ = amplitude of moment M_ζ

m = mass of footing

m_e = unbalanced rotating mass

N_ζ = torsional reaction of layer adjacent to footing

Q = horizontal excitation force

Q_0 = amplitude of horizontal excitation force

r = cylindrical co-ordinate

r_e = lever arm of horizontal excitation

r_0 = radius of footing base; equivalent radius of footing base

R = function of r

R_ζ = torsional reaction of soil in footing base

S_{u1}, S_{u2} = side layer stiffness and damping parameters for horizontal translation

$S_{\psi1}, S_{\psi2}$ = side layer stiffness and damping parameters for rocking

$S_{\zeta1}, S_{\zeta2}$ = side layer stiffness and damping parameters for torsion

$\bar{S}_{\zeta1}, \bar{S}_{\zeta2}$ = frequency independent side layer stiffness and damping parameters for torsion

t = time

u = complex horizontal displacement of footing; radial displacement of medium

u_c = complex amplitude of horizontal displacement

u_0 = real amplitude of horizontal displacement

$u_{1,2}$ = real and imaginary parts of u_c

v = horizontal displacement of medium perpendicular to r

X = horizontal axis

$Y_{0,1}$ = Bessel functions of second kind of order 0 and 1 respectively

z = vertical co-ordinate

z_c = height of centre of gravity above footing base

z_e = height of horizontal excitation force above centre of gravity

w = vertical displacement of medium

α = constant

Δ = relative volume change

$\delta = l/r_0$ = embedment ratio

ζ = torsional vibration

$\zeta_c = \zeta_1 + i\zeta_2$ = complex amplitude of torsional component

ζ_0 = real amplitude of torsional vibration

ζ_1, ζ_2 = real and imaginary parts of ζ_c

$\eta = \rho_s/\rho$ = density ratio

θ = cylindrical co-ordinate

λ = Lamé's constant

ρ = mass density of elastic medium; mass density of undisturbed soil

ρ_s = mass density of side layer; mass density of backfill

σ = normal stress

τ = shear stress

Φ = function of θ

ϕ = phase shift

ψ = rocking component of vibration

$\psi_c = \psi_1 + i\psi_2$ = complex amplitude of rocking component of vibration

ψ_0 = real amplitude of rocking component

$\psi_{1,2}$ = real and imaginary parts of complex amplitude of rocking ψ_c

ω = circular excitation frequency

ω_m = circular frequency at maximum amplitude

ω_0 = natural circular frequency

$\omega_{r,\theta,z}$ = components of rotational vector

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Coupled Horizontal and Rocking Vibration of Embedded Footings

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The prediction of the coupled response to horizontal forces is of major importance for the design of footings exposed to dynamic effects. The theory of surface footings greatly overestimates the real response and neglects the fact that the footings are founded beneath the surface of the ground, i.e. partly embedded. This latter factor considerably affects the footing vibrations in that it reduces the resonant amplitudes and increases the resonant frequencies.

An approximate analytical solution was developed in this paper and the theoretical response curves were compared with field experiments. Closed form formulas were obtained that are simple enough to be directly used in design. The approximate analytical solution is better able to predict the coupled response of embedded footings than that of surface footings. The response is usually dominated by the first resonant peak with the second resonant peak entirely suppressed. The derived formulas for equivalent stiffnesses and damping coefficients due to soil can be introduced into the solution of any structure.

La prédiction de la réponse combinée à des forces horizontales est d'importance majeure pour le calcul des empattements soumis à des efforts dynamiques. La théorie des empattements de surface surestime la réponse réelle et néglige le fait que les empattements sont placés sous la surface du sol, i.e. ils sont partiellement enfouis. Ce dernier facteur affecte considérablement les vibrations de l'empattement en réduisant les amplitudes de résonance et en augmentant leurs fréquences.

Une solution analytique approximative est développée dans cet article et les courbes théoriques de réponse sont comparées avec des mesures effectuées sur le terrain. On obtient des formules finies qui sont assez simples pour être utilisées directement dans le calcul des fondations. Cette solution analytique approximative est meilleure pour prédire la reponse combinée des empattements enfouis que celle des empattements de surface. Le comportement est habituellement gouverné par la première pointe de résonance alors que la seconde pointe disparaît complètement. Pour tenir compte du sol autour de la fondation, on propose des formules avec des coefficients de rigidité et d'amortissement équivalents, formules qui peuvent servir à la solution des calculs pour n'importe quelle structure.

Introduction

It has been recognized for many years that the vibrations of footings can be greatly affected by their partial embedment into the soil. Nevertheless, very little quantitative information is available because of the difficulties associated with a rigorous theoretical solution and with the generalization of experimental findings. *Uncoupled vibration* of embedded footings was solved by Lysmer and Kuhlemeyer (1969, 1971), who investigated vertical motion using the finite element method. Novak and Beredugo (1971) have presented approximate formulas for vertical, horizontal, and rocking vibrations. *Coupled horizontal and rocking vibration* of embedded footings, most important in design of machine foundations, nuclear power plants, etc., was

first investigated by Baranov (1967), who formulated an approximate analytical solution; however, he presented numerical results only for pure rocking. Tajimi (1969) analyzed the response of a structure partially embedded in an elastic stratum and attached at its base to a rigid half space underlying the elastic stratum.

In this paper, an approximate theory is used based on the assumption that the dynamic reactions in the footing base are equal to those of an elastic half space and that the reactions acting on the footing sides are equal to those of an overlying independent elastic layer. This approach, employed first by Baranov (1967), is extended here to yield closed form formulas and graphs that can be directly used for design purposes. Also, the

effect of backfill is incorporated. The major advantages of this approach are its relative simplicity and great versatility. The known solutions for coupled motion of surface footings (Richart *et al.* 1970; Ratay 1971) are a special case of the more general approach presented herein.

To provide more experimental data, field tests were conducted with concrete blocks subjected to horizontal excitation. The block bases were either square or rectangular in accordance with shapes most often used in practice. The reported test results are compared with the theoretical solution in order to assess the validity of the theory.

Equations of Motion

With the notation given in Fig. 1, the equations of vibration of a footing in coupled horizontal translation $u(t)$ and rotation $\psi(t)$ about a horizontal axis passing through the center of gravity are:

[1]
$$\begin{cases} m\ddot{u}(t) = Q(t) - R_x(t) - N_x(t) \\ I\ddot{\psi}(t) = M(t) - R_\psi(t) - N_\psi(t) \end{cases}$$

in which m = total mass of footing, I = mass moment of inertia about a horizontal axis passing through the center of gravity, $R_x(t)$ = horizontal reaction at the footing base, $N_x(t)$ = resultant horizontal reaction acting on embedded surfaces (sides) of footing, $R_\psi(t)$ = reactive moment of forces acting at footing base about center of gravity, $N_\psi(t)$ = reactive moment of forces acting on footing sides about center of gravity and t = time. The dots represent differentiation with respect to time. $Q(t)$ = horizontal exciting force acting at height z_e. If $M_e(t)$ = exciting moment, the total moment of excitation is:

[2]
$$M(t) = Q(t)z_e + M_e(t)$$

No rigorous expressions for reactions R and N are known; however, they can be ap-

proximately described if the following assumptions are adopted:

(*i*) the footing is of cylindrical shape,

(*ii*) the reactions in the base are the same as those of a surface footing and thus can be taken from the solutions of the elastic half-space or stratum, and

(*iii*) the side reactions are produced by an independent layer lying above the level of the footing base.

For this layer, various assumptions can be made. If it is assumed that the layer is composed of independent infinitesimally thin layers, then the side reactions derived by Baranov (1967) can be used. Baranov's original solution can be extended to yield formulas, directly applicable in foundation design and analysis, and to involve the effect of soil layering, backfill, and various stress distributions.

In accepting the assumptions (*ii*) and (*iii*), the compatability condition is satisfied only at the footing and far from it. Despite this, the comparison with experiments and with finite element solutions for vertical motion (Novak and Beredugo 1972) and torsion indicates that the approximate theory yields quite reasonable results.
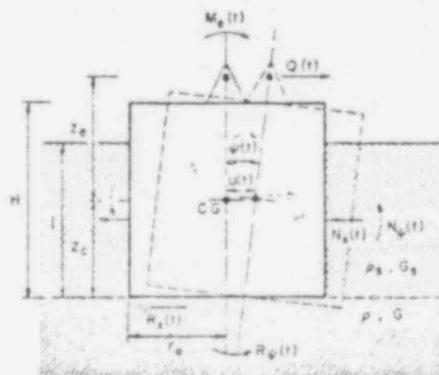
FIG. 1. Mathematical model of embedded footing in coupled motion.

With the above assumptions, the soil reactions can be formulated as follows.

The horizontal and moment reactions in the footing base are:

[3]
$$\begin{cases} R_x(t) = Gr_o(C_{u1} + i\,C_{u2})\,[u(t) - z_c\,\psi(t)] \\ R_\psi(t) = Gr_o^3\,(C_{\psi 1} + iC_{\psi 2})\,\psi(t) - Gr_o(C_{u1} + iC_{u2})\,[u(t)z_c - z_c^2\psi(t)] \end{cases}$$

in which

$$[4] \quad \begin{cases} C_{u1} = \dfrac{-f_{u1}}{f_{u1}^2 + f_{u2}^2} \\[2mm] C_{u2} = \dfrac{f_{u2}}{f_{u1}^2 + f_{u2}^2} \\[2mm] C_{\psi 1} = \dfrac{-f_{\psi 1}}{f_{\psi 1}^2 + f_{\psi 2}^2} \\[2mm] C_{\psi 2} = \dfrac{f_{\psi 2}}{f_{\psi 1}^2 + f_{\psi 2}^2} \end{cases}$$

Here, G = shear modulus, r_o = base radius. $i = \sqrt{-1}$, f_{ij} = displacement functions of dimensionless frequency $a_o = \omega r_o \sqrt{\rho/G}$ and Poisson's ratio v. ω = frequency of excitation and ρ = mass density of the medium below the base. Functions f_{ij} can be found, e.g. in Bycroft (1956) or Luco and Westmann (1971) and are used here with the original signs.

The horizontal and moment side reactions can be written as

$$[5] \quad \begin{cases} N_x(t) = G_s r_o \delta (S_{u1} + iS_{u2}) \left[u(t) + \left(\dfrac{l}{2} - z_c \right) \psi(t) \right] \\[3mm] N_\psi(t) = G_s r_o^3 \delta \left\{ \left[(S_{\psi 1} + iS_{\psi 2}) + \left(\dfrac{\delta^2}{3} - \delta \dfrac{z_c}{r_o} + \dfrac{z_c^2}{r_o^2} \right)(S_{u1} + iS_{u2}) \right] \psi(t) \right. \\[3mm] \left. + \dfrac{1}{r_o} \left(\dfrac{\delta}{2} - \dfrac{z_c}{r_o} \right)(S_{u1} + iS_{u2})\, u(t) \right\} \end{cases}$$

in which G_s = shear modulus of the overlying layer. $\delta = l/r_o$ = relative embedment and l = embedment depth.

Functions S_ψ are independent of v and are

$$[6] \quad \begin{cases} S_{\psi 1} = \pi \left[1 - a_o \dfrac{J_o(a_o)J_1(a_o) + Y_o(a_o)Y_1(a_o)}{J_1^2(a_o) + Y_1^2(a_o)} \right] \\[3mm] S_{\psi 2} = \dfrac{2}{J_1^2(a_o) + Y_1^2(a_o)} \end{cases}$$

In Eq. [6] $J_n(a_o)$ and $Y_n(a_o)$ are Bessel functions of the first and second kinds. respectively. of order n.

Functions S_{u1} and S_{u2} depend on Poisson's ratio v and are the real and imaginary parts of the complex function:

$$[7] \quad S_u(a_o, v) = G_s [S_{u1}(a_o, v) + iS_{u2}(a_o, v)]$$

$$= 2\pi G_s a_o \frac{\dfrac{1}{\sqrt{q}} H_2^{(2)}(a_o) H_1^{(2)}(x_o) + H_1^{(2)}(x_o) H_1^{(2)}(a_o)}{H_0^{(2)}(a_o) H_1^{(2)}(x_o) + H_0^{(2)}(x_o) H_2^{(2)}(a_o)}$$

Here. $q = (1 - 2v)/2(1 - v)$. $x_o = a_o \sqrt{q}$ and $H_n^{(2)}$ = Hankel functions of the second kind of order n. Baranov evaluated functions S_{u1} and S_{u2} only for Poisson's ratio $v = 0.5$ with which Eq. [7] simplifies. For several other values of v these parameters were computed from Eq. [7]

and approximated by polynomials. The results are given in Appendix I together with polynomial descriptions of other parameters C and S to facilitate the computation. The parameters are shown in Fig. 2.

Substitution of Eqs. [3] and [5] into Eq. [1] yields the differential equations of coupled vibration of embedded footings:

$$[8] \begin{cases} m\ddot{u}(t) + r_o\left[G(C_{u1} + iC_{u2}) + G_s\delta(S_{u1} + iS_{u2})\right]u(t) \\ \quad + r_o\left[-G z_c (C_{u1} + iC_{u2}) + G_s\delta\left(\frac{1}{2}l - z_c\right)(S_{u1} + iS_{u2})\right]\psi(t) = Q(t) \\ I\ddot{\psi}(t) + r_o^2\left[G_s\delta\left(\frac{1}{2}\delta - \frac{z_c}{r_o}\right)(S_{u1} + iS_{u2}) - G\frac{z_c}{r_o}(C_{u1} + iC_{u2})\right]u(t) \\ \quad + r_o^3\Bigg\{G(C_{\psi1} + iC_{\psi2}) + G_s\delta\left[(S_{\psi1} + iS_{\psi2}) + \left(\frac{\delta^2}{3} - \delta\frac{z_c}{r_o} + \frac{z_c^2}{r_o^2}\right)\right. \\ \quad \left. (S_{u1} + iS_{u2})\right] + G\frac{z_c^2}{r_o^2}(C_{u1} + iC_{u2})\Bigg\}\psi(t) = M(t) \end{cases}$$

With complex excitation

$$[9] \begin{cases} Q(t) = Q_o\exp(i\omega t) = Q_o(\cos \omega t + i \sin \omega t) \\ M(t) = M_o\exp(i\omega t) = M_o(\cos \omega t + i \sin \omega t) \end{cases}$$

in which Q_o and M_o are real excitation amplitudes, the particular solutions describing the steady state motions are:

$$[10] \begin{cases} u(t) = u_c \exp(i\omega t) \\ \psi(t) = \psi_c \exp(i\omega t) \end{cases}$$

where u_c and ψ_c are complex displacement amplitudes. Substitution of Eqs. [10] into Eqs. [8] provides the following equations for the complex amplitudes of coupled motion:

$$[11] \begin{cases} [(k_{xx} - m\omega^2) + i\omega c_{xx}]u_c + (i\omega c_{x\psi} + k_{x\psi})\psi_c = Q_o \\ [(k_{\psi\psi} - I\omega^2) + i\omega c_{\psi\psi}]\psi_c + (i\omega c_{x\psi} + k_{x\psi})u_c = M_o \end{cases}$$
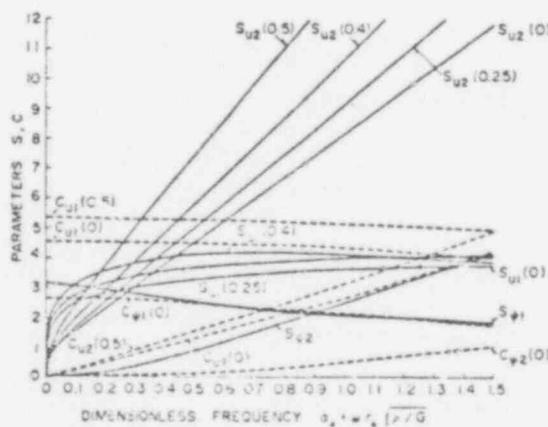
FIG. 2. Half-space stiffness and damping parameters $C_{1,2}$ (dashed lines) and Baranov's layer stiffness and damping parameters $S_{1,2}$ (full lines). (Poisson's ratio shown in parenthesis).

in which frequency dependent stiffness constants

$$[12] \begin{cases} k_{xx} = Gr_o \left(C_{u1} + \dfrac{G_s}{G} \delta S_{u1} \right) \\[2mm] k_{\psi\psi} = Gr_o^3 \left[C_{\psi 1} + \left(\dfrac{z_c}{r_o} \right)^2 C_{u1} + \dfrac{G_s}{G} \delta S_{\psi 1} + \dfrac{G_s}{G} \delta \left(\dfrac{\delta^2}{3} + \dfrac{z_c^2}{r_o^2} - \delta \dfrac{z_c}{r_o} \right) S_{u1} \right] \\[2mm] k_{x\psi} = -Gr_o \left[z_c C_{u1} + \dfrac{G_s}{G} \delta \left(z_c - \dfrac{1}{2} l \right) S_{u1} \right] \end{cases}$$

and frequency dependent damping constants

$$[13] \begin{cases} c_{xx} = \dfrac{Gr_o}{\omega} \left(C_{u2} + \dfrac{G_s}{G} \delta S_{u2} \right) \\[2mm] c_{\psi\psi} = \dfrac{Gr_o^3}{\omega} \left[C_{\psi 2} + \left(\dfrac{z_c}{r_o} \right)^2 C_{u2} + \dfrac{G_s}{G} \delta S_{\psi 2} + \dfrac{G_s}{G} \delta \left(\dfrac{\delta^2}{3} + \dfrac{z_c^2}{r_o^2} \right. \right. \\[4mm] \left. \left. \qquad\qquad\qquad\qquad\qquad\qquad\qquad - \delta \dfrac{z_c}{r_o} \right) S_{u2} \right] \\[2mm] c_{x\psi} = -\dfrac{Gr_o}{\omega} \left[z_c C_{u2} + \dfrac{G_s}{G} \delta \left(z_c - \dfrac{1}{2} l \right) S_{u2} \right] \end{cases}$$

Eqs. [11] are formally equal to the equations of motion of a two-degrees-of-freedom system and therefore, k_{xx}, $k_{\psi\psi}$, and $k_{x\psi}$ are the frequency dependent spring constants, and c_{xx}, $c_{\psi\psi}$, and $c_{x\psi}$ are the frequency dependent damping coefficients. Thus, further solution does not represent any difficulties.

Calculation of Vibration Amplitudes

Forced Vibration

Constant force excitation is considered first. Note

$$[14] \begin{cases} \alpha_1 = k_{\psi\psi} - I\omega^2 - \dfrac{M_o}{Q_o} k_{x\psi} \\[2mm] \alpha_2 = \left(c_{\psi\psi} - \dfrac{M_o}{Q_o} c_{x\psi} \right) \omega \\[2mm] \beta_1 = k_{xx} - m\omega^2 - \dfrac{Q_o}{M_o} k_{x\psi} \\[2mm] \beta_2 = \left(c_{xx} - \dfrac{Q_o}{M_o} c_{x\psi} \right) \omega \\[2mm] \varepsilon_1 = mI\omega^4 - (mk_{\psi\psi} + Ik_{xx} + c_{xx}c_{\psi\psi} - c_{x\psi}^2)\omega^2 + (k_{xx}k_{\psi\psi} - k_{x\psi}^2) \\[2mm] \varepsilon_2 = -(mc_{\psi\psi} + Ic_{xx})\omega^3 + (c_{xx}k_{\psi\psi} + c_{\psi\psi}k_{xx} - 2c_{x\psi}k_{x\psi})\omega \end{cases}$$

Then, the complex vibration amplitudes are from Eqs. [11],

[15]
$$\begin{cases} u_c = Q_o \dfrac{\alpha_1 + i\alpha_2}{\varepsilon_1 + i\varepsilon_2} \\[2mm] \psi_c = M_o \dfrac{\beta_1 + i\beta_2}{\varepsilon_1 + i\varepsilon_2} \end{cases}$$

or

[16]
$$\begin{cases} u_c = u_1 + iu_2 = Q_o \dfrac{\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2} + iQ_o \dfrac{\alpha_2\varepsilon_1 - \alpha_1\varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2} \\[2mm] \psi_c = \psi_1 + i\psi_2 = M_o \dfrac{\beta_1\varepsilon_1 + \beta_2\varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2} + iM_o \dfrac{\beta_2\varepsilon_1 - \beta_1\varepsilon_2}{\varepsilon_1^2 + \varepsilon_2^2} \end{cases}$$

The real vibration amplitudes u_o and ψ_o from Eqs. [16] are

[17]
$$\begin{cases} u_o = \sqrt{u_1^2 + u_2^2} = Q_o \sqrt{\dfrac{\alpha_1^2 + \alpha_2^2}{\varepsilon_1^2 + \varepsilon_2^2}} \\[2mm] \psi_o = \sqrt{\psi_1^2 + \psi_2^2} = M_o \sqrt{\dfrac{\beta_1^2 + \beta_2^2}{\varepsilon_1^2 + \varepsilon_2^2}} \end{cases}$$

When the motion is excited by a moment alone, $Q_o = 0$ and Eqs. [17] simplify to:

[18]
$$\begin{cases} u_o = M_o \sqrt{\dfrac{k_{x\psi}^2 + c_{x\psi}^2\omega^2}{\varepsilon_1^2 + \varepsilon_2^2}} \\[2mm] \psi_o = M_o \sqrt{\dfrac{(k_{xx} - m\omega^2)^2 + c_{xx}^2\omega^2}{\varepsilon_1^2 + \varepsilon_2^2}} \end{cases}$$

The phase angles are:

[19]
$$\begin{cases} \phi_x = \text{Arc tan} \dfrac{u_2}{u_1} = -\text{Arc tan}\left(\dfrac{\alpha_1\varepsilon_2 - \alpha_2\varepsilon_1}{\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2}\right) \\[2mm] \phi_\psi = \text{Arc tan} \dfrac{\psi_2}{\psi_1} = -\text{Arc tan}\left(\dfrac{\beta_1\varepsilon_2 - \beta_2\varepsilon_1}{\beta_1\varepsilon_1 + \beta_2\varepsilon_2}\right) \end{cases}$$

and the real motion of the center of gravity is:

[20]
$$\begin{cases} u(t) = u_o \cos(\omega t + \phi_x) \\[2mm] \psi(t) = \psi_o \cos(\omega t + \phi_\psi) \end{cases}$$

As in the case of uncoupled modes, dimensionless amplitudes $A_c = u_o Gr_o/Q_o$ and $A_\psi = \psi_o Gr_o^3/M_o$ may be introduced to facilitate the presentation and analysis of the results.

Frequency variable excitation, often encountered in practical cases, can be easily introduced into the above formulae. Assume a frequency variable horizontal excitation, caused by an unbalanced rotating mass m_e acting at a height z_e above the center of gravity. Then, in Eqs. [14]–[20]:

[21] $\qquad Q_o = m_e\, e\, \omega^2, \qquad M_o = m_e\, e\, \omega^2\, z_e \quad$ and $\quad \dfrac{M_o}{Q_o} = z_e$

in which e = rotating mass eccentricity.

The dimensionless vibration amplitudes are, in the case of frequency variable excitation $A_u = u_o m/m_e e$ and $A_\psi = \psi_o I/m_e e z_e$.

From Eqs. [17] and [18], the complete response of embedded footings in coupled motion can be computed directly and relatively easily. With the use of approximate expressions for the side reactions S given in Appendix I, the computation becomes even simpler.

The uncoupled modes of vibration (Novak and Beredugo 1971) and the known solutions of surface footings (Richart *et al.* 1970; Ratay 1971) are special cases of the solution described.

It may be noted that an alternative direct calculation may be used in which the complex amplitudes u_c and ψ_c are separated into their real and imaginary parts beforehand. This approach leads to four simultaneous equations with real coefficients; however, the computing requires more time.

From the motion of the center of gravity, the horizontal and vertical components of the motion experienced by the surface (edges) of the footing can be computed. The upper edge of the footing experiences vertical amplitude w_e and horizontal amplitude u_e that are:

[22]
$$\begin{cases} w_e = r_o \psi_o \\ u_e = u_o + (H - z_c)\psi_o \end{cases}$$

(In the last formula, the phase difference is neglected between u and ψ.)

Natural Frequencies and Modes

In addition to the computation of the complete response, the natural undamped frequencies and modes of free vibrations can be of interest and are useful in the direct resonant amplitude calculation described later herein.

The equations for the natural frequencies and modes follow from Eqs. [11] by putting the damping coefficients c_{xx}, $c_{\psi\psi}$, and $c_{x\psi}$ as well as Q_o and M_o equal to zero, which yields, in terms of real amplitudes.

[23]
$$\begin{bmatrix} k_{xx} - m\omega^2 & k_{x\psi} \\ k_{x\psi} & k_{\psi\psi} - I\omega^2 \end{bmatrix} \begin{Bmatrix} u_o \\ \psi_o \end{Bmatrix} = 0$$

The two natural undamped frequencies ω_1 and ω_2 are found from the condition that the determinant of the coefficients must be equal to zero, which yields:

[24]
$$\omega^2_{1,2} = \frac{1}{2}\left(\frac{k_{xx}}{m} + \frac{k_{\psi\psi}}{I} \right) \mp \sqrt{ \frac{1}{4}\left(\frac{k_{xx}}{m} - \frac{k_{\psi\psi}}{I} \right)^2 + \frac{k^2_{x\psi}}{mI} }$$

From this equation, two natural undamped frequencies can be found by a trial and error procedure because coefficients k are frequency dependent according to Eqs. [12]. With these two natural frequencies ω_j $(j = 1,2)$ the two vibration modes (eigenvectors) are, from Eqs. [23]:

[25]
$$\left(\frac{u_o}{\psi_o} \right)_j = \frac{-k_{x\psi}(\omega_j)}{k_{xx}(\omega_j) - m\omega_j^2} = \frac{k_{\psi\psi}(\omega_j) - I\omega_j^2}{-k_{x\psi}(\omega_j)}$$

with $j = 1$ or 2. (These equations provide a quick check of ω_j. With correct values of $\omega_{1,2}$ both equations give the same results).

The first mode represents rotation about a center lying under the footing base. In the second mode the rotation takes place about a point lying above the center of gravity (Fig. 3). The modes actually have a meaning of radii of rotation.

Examples of Theoretical Response Curves

Several examples of theoretical response curves computed from Eqs. [17] with quadratic excitation according to Eqs. [21] are plotted in Figs. 4–6.
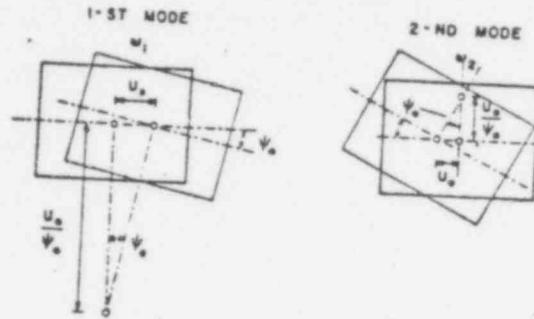
FIG. 3. Modes of free vibrations.
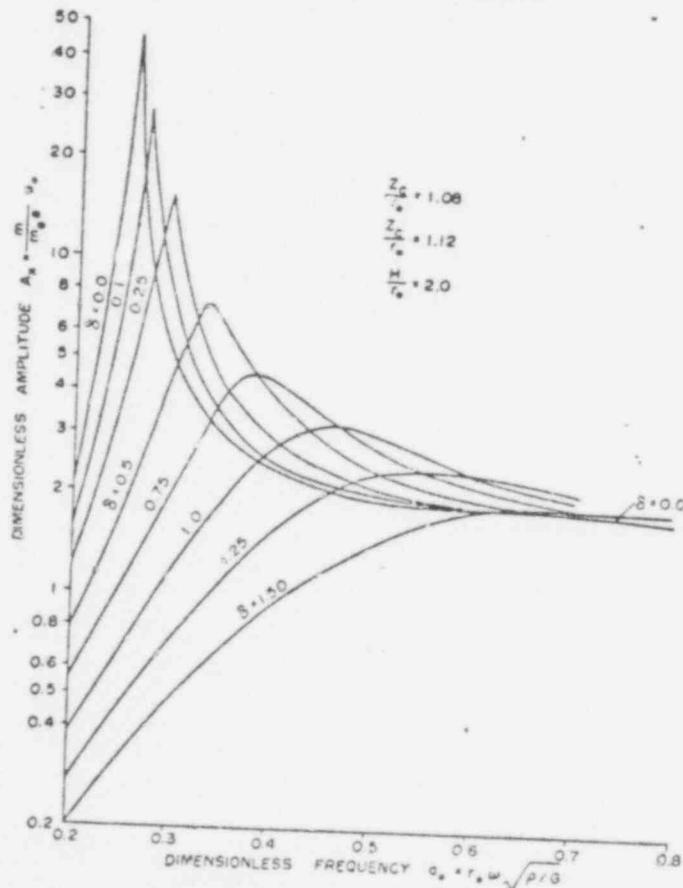
FIG. 4. Theoretical response curves for horizontal translation of footing in coupled motion ($r = 1.0$, $r = 0$, $B_x = 4.0$ $B_\psi = 4.35$, various embedments).
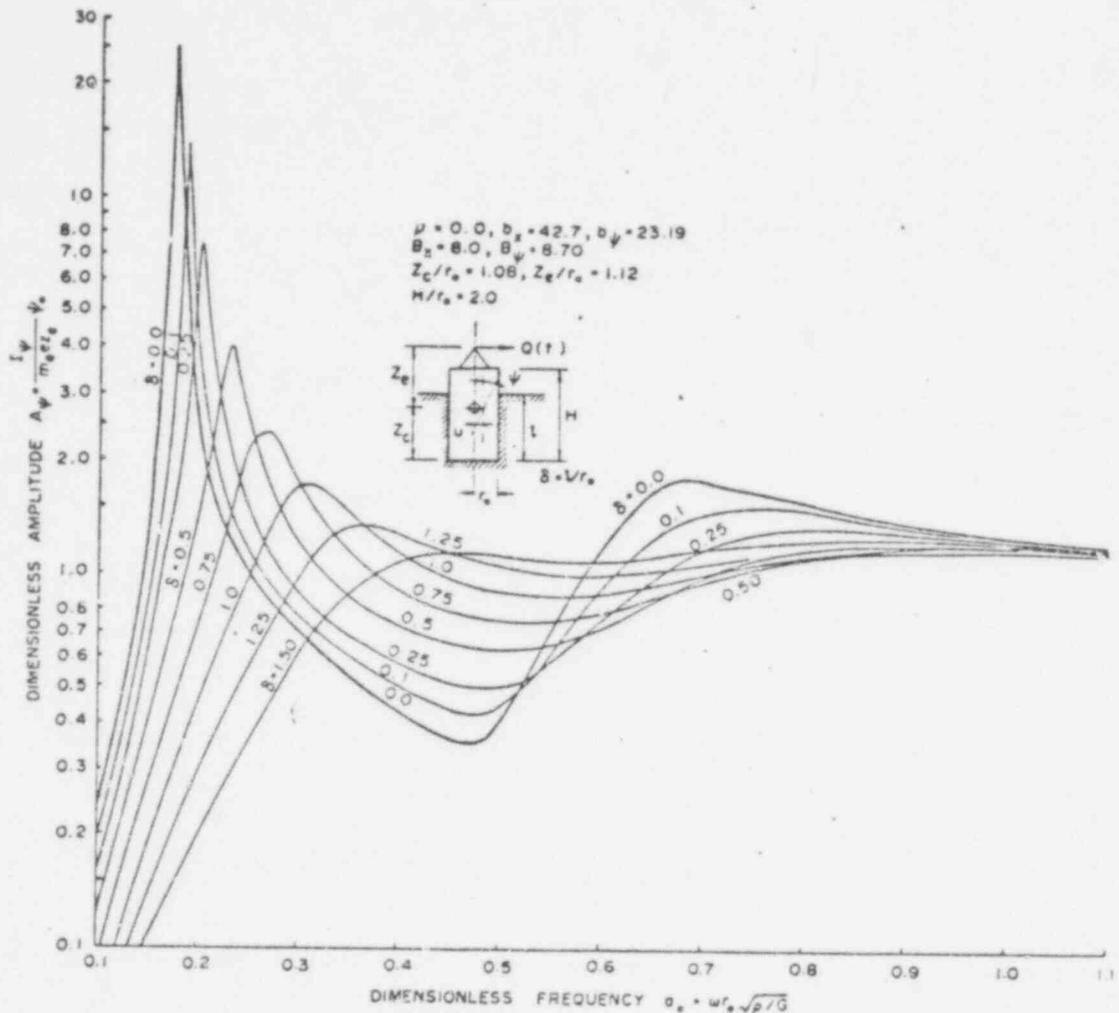
FIG. 5. Theoretical response curves for rocking component of coupled motion ($\eta = 1.0$, $B_x = 8.0$, $\_\_ = 8.70$, various embedments).

In these figures the horizontal translation and rocking components are plotted for various relative embedments $\delta = l/r_o$. The mass parameters used are shown in both the standard $b_{,\varphi}$ and modified $B_{,\varphi}$ shapes (Richart *et al.* 1970). The modified mass parameters are shown here to reduce the dependence of the results on Poisson's ratio. Two sets of mass parameters were used to indicate their effects on the character of the response. Figures 4 and 5 illustrate the steady response of footings embedded in undisturbed soil. Figure 6 illustrates the effects of backfill. The properties of backfill G_s, ρ_s were introduced using an approximate expression (Beredugo 1971) $G_s/G \simeq (\rho_s/\rho)^1$. Thus, ratio $\eta = \rho_s/\rho = 1$ denotes embedment in undisturbed soil.

Several conclusions can be drawn from Figs. 4–6[1]. The response of embedded footings is dominated by the first resonance peak, which is thus of major importance. (For surface footings this was also observed by Ratay (1971). The response in the region of the second resonance is, in general, much less pronounced and does not vary too much with embedment.

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[1]See also Fig. 9.

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In most practical cases, the second resonant peak is entirely suppressed: it can only be recognized in the rocking components with very high mass ratios (Fig. 5).

The increase in the first resonant frequency and the decrease in the corresponding resonant amplitude due to embedment are quite drastic. Both of these effects are smaller in the case of backfill.

The variations of the resonant amplitudes and resonant frequencies with relative embedment and mass ratios are further illustrated in Figs. 7 and 8. Resonant amplitude ratio R_a and resonant frequency ratio R_f represent the relative variations of resonant amplitudes and frequencies due to embedment related to amplitudes and frequencies of a surface footing. These figures apply exactly just for the parameters used in the computing: however, they indicate the trends to be expected in any particular case. Further parameter studies revealed that the resonant amplitude ratio is practically independent of the relative height of horizontal excitation $z_e H$ and only weakly dependent on the modified mass ratio with any particular embedment. Comparison with pure rocking and horizontal translation (uncoupled motions) indicates that coupling increases the resonant horizontal amplitudes and decreases the rocking amplitudes and resonant frequencies. (Similar observations were made with surface footings by Ratay 1971).

Simplified Design Analysis

The calculation of natural frequencies, vibration modes and amplitudes of forced oscillations from the above formulas can be considerably simplified if the stiffness parameters C_1 and S_1 are assumed to be constant and if the damping parameters C_2, S_2 are assumed to be proportional
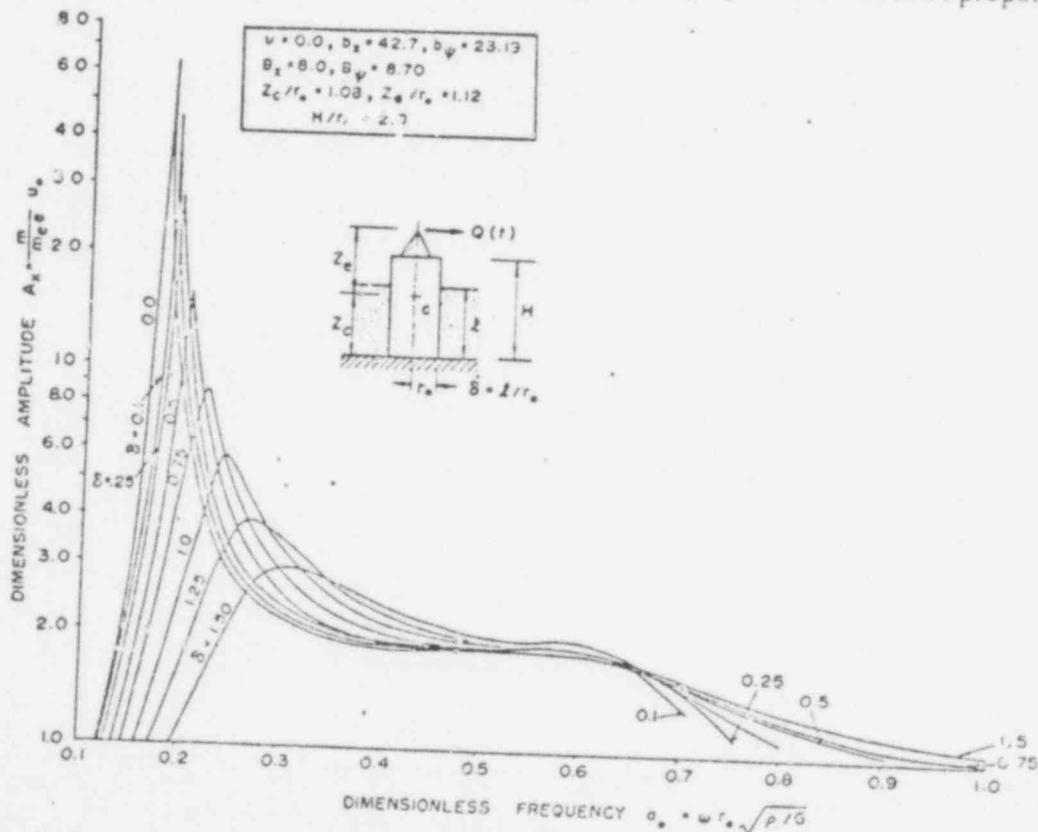
FIG. 6. Theoretical response curves for horizontal translation in coupled motion of footing (backfill with η 0.75, $B_x = 8.0$, $B_\psi = 8.70$).
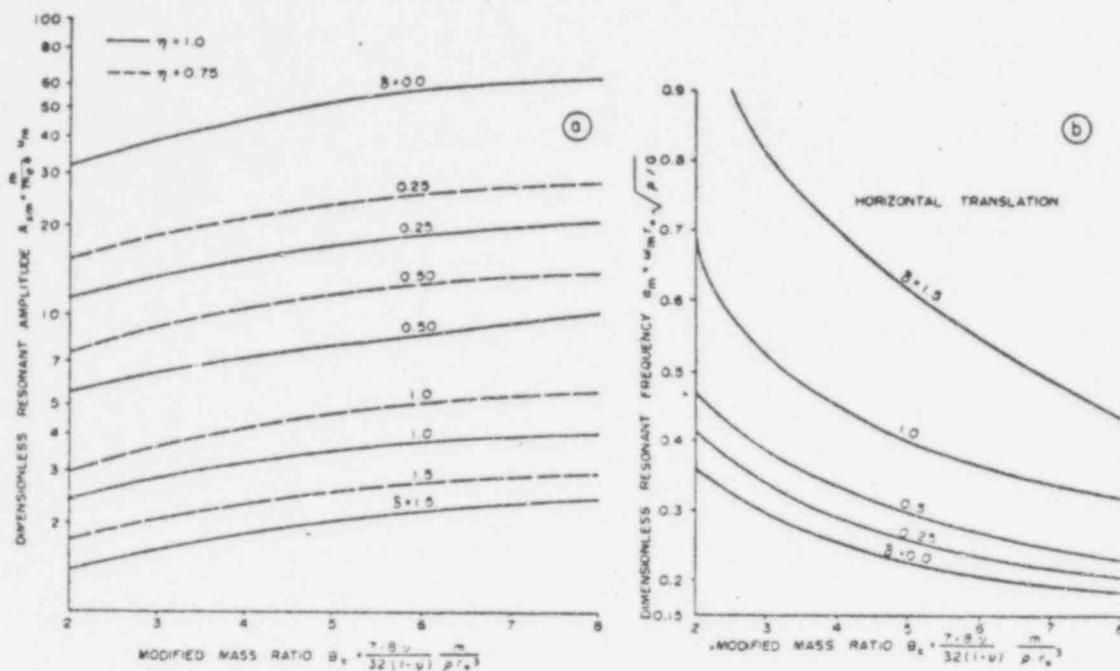
FIG. 7a. Effect of mass ratio on theoretical resonant amplitude for horizontal translation in coupled motion.
FIG. 7b. Effect of mass ratio on theoretical resonant frequency of footing in coupled motion ($\eta = 1.0$).
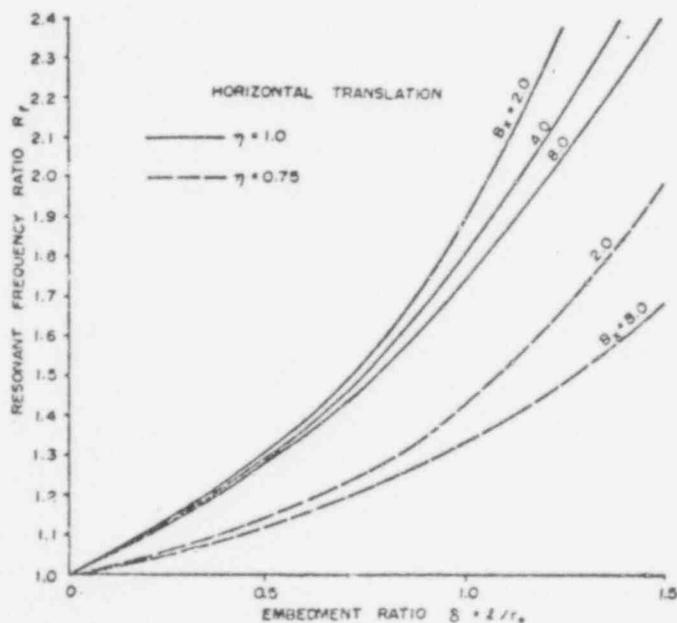
FIG. 8. Theoretical resonant frequency ratio vs. embedment for footing in coupled motion.

to dimensionless frequency a_o. It can be seen from Fig. 2 that these assumptions are quite accept-able for practical applications except for some side reactions at very low frequencies. Thus, stiffness parameters can be considered approximately constant:

[26] $C_{u1} = \overline{C}_{u1}, C_{\psi1} = \overline{C}_{\psi1}, S_{u1} = \overline{S}_{u1}, S_{\psi1} = \overline{S}_{\psi1}$

and with these values the stiffness constants, given by Eqs. [12], become frequency independent. The damping parameters in Eqs. [13] can be approximately taken as:

[27] $C_{u2} = \overline{C}_{u2}a_o, C_{\psi2} = \overline{C}_{\psi2}a_o, S_{u2} = \overline{S}_{u2}a_o, S_{\psi2} = \overline{S}_{\psi2}a_o$

The parameters denoted by bars are constants. Their suitable values are given in Tables 1 and 2 in Appendix I.

Substitution of Eqs. [27] into Eqs. [13] yields the frequency independent damping constants for embedded footings:

$$[28] \begin{cases} c_{xx} = \sqrt{\rho G}\, r_o^2 \left(\overline{C}_{u2} + \delta\sqrt{\frac{\rho_s}{\rho}\frac{G_s}{G}}\; \overline{S}_{u2} \right) \\[2mm] c_{\psi\psi} = \sqrt{\rho G}\, r_o^4 \left\{ \overline{C}_{\psi2} + \left(\frac{z_c}{r_o}\right)^2 \overline{C}_{u2} + \delta\sqrt{\frac{\rho_s}{\rho}\frac{G_s}{G}} \left[\overline{S}_{\psi2} + \right. \right. \\[2mm] \qquad\qquad\qquad \left. \left. \left(\frac{\delta^2}{3} + \frac{z_c^2}{r_o^2} - \delta\frac{z_c}{r_o}\right) \overline{S}_{u2} \right] \right\} \\[2mm] c_{x\psi} = -\sqrt{\rho G}\, r_o^3 \left[z_c\overline{C}_{u2} + \delta\sqrt{\frac{\rho_s}{\rho}\frac{G_s}{G}} \left(z_c - \frac{1}{2}l\right) \overline{S}_{u2} \right] \end{cases}$$

With frequency independent stiffness and damping parameters, the calculation of vibration amplitudes is as follows: (i) with values of $\overline{C}_1$ and $\overline{S}_1$ taken from Tables 1 and 2 (or read from Fig. 2) stiffness constants are obtained from Eqs. [12] and the two natural frequencies from Eq. [24]; (ii) damping coefficients are computed from Eqs. [28] with $\overline{C}_2$ and $\overline{S}_2$ taken from Tables 1 and 2, and α, β, and c obtained from Eqs. [14] for any excitation frequency of interest. Substitution of these values into Eqs. [17] yields the amplitudes of horizontal translation and rotation at the center of gravity. Eqs. [22] give the motions of footing edges.

Very often only the amplitudes at the first resonance with ω_1 need to be found because these resonant vibrations cause frequent difficulties. (The amplitudes at ω_1 are slightly smaller than the maximum).

Examples of response curves computed with constant parameters are shown in Fig. 9. The agreement with the results obtained with variable parameters is satisfactory for em-

bedded footings. For surface footings, the constant parameters given in Tables 1 and 2 yield smaller resonant amplitudes. This is desirable for reasons discussed in the next section. (If desired, a perfect fit can also be obtained for surface footings by the proper choice of constant parameters).

The effect of embedment may be reduced by an imperfect bond between the footing and the soil and by backfill. These effects can be accounted for by considering $G_s < G$ and $\rho_s < \rho$ in Eqs. [12] and [28].

The choice of equivalent radius r_o for rectangular footings is rather uncertain. Some indications of the possible differences are mentioned in the next paragraph.

Comparison with Experiments

Relatively few experiments have been conducted with embedded footings subjected to horizontal excitation despite the fact that this is the case of major practical importance. To provide more data, a series of field tests has been carried out at The University of
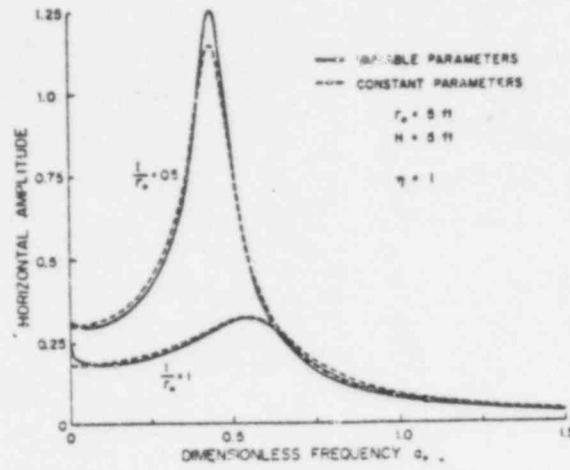
FIG. 9. Comparison of response curves computed with variable and constant parameters ($\nu = 0$, $\eta = 1$).
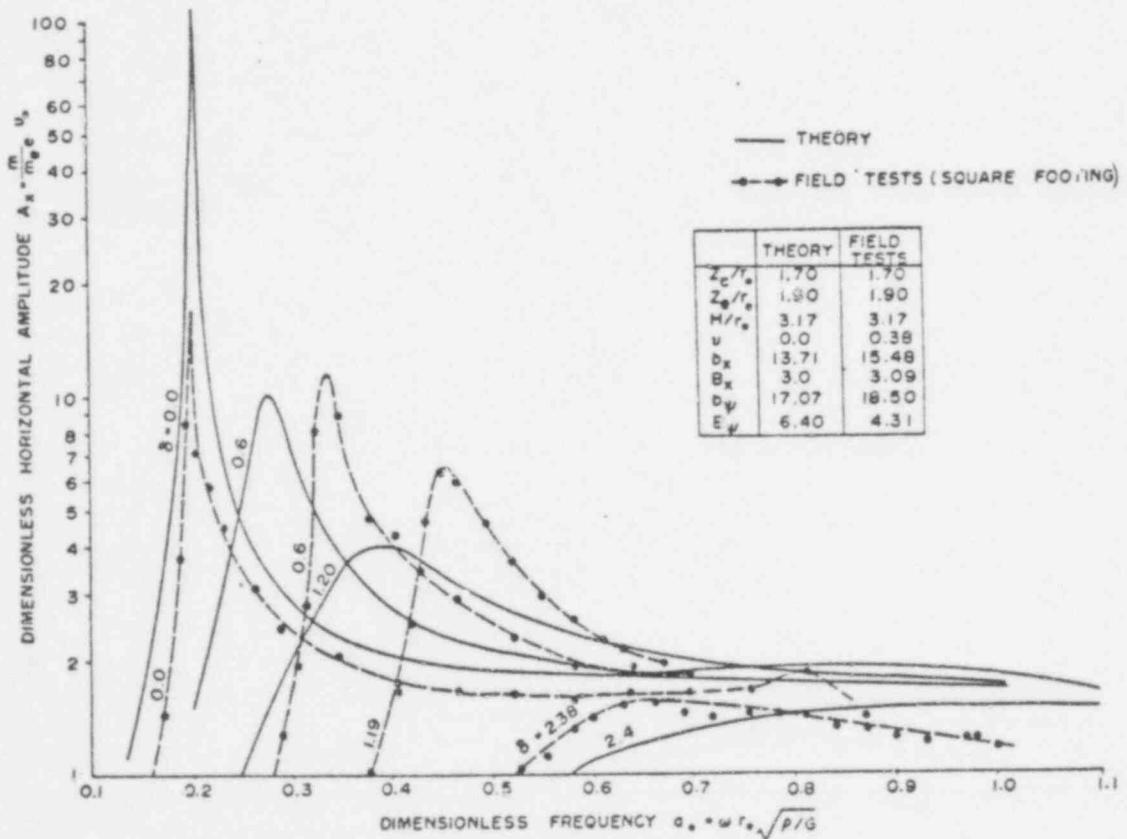
FIG. 10. Comparison of theoretical and measured response curves for horizontal translation in coupled motion of square footing (undisturbed soil, $eg_0 = 1.71$ lb-in.).

Western Ontario with two concrete blocks, one having a square base, the other featuring a rectangular base with a side ratio of 2/1. The base area was 5 ft² (0.465 m²) in both cases. The blocks were cast directly into neatly cut excavations. The embedment depth was changed by removing the soil in several steps. The effect of embedment into backfill was investigated by stepwise backfilling of the soil and by tamping to two different densities. The subsoil was composed of about 5 ft (1.5 m) of brown, silty clay underlain by a glacial till of considerable thickness. Shear modulus of undisturbed soil was found to be 6.6×10^5 lb/ft² (320 kg/cm²) and Poisson's ratio was 0.38. Further details can be found in Beredugo (1971) and Novak and Beredugo (1971).

The comparison of the theory with the experiments is complicated by distinct non-linearities. This effect is accepted as a scatter in resonant frequencies and amplitudes in this paper. (Some other ways of dealing with nonlinearities observed in experiments are described in Novak (1971). Another difficulty is to choose an equivalent radius r_o for rectangular embedded footings vibrating in a coupled mode.

Figures 10–15 indicate the suitability of the theory and the differences in response of the rectangular footings in the two major directions.

Figure 10 shows the comparison of theoretical and measured response curves of a *square footing*. The equivalent radius r_o = 1.26 ft (0.38 m) was derived from the equality of footing bases. In Figs. 11 and 12 the first resonant amplitudes and frequencies are compared. It can be seen that the resonant amplitudes are predicted much better for
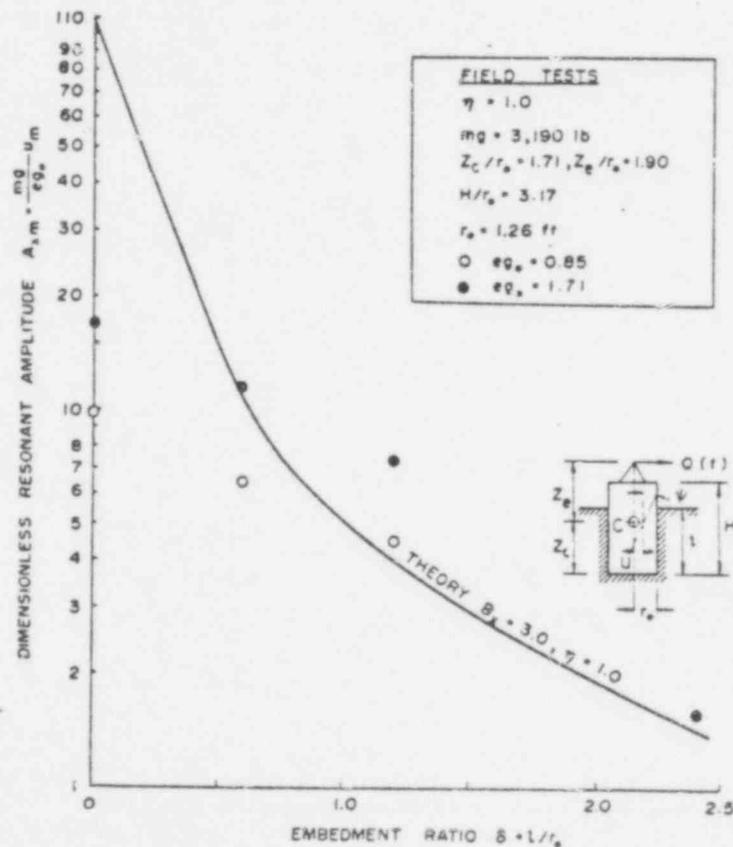
FIG. 11. Comparison of theoretical and measured resonant amplitudes for horizontal translation in coupled motion (undisturbed soil, square footing, ν = 0.38, b_c = 15.48, b_ψ = 18.50).
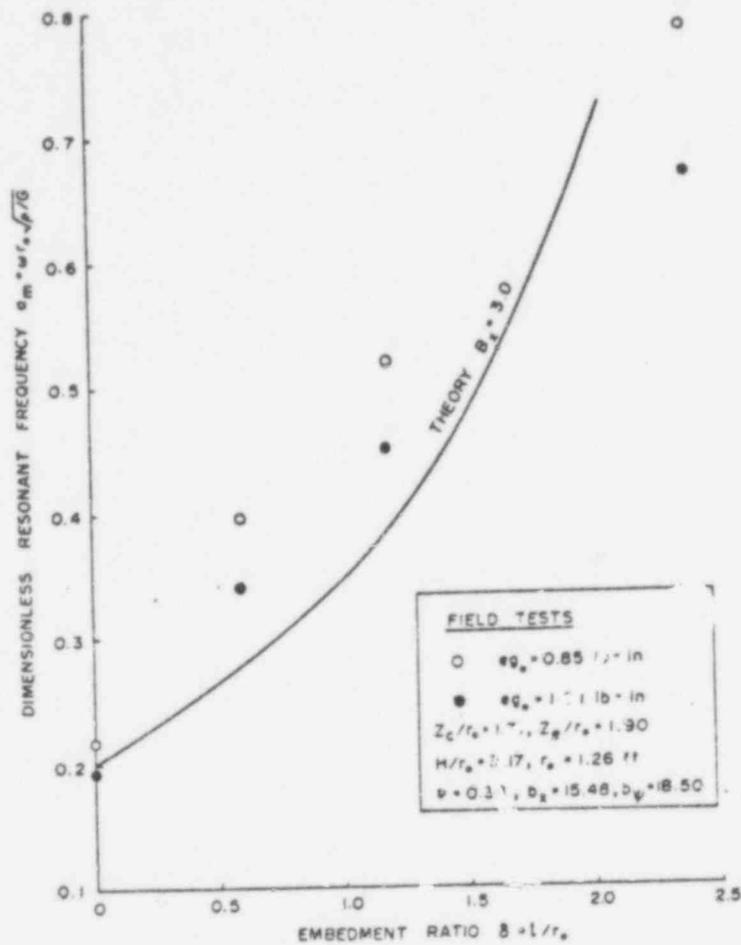
FIG. 12. Comparison of theoretical and measured resonant frequencies in coupled motion ($\eta = 1.0$, square footing, $\nu = 0.38$, $b_\varepsilon = 15.48$, $b_\psi = 18.50$).

the embedded footings than for the surface ones. This important observation probably can be attributed to the overwhelming effect of geometric damping with embedded footings. The first resonant amplitudes of surface footings seem considerably overestimated because the geometric damping is very small and the hysteretic damping is omitted in the theory. The prediction of resonant frequencies appears quite reasonable too. Rectangular footings also show a better agreement in resonant amplitudes in the case of embedment (Fig. 13). The choice of equivalent radii for embedded rectangular footings is, of course, questionable. Figures 14 and 15

give some idea about the differences in response in both major directions. Smaller amplitudes in the direction of the longer axis can be recognized despite the scatter due to nonlinearity (Fig. 14), while a somewhat smaller relative increase in stiffness due to embedment can be seen in the same direction (Fig. 15).

Until better guidelines are found it seems that the equivalent ratios for rectangular footings can be derived from equality of base areas for translation constants k_{xx} and c_{xx} in Eqs. [12] and [13], and from equality of base moments of inertia for the other constants. Useful data on rectangular surface
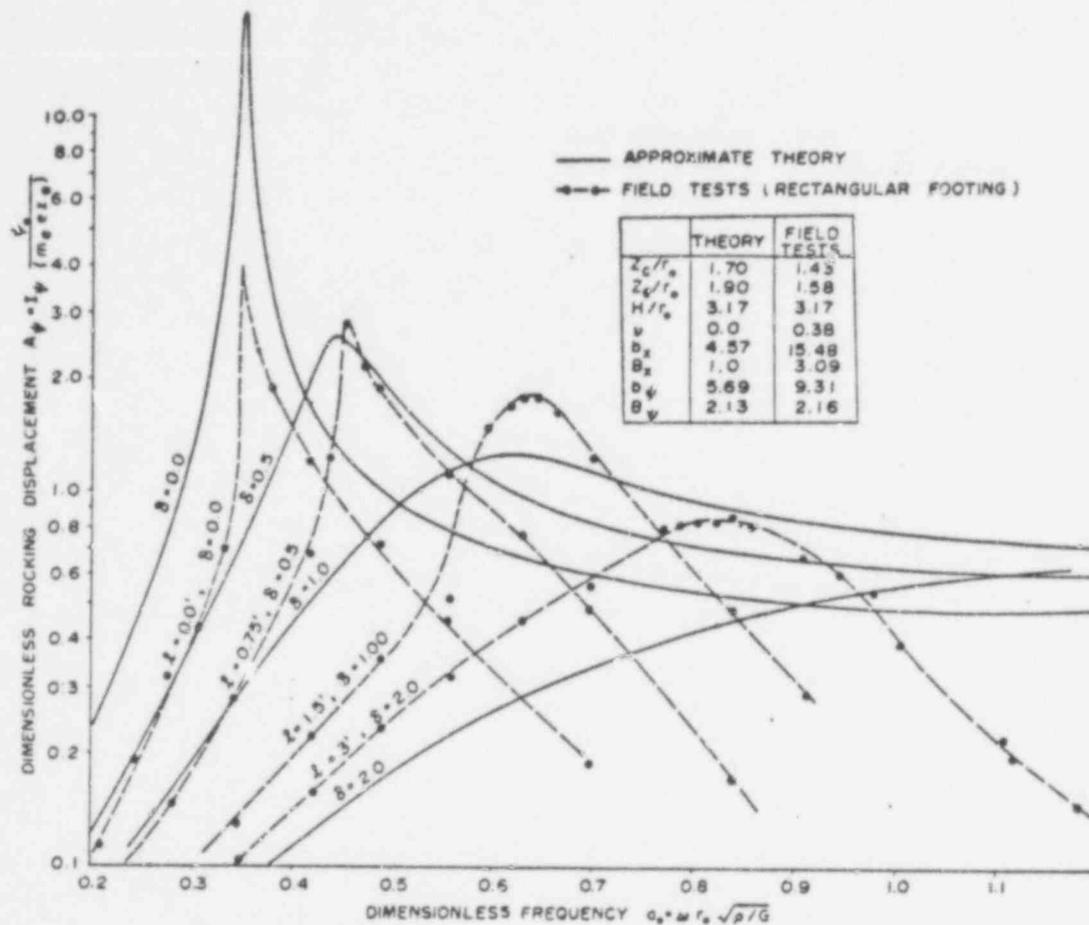
FIG. 13. Comparison of theoretical and measured response curves for rocking in coupled motion of rectangular footing (undisturbed soil, $rg_0 = 1.71$ lb-in.).

footings were derived by Kobori *et al.* (1971) and may be helpful together with the intuition of the designer. Equivalent base radii for surface footings can be found in Richart *et al.* (1970).

Summary and Conclusions

Coupled forced vibration in horizontal translation and rocking of partially embedded rigid footings was investigated both theoretically and experimentally.

An approximate analytical solution was used to derive directly usable formulas and graphs, information about embedment into backfill and relations between uncoupled and coupled motions. The major advantages of

this approach are its simplicity, the ease with which parameters can be changed and the ability to consider layering and to introduce the soil reactions into the solution of any structure.

Field experiments were carried out with concrete embedded footings subjected to horizontal excitation. The theoretical and experimental results were compared.

The major findings can be summarized as follows:

(1) The response is usually dominated by the first resonant peak and the second resonant peak is entirely suppressed. Despite this, the omission of coupling leads to considerable errors in both resonant frequencies and amplitudes.
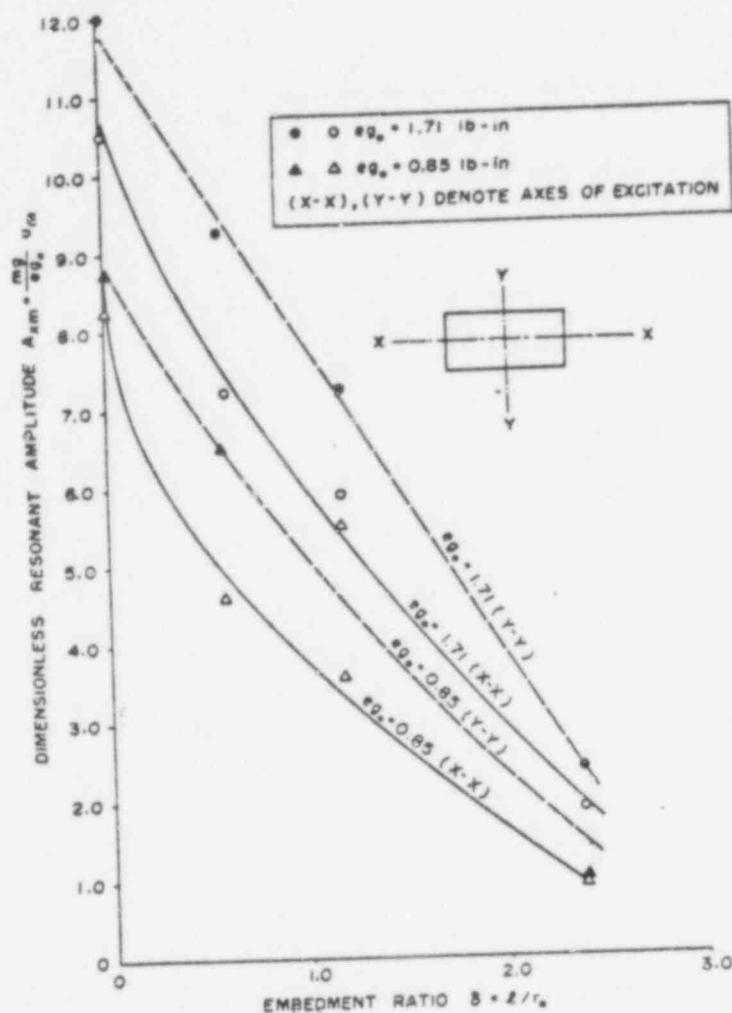
FIG. 14. Variation of measured horizontal resonant amplitude with embedment for rectangular footing in coupled motion (undisturbed soil, eg_o = 0.85 and 1.71 lb-in.)

(2) _Embedment substantially affects the response in that it increases the resonant frequencies and reduces the resonant amplitudes._ This effect is much more pronounced in coupled motion than in vertical translation.

(3) Backfill reduces the effect of embedment. This can be accounted for in the theory.

(4) The approximate theory is better able to predict the coupled response of embedded footings than of surface footings. The theory of surface footings considerably overestimates the resonant amplitudes.

(5) Equivalent frequency independent damping and stiffnesses were derived for design purposes. They facilitate the prediction of resonant frequencies and amplitudes from closed form formulas.

Acknowledgments
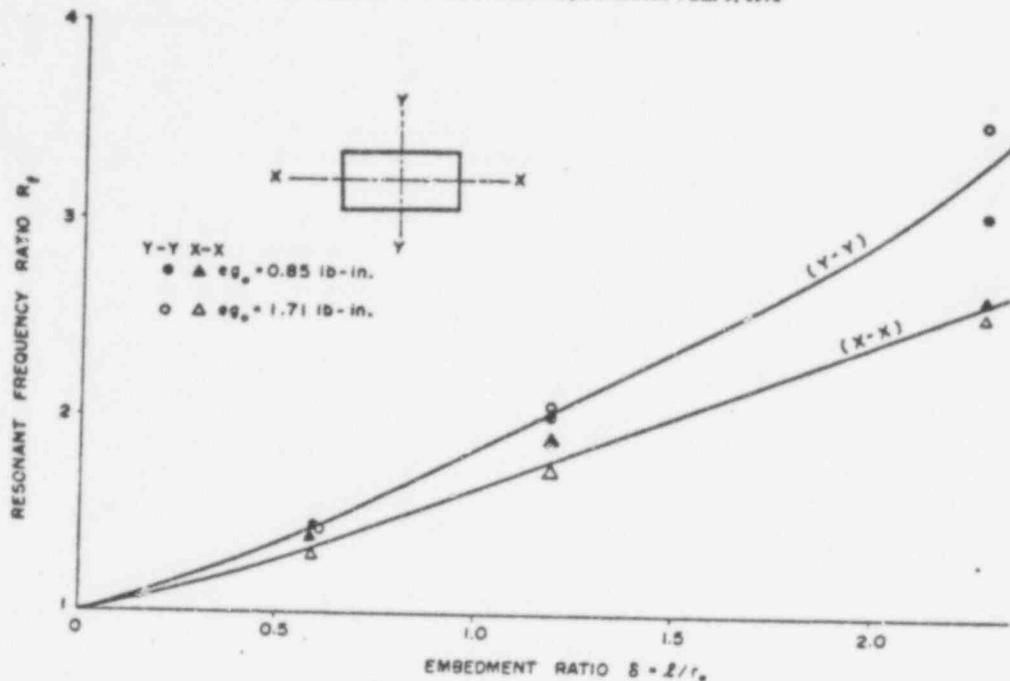
FIG. 15. Variation of measured resonant frequency ratio with embedment for rectangular footing in coupled motion ($\eta = 1.0$).

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Appendix I. – Stiffness and Damping Parameters C and S

To facilitate the computation, the stiffness and damping functions C and S were approximated by the expressions given below. Their accuracy is sufficient for practical applications. The elastic half space formulas for C were computed with Bycroft's (1956) functions $f_{1,2}$. Functions S for the layer were calculated from general equations derived by Baranov (1967).

In all formulas dimensionless frequency $a_o = \omega r_o \sqrt{\rho/G}$. Constant parameters $\overline{C}$ and $\overline{S}$ can be used in frequency range $0 < a_o < 2$. The parameter S_{u1} exhibits pronounced variations with a_o for Poisson's ratio $v \gtrsim 0.43$.

TABLE 1

v	Half space functions	Validity range	Constant parameters
	Horizontal translation		
0.0	$C_{u1} = 4.571 - 4.653a_o + \dfrac{89.09a_o}{a_o + 19.14}$	$0 \leq a_o \leq 2.0$	$\overline{C}_{u1} = 4.30$
	$C_{u2} = 2.536a_o - \dfrac{0.1345a_o}{a_o + 1.923}$		$\overline{C}_{u2} = 2.70$
0.5	$C_{u1} = 5.333 - 1.584a_o + \dfrac{10.39a_o}{a_o + 6.552}$	$0 \leq a_o \leq 2.0$	$\overline{C}_{u1} = 5.10$
	$C_{u2} = 2.923a_o - \dfrac{0.1745a_o}{a_o + 1.927}$		$\overline{C}_{u2} = 3.15$
	Rotation about horizontal axis (rocking)		$\overline{C}_{\psi 1} = 2.50$
0.0	$C_{\psi 1} = 2.654 + 0.1962a_o - 1.729a_o^2 - 1.485a_o^3 - 0.4881a_o^4 - 0.03498a_o^5$	$0 \leq a_o \leq 1.5$	
	$C_{\psi 2} = 0.008025a_o - 0.01583a_o^2 - 0.2035a_o^3 - 1.202a_o^4 - 1.448a_o^5 + 0.4491a_o^6$		$\overline{C}_{\psi 2} = 0.43$

TABLE 2

v	Side layer functions	Validity range	Constant parameters
	Horizontal translation		
	$S_{u1} = 0.2328a_o - \dfrac{3.609a_o}{a_o + 0.06159}$	$0.2 \leq a_o \leq 1.5$	
0.0	$S_{u1} = 150.3a_o - 3630a_o^2 - 3948a_o^3 - 1934a_o^4 - 3488a_o^5$	$0 \leq a_o \leq 0.2$	$\overline{S}_{u1} = 3.60$
	$S_{u2} = 7.334a_o - \dfrac{0.8652a_o}{a_o + 0.00874}$	$0 \leq a_o \leq 1.5$	$\overline{S}_{u2} = 8.20$
	$S_{u1} = 2.474 - 4.119a_o - 4.320a_o^2 - 2.057a_o^3 - 0.362a_o^4$	$0.2 \leq a_o \leq 2.0$	$\overline{S}_{u1} = 4.00$
0.25	$S_{u1} = -1.468 \sqrt{a_o} + 5.662 \sqrt[4]{a_o}$	$0 \leq a_o \leq 0.2$	
	$S_{u2} = 0.83a_o + \dfrac{41.59a_o}{3.90 + a_o}$	$0 \leq a_o \leq 1.5$	$\overline{S}_{u2} = 9.10$
	$S_{u1} = 2.824 - 4.776a_o - 5.539a_o^2 - 2.445a_o^3 - 0.394a_o^4$	$0.2 \leq a_o \leq 2.0$	$\overline{S}_{u1} = 4.10$
0.4	$S_{u1} = -1.796 \sqrt{a_o} + 6.539 \sqrt[4]{a_o}$	$0 \leq a_o \leq 0.2$	
	$S_{u2} = 0.96a_o + \dfrac{56.55a_o}{4.68 + a_o}$	$0 \leq a_o \leq 1.5$	$\overline{S}_{u2} = 10.60$
	Rotation about a horizontal axis (rocking)		$S_{\psi 1} = 2.50$
any value	$S_{\psi 1} = 3.142 - 0.4215a_o - 4.209a_o^2 - 7.165a_o^3 - 4.667a_o^4 - 1.093a_o^5$	$0 \leq a_o \leq 1.5$	
	$S_{\psi 2} = 0.0144a_o - 5.263a_o^2 - 4.177a_o^3 - 1.643a_o^4 - 0.2542a_o^5$		$S_{\psi 2} = 1.80$

Appendix II. – Notation

$A_x = \dfrac{m}{m_e e} u_o$ – dimensionless amplitude of horizontal vibration

$A_{xm} = \dfrac{m}{m_e e} u_m$ – dimensionless resonant (maximum) amplitude of horizontal vibration

$A_\psi = \dfrac{I}{m_e e z_e} \psi_o$ – dimensionless angular amplitude of rocking

$A_{\psi m} = \dfrac{I}{m_e e z_e} \psi_m$ – dimensionless resonant (maximum) amplitude of rocking

$a_o = \omega r_o \sqrt{p/G}$ – dimensionless excitation frequency
a_m – dimensionless resonant frequency (at maximum amplitude)
$B_x = b_x(7-8v)/32(1-v)$ – modified mass ratio for horizontal vibration
$B_\psi = 3b_\psi(1-v)/8$ – modified mass ratio for rocking motion
$b_x = m/pr_o^3$ – mass ratio for horizontal vibration
$b_\psi = I/pr_o^5$ – mass ratio for rocking motion
C_{u1}, C_{u2} – elastic half space stiffness and damping parameters for horizontal translation
$\bar{C}_{u1}, \bar{C}_{u2}$ – frequency independent half space stiffness and damping parameters for horizontal translation
$C_{\psi1}, C_{\psi2}$ – elastic half space stiffness and damping parameters for rocking
$\bar{C}_{\psi1}, \bar{C}_{\psi2}$ – frequency independent half space stiffness and damping parameters for rocking
c_{xx} – equivalent damping constant for horizontal component of coupled motion
$c_{x\psi}$ – equivalent "cross" damping constant for coupled motion
$c_{\psi\psi}$ – equivalent damping constant for rocking in coupled motion
e – eccentricity of rotating mass
f_{u1}, f_{u2} – components of Reissner's displacement function for horizontal vibration
$f_{\psi1}, f_{\psi2}$ – components of Reissner's displacement function for rocking
G – shear modulus of elastic half space; shear modulus of undisturbed soil beneath footing
G_s – shear modulus of side layers; shear modulus of backfill
g – acceleration of gravity
$g_o = m_e g$ – weight of rotating mass
H – height of footing
$H_n^{(2)}$ – Hankel function of the second kind of order $n = J_n - iY_n$
h – thickness of elastic stratum
I – mass moment of inertia about horizontal axis passing through center of gravity
$i = \sqrt{-1}$
J_0, J_1, J_2 – Bessel functions of first kind of orders 0, 1, and 2 respectively
k_{xx} – equivalent spring constant for horizontal component of coupled motion
$k_{x\psi}$ – equivalent "cross" spring constants for coupled motion
$k_{\psi\psi}$ – equivalent spring constant for rocking component of coupled motion
L – length of footing
l – depth of embedment of footing
M_o – amplitude of excitation moment
$M(t)$ – excitation moment about horizontal axis
m – mass of footing; mass of footing and oscillator
m_e – unbalanced rotating mass
$N_x(t)$ – horizontal side reaction due to embedment in coupled motion

$N_\psi(t)$ = reactive torque due to embedment in coupled motion

Q_o = amplitude of horizontal excitation force

$Q(t)$ = horizontal excitation force

$q = (1 - 2v)/2(1 - v)$ = function of Poisson's ratio

R_a = resonant amplitude ratio (relative decrease in resonant amplitude)

R_f = resonant frequency ratio (relative increase in resonant frequency)

$R_\lambda(t)$ = horizontal reaction in the footing base

$R_\psi(t)$ = half space reactive torque in coupled motion

r_o = radius of cylindrical footing; equivalent radius of rectangular footing

S_x, S_ψ = Baranov's rigidity parameters

S_{u1}, S_{u2} = side layer stiffness and damping parameters for horizontal translation

S_{n1}, S_{n2} = side layer stiffness and damping parameters for rocking

$S_{\psi1}, S_{\psi2}$ = side layer stiffness and damping parameters for translation

$\bar{S}_{u1}, \bar{S}_{n2}$ = frequency independent side layer stiffness and damping parameters for translation

$\bar{S}_{\psi1}, \bar{S}_{\psi2}$ = frequency independent side layer stiffness and damping parameters for rocking

t = time

$u_c = u_1 + iu_2$ = complex amplitude of horizontal displacement

u_m = resonant (maximum) amplitude of horizontal displacement

u_o = real amplitude of horizontal displacement

u_1, u_2 = real and imaginary parts of complex amplitude u_c

$u(t)$ = horizontal displacement

$x_o = a_o\sqrt{q}$ parameter depending on ω and v

z_b = height of horizontal excitation force above base of footing

z_c = height of center of gravity above footing base

Y_n = Bessel functions of the second kind of order n

z_e = height of horizontal exciting force above center of gravity

$\delta = l/r_o$ = embedment ratio

$\eta = \rho_s/\rho$ = density ratio

v = Poisson's ratio

ω = circular excitation frequency

ω_m = frequency at maximum amplitude

ω_j = jth undamped natural frequency

$\omega_{1,2}$ = first and second undamped natural frequency

ϕ_x, ϕ_ψ = phase angles

$\psi_c = \psi_1 + i\psi_2$ = complex amplitude of angular (rocking) displacement

ψ_m = reasonant amplitude of angular displacement

ψ_0 = real amplitude of angular displacement

ψ_1, ψ_2 = real and imaginary parts of ψ_c

$\psi(t)$ = angular displacement

ρ = mass density of elastic half space; mass density of undisturbed soil

ρ_s = mass density of side layer; mass density of backfill