APPENDIX E

1.1 Nested Component Stiffness and Strength Parameters

The nested component parameters $G_i^{p_{ref}}$ and $\tau_i^{p_{ref}}$ are determined from discrete backbone curve ($F_{bb}^{p_{ref}}$) provided at a given p_{ref} as input with n number of shear stress – shear strain points. The back substitution for reference shear modulus is executed automatically by I-soil using following formulas:

$$G_n^{p_{ref}} = \frac{\left(\tau_n^{F_{bb}^{p_{ref}}} - \tau_{n-1}^{F_{bb}^{p_{ref}}}\right)}{(\gamma_n^{F_{bb}^{p_{ref}}} - \gamma_{n-1}^{F_{bb}^{p_{ref}}})} \tag{E-1}$$

$$G_i^{p_{ref}} = \frac{\left(\tau_i^{F_{bb}^{p_{ref}}} - \tau_{i-1}^{F_{bb}^{p_{ref}}}\right)}{(\gamma_i^{F_{bb}^{p_{ref}}} - \gamma_{i-1}^{F_{bb}^{p_{ref}}})} - \sum_{i^*=i+1}^{n} G_{i^*}\,;\ where\ i = from\ (n-1)\ to\ 2 \tag{E-2}$$

$$G_1^{p_{ref}} = \frac{\left(\tau_1^{F_{bb}^{p_{ref}}}\right)}{(\gamma_1^{F_{bb}^{p_{ref}}})} - \sum_{i^*=2}^{n} G_{i^*} \tag{E-3}$$

where $\tau_i^{F_{bb}^{p_{ref}}}$ and $\gamma_i^{F_{bb}^{p_{ref}}}$ are ith shear stress and shear strain points in a given reference backbone curve and, $G_n^{p_{ref}}$ is the n^{th} nested component of which the reference shear modulus is determined first using equation (E-1). Then, subsequently, equation (E-2) and (E-3) are executed to determine the shear moduli for remaining components. Once the shear modulus for each component is determined, the reference yield strength for each component is calculated via $\tau_i^{p_{ref}} = G_i^{p_{ref}}\gamma_i$.

1.2 Radial Return Algorithm

I-soil model takes advantage of *J2* (second principal invariant of the deviatoric stress tensor) plasticity theory. Even though direct adoption of this approach constraints constitutive models to behave isotopically, it provides decent amount of versatility for modeling the cyclic behavior of soils. I-soil uses one step forward (explicit) Euler radial return algorithm executing closest projection of trial deviatoric stress onto deviatoric yield surface. Stress tensor is generalized and collapsed to scalar forms using two main components:

$$J2 = \left[\frac{1}{2}\boldsymbol{s}\right]^{\mathbf{1/2}} = \left[\frac{1}{2}s_{\boldsymbol{ij}}s_{\boldsymbol{ij}}\right]^{1/2} \tag{E-4}$$

and

$$p = tr\,(\boldsymbol{\sigma}) = \frac{\sigma_{\boldsymbol{ii}}}{3} \tag{E-5}$$

Strain increments are decomposed to their elastic and plastic part as following:

$$\Delta\varepsilon_q = \Delta\varepsilon_q^e + \Delta\varepsilon_q^p \tag{E-6}$$

$$\Delta\varepsilon_v = \Delta\varepsilon_v^e + \Delta\varepsilon_v^p \tag{E-7}$$

where superscripts e and p denote elastic and plastic respectively. Once the elastic predictor stresses of a component leads to $\mathrm{f_c} \geq 0$ (shear failure), radial return algorithm is triggered to bring the stress state to yield surface to satisfy admissible stresses via $\mathrm{f_c} = 0$. The plastic strains are calculated as:

$$\Delta\varepsilon_q^p = \lambda\frac{\partial f_c}{\partial J_2} \tag{E-8}$$

where λ = positive multiplier to e determined and:

$$\Delta\varepsilon_v^p = A_0(\eta_{pt} - \eta)\Delta\varepsilon_q^p \tag{E-9}$$

Once the yielding occurs, admissible stress state should obey $\mathrm{f_c} = 0$. Thus:

$$f_c = [J_2]^{t+\Delta t} - k_c^* = 0 \tag{E-10}$$

where $k_c^* = g\left(k_c^{p_{ref}}, p_{ref}, p^t, a_0, a_1, a_2, p_0\right)$. Plugging in equation (E-10) in (E-8) gives $\Delta\varepsilon_q^p = \lambda$ and $\Delta\varepsilon_v^p = A_0(\eta_{pt} - \eta)\lambda$. Using equation (E-6) and (E-7) as well as $\dot{\boldsymbol{\sigma}} = \boldsymbol{E}\,(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}^p})$ it is deduced that:

$$\Delta J_2 = G\,\Delta\varepsilon_q - G\,\lambda \tag{E-11}$$

and

$$\Delta p = K\,\Delta\varepsilon_v - KA_0\left(\eta_{pt} - \eta\right)\lambda \tag{E-12}$$

Now the condition $f_c = 0$ can be written as:

$$f_c = [J_2]^t + \Delta J_2 - k_c^* = 0 \qquad \text{(E-13)}$$

Plugging equation (E-11) into (E-13) one can obtain the following form:

$$[J_2]^t + G\,\Delta\varepsilon_q - G\,\lambda - k_c^* = 0 \qquad \text{(E-14)}$$

where $[J_2]^t + G\,\Delta\varepsilon_q = [J_2]^{t+\Delta t(trial)}$, thus:

$$[J_2]^{t+\Delta t(trial)} - G\,\lambda - k_c^* = 0 \qquad \text{(E-15)}$$

Combination of first and third term of the left hand-side of (E-15) is equal to $[f_c]^{t+\Delta t(trial)}$. This leads to:

$$\lambda = \frac{[f_c]^{t+\Delta t(trial)}}{G} \qquad \text{(E-16)}$$

Equation (E-16) determines the positive scalar λ. Thus all the scalar unknowns can be found. Now, new stresses in tensorial form can be written as:

$$[\sigma_{ij}^c]^{t+\Delta t} = \left\{[\sigma_{ij}^c]^{t+\Delta t(trial)} - \frac{[\sigma_{kk}^c]^{t+\Delta t(trial)}}{3}\delta_{ij}\right\}\left\{\frac{[J_2]^{t+\Delta t}}{[J_2]^{t+\Delta t(trial)}}\right\} + \frac{[\sigma_{kk}^c]^{t+\Delta t}}{3}\delta_{ij} \qquad \text{(E-17)}$$

Accurate use of stress integration via forward Euler method requires small strain increments. Time step determined for a stable explicit solver generally satisfies the accuracy due to time step and thus strain increment being small for most of the geotechnical earthquake engineering related applications. For implicit solver, most of the time, time step of the ground motion and thus the strain increments are small enough for forward Euler type integration. However, to assure the accuracy of the solution, it is useful to conduct convergence analysis.

1.3 MRDF type Non-Masing Formulation

Masing-type hysteretic behavior is defined by a set of rules commonly stated as:

1. For initial loading, the stress-strain curve follows the backbone curve:

   $$\tau = F_{bb}(\gamma) \qquad \text{(E-18)}$$

   where τ is the shear stress and $F_{bb}(\gamma)$ is the backbone curve function as a function of shear strain, γ.

2. If a stress reversal occurs at a point $(\gamma_{rev}, \tau_{rev})$, the stress-strain curves follows a path defined by:

   $$\frac{\tau - \tau_{rev}}{2} = F_{bb}\left(\frac{\gamma - \gamma_{rev}}{2}\right) \qquad \text{(E-19)}$$

3. If an unloading or reloading curve intersects the backbone curve, it follows the backbone until the next stress reversal.

4. If an unloading or reloading curve crosses an unloading or reloading curve from the previous cycle, it follows the stress-strain curve of that previous cycle.

To better capture the hysteretic behavior at moderate-to-large strain levels, the hysteretic loops obtained from the second Masing rule using F_{bb} should be reduced in size. Phillips and Hashash (2009) derived an unloading-reloading stress-strain path for the MKZ model as:

$$\tau = F(\gamma_m)\left[\frac{2G_0\left(\frac{\gamma-\gamma_{rev}}{2}\right)}{1+\beta\left(\frac{\gamma-\gamma_{rev}}{2\gamma_r}\right)^s} - \frac{G_0(\gamma-\gamma_{rev})}{1+\beta\left(\frac{\gamma_m}{\gamma_r}\right)^s}\right] + \frac{G_0(\gamma-\gamma_{rev})}{1+\beta\left(\frac{\gamma_m}{\gamma_r}\right)^s} + \tau_{rev} \qquad \text{(E-20)}$$

where G_0 is the initial shear modulus, γ_r is the pseudoreference shear strain, β and s are curve fitting parameters, γ_m is the current maximum shear strain, γ_{rev} is the current reversal shear strain, γ is the current shear strain, and τ_{rev} is the current reversal shear stress, and $F(\gamma_m)$ is the reduction factor as defined by Darendeli (2001) in equation (E-21) or Phillips and Hashash (2009) in equation (E-22) as:

$$F(\gamma_m) = p_1\left(\frac{G_{\gamma_m}}{G_0}\right)^{p_2} \qquad \text{(E-21)}$$

$$F(\gamma_m) = p_1 - p_2\left(1-\frac{G_{\gamma_m}}{G_0}\right)^{p_3} \qquad \text{(E-22)}$$

where G_{γ_m} is the secant shear modulus at the maximum strain experienced by the soil, G_0 is the initial shear modulus, and p_1, p_2, and p_3 are non-dimensional curve-fitting parameters. Additional functional forms of the reduction factor may be used interchangeably.

Equation (E-20) can be simplified to:

$$\tau = F(\gamma_m)\left(\frac{2G_0\left(\frac{\gamma-\gamma_{rev}}{2}\right)}{1+\beta\left(\frac{\gamma-\gamma_{rev}}{2\gamma_r}\right)^s}\right) + [1-F(\gamma_m)]\left(\frac{G_0(\gamma-\gamma_{rev})}{1+\beta\left(\frac{\gamma_m}{\gamma_r}\right)^s}\right) + \tau_{rev} \qquad \text{(E-23)}$$

Equation (E-23) defines an unloading/reloading stress path for a hysteresis loop starting at $(\gamma_{rev}, \tau_{rev})$. The first term on the right-hand side of equation (E-23) scales the Masing unloading-reloading stress path vertically while the second term skews the Masing unloading-reloading stress path causing the tangent shear modulus to approach the secant shear modulus corresponding to the maximum shear strain.

The backbone formulation of the MKZ model is given by:

$$F_{bb}(\gamma) = \frac{\gamma G_0}{1 + \beta\left(\frac{\gamma}{\gamma_r}\right)^s} \tag{E-24}$$

Substituting equation (E-24) into equation (E-23), the unload-reload stress-strain path simplifies to:

$$\tau = 2[F(\gamma_m)]\left[F_{bb}\left(\frac{\gamma - \gamma_{rev}}{2}\right)\right] + [1 - F(\gamma_m)]\left[G_{\gamma_m}\right][\gamma - \gamma_{rev}] + \tau_{rev} \tag{E-25}$$

Equation (E-25) presents a generalized unloading-reloading model with MRDF-type non-Masing hysteretic behavior independent of the functional form of F_{bb}. Equation (E-25) can be used to calculate the shear stress during unloading-reloading so long as the reduction factor and the F_{bb} function or discrete points are defined. Equation (E-25) reduces to the second Masing rule applied using F_{bb} when the reduction factor is unity. Moreover, equation (E-25) can be further simplified by undoing the coordinate transformation and second Masing rule on F_{bb}. This operation maps F_{bb} to a separate backbone curve termed the mapped backbone, F_{bb}':

$$F_{bb}'(\gamma) = [F(\gamma_m)][F_{bb}(\gamma)] + [1 - F(\gamma_m)]\left[G_{\gamma_m}\right][\gamma] \tag{E-26}$$

The mapped backbone curve can be used directly with the second Masing rule in place of the backbone curve $F_{bb}(\gamma)$ during unloading-reloading to obtain MRDF-type non-Masing hysteretic behavior. The second Masing rule can be directly applied to F_{bb}' and results in MRDF-type non-Masing hysteretic behavior. Equation (E-26) can be used for both 3D elasto-plastic models and 1D hyperbolic models as long as a backbone and reduction factor are defined. Since I-soil is a backbone driven model, it conveniently uses the backbone mapping function defined in equation (E-26) at strain reversals to achieve MRDF type hysteretic behavior.