Derivation of the Currie equations

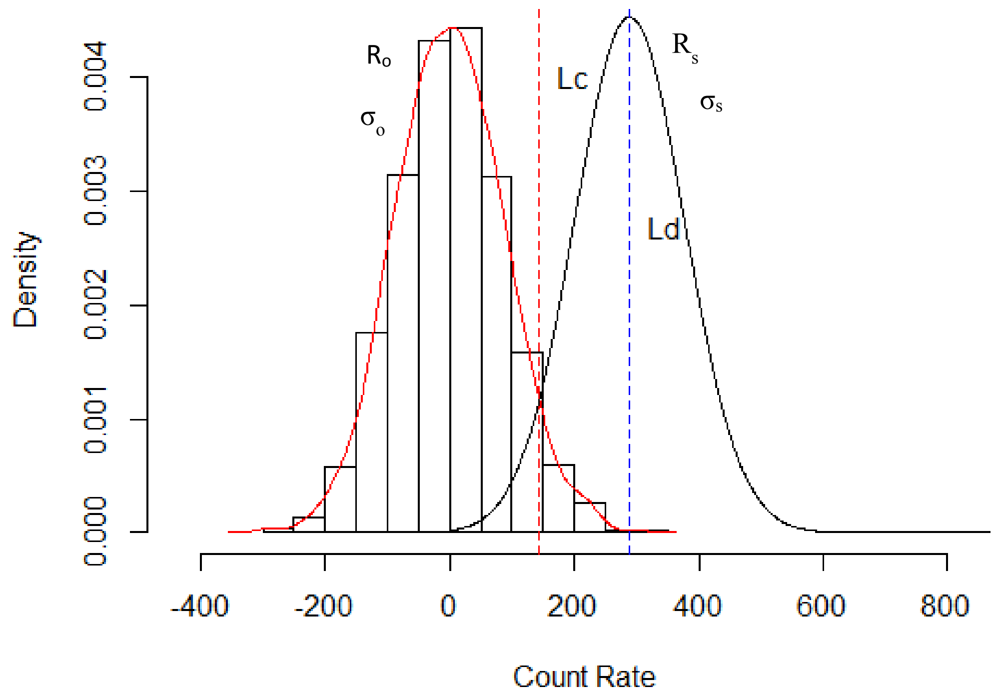

Figure 1: Relationship between Lc and Ld

- R_o = Background count rate

$$R_o = C_o/T_o$$

Co = The number of counts due to background
To = the background count time

- R_s = Source Count Rate

$$R_s = C_s/T_s$$

Cs = The number of counts due to the source
To = the source count time

The net count rate, as measured by the instrument, is given by:

$$N = R_s - R_o$$

The uncertainty of this value is:

$$\delta N^2 = \delta R_s^2 + \delta R_o^2$$

$$\delta N^2 = \left(\frac{\delta C_s}{T_s}\right)^2 + \left(\frac{\delta C_o}{To}\right)^2$$

$$\delta N^2 = \left(\frac{\sqrt{C_s}}{T_s}\right)^2 + \left(\frac{\sqrt{C_o}}{To}\right)^2$$

The variance of the source signal is a function of the total measured counts, from the source plus the background. Therefore, the total number of counts in the source signal is given by the sum of the counts due to the source activity plus the counts from background during period T_s.

$$C_s = C_s' + R_o \cdot T_s$$

where C_s' is the number of counts due to the source activity alone.

$$\delta N^2 = \left(\frac{\sqrt{C_s' + R_o T_s}}{T_s}\right)^2 + \frac{R_o}{T_o}$$

$$\delta N^2 = \frac{C_s'}{T_s^2} + \frac{R_o}{T_s} + \frac{R_o}{T_o}$$

Therefore variation in the net count rate is given by,

$$\sigma_s = \delta N = \sqrt{\frac{C_s'}{T_s^2} + \frac{R_o}{T_s} + \frac{R_o}{T_o}} \qquad \textbf{(1)}$$

Critical Level

The critical level (L_c), is the net count rate which must be exceeded before the sample can be said, with a certain level of confidence, to contain measureable radioactivity above background. It is linked to the Type-I error rate, or the rate of false positives. Stated another way, a net count rate greater than L_c has a probability less than α of being due to background alone. (false positive).

Conclusion

		Source	**No Source**
Reality	**Source**	Correct	Incorrect (False Negative) $P = \beta$
	No Source	Incorrect (false positive) $P = \alpha$	Correct

From Figure 1, the critical level can be defined as:

$$L_c = k_\alpha \sigma_o$$

Where σ_o is the standard deviation of the background counts, and k_α is the one-sided confidence factor for a normal distribution for $P=\alpha$. The standard deviation of the background counts is given by equation (1), and the activity due to the source $C_s' = 0$.

$$L_c = k_\alpha \sigma_o$$

$$L_c = k_\alpha \sqrt{\frac{R_o}{T_s} + \frac{R_o}{T_o}}$$

$$L_c = k_\alpha \sqrt{\frac{R_o}{T_o}\left(1 + \frac{T_o}{T_s}\right)} \qquad \textbf{(2)}$$

Equation (2) is the usual Currie equation for the critical level for the net count signal.

Detection Limit

The detection limit is defined as the smallest quantity of activity present that can be detected with a specified degree of confidence. The detection limit is linked to the Type-II error rate (false negatives), and limits the probability of concluding there is no activity present, when in fact there was, to a probability P=β.

From Figure 1, the detection limit is given by:

$$L_d = L_c + k_\beta \sigma_D$$

where σ_D is the standard deviation of the signal counts when the signal count rate is equal to the detection limit, and k_β is the one-sided confidence factor for a normal distribution for P=β. Again, the variance of the net counts is a function of the total counts. The standard deviation of the signal counts is given by equation (1), this time with the activity due to the source being at the detection limit (i.e. $L_d = C_s'/T_s$) .

Therefore,

$$L_d = L_c + k_\beta \sqrt{\frac{L_d}{T_s} + \frac{R_o}{T_s} + \frac{R_o}{T_o}}$$

$$(L_d - L_c)^2 = k_\beta^2 \left(\frac{L_d}{T_s} + \frac{R_o}{T_s} + \frac{R_o}{T_o}\right)$$

$$(L_d - L_c)^2 = \frac{k_\beta^2}{T_s} L_d + \frac{k_\beta^2}{k_\alpha^2} L_c^2$$

$$L_d^2 - \left(2L_c + \frac{k_\beta^2}{T_s}\right) L_d + \left(1 - \frac{k_\beta^2}{k_\alpha^2}\right) L_c^2 = 0$$

If one chooses $k_\alpha = k_\beta = k$ then this reduces to:

$$L_d = \frac{k^2}{T_s} + 2k \sqrt{\frac{R_o}{T_o}\left(1 + \frac{T_o}{T_s}\right)} \qquad \textbf{(3)}$$

Equation (3) is again the typical detection limit defined by Currie.

It is often advantageous to express the detection limit in the form of a minimum detectable activity. If the count rates and count times in Eq. (3) are expressed in CPM and minutes respectively, then the MDA is:

$$MDA\ (\frac{Bq}{cm^2}) = \frac{\frac{k^2}{T_s} + 2k\sqrt{\frac{R_o}{T_o}\left(1+\frac{T_o}{T_s}\right)}}{60 \cdot \epsilon \cdot A \cdot f}$$

$$MDA\ (\frac{Bq}{cm^2}) = \frac{k^2 + 2k\sqrt{R_o T_s\left(1+\frac{T_s}{T_o}\right)}}{60 \cdot \epsilon \cdot A \cdot T_s \cdot f}$$

Where ε is the efficiency of the detector, A is the area of the detector in cm^2, and f is the appropriate branching ratio for the isotope of interest. If we choose α=β=0.05 then k = 1.645.

$$MDA\ (\frac{Bq}{cm^2}) = \frac{2.71 + 3.29\sqrt{R_o T_s\left(1+\frac{T_s}{T_o}\right)}}{60 \cdot \epsilon \cdot A \cdot T_s \cdot f} \quad \textbf{(4)}$$

Maximum Count Rates

Often, there is an upper specification limit on the maximum allowable activity, or activity concentration present in a sample. For example, the free release limit for alpha emitting radionuclides is 0.4 Bq/cm^2.

We can now ask the question: what is the highest signal count rate that would ensure the measured result is lower than the upper specification limit (USL) with a specified level of confidence? In essence, we are limiting the probability that a measured result exceeds the USL to δ.

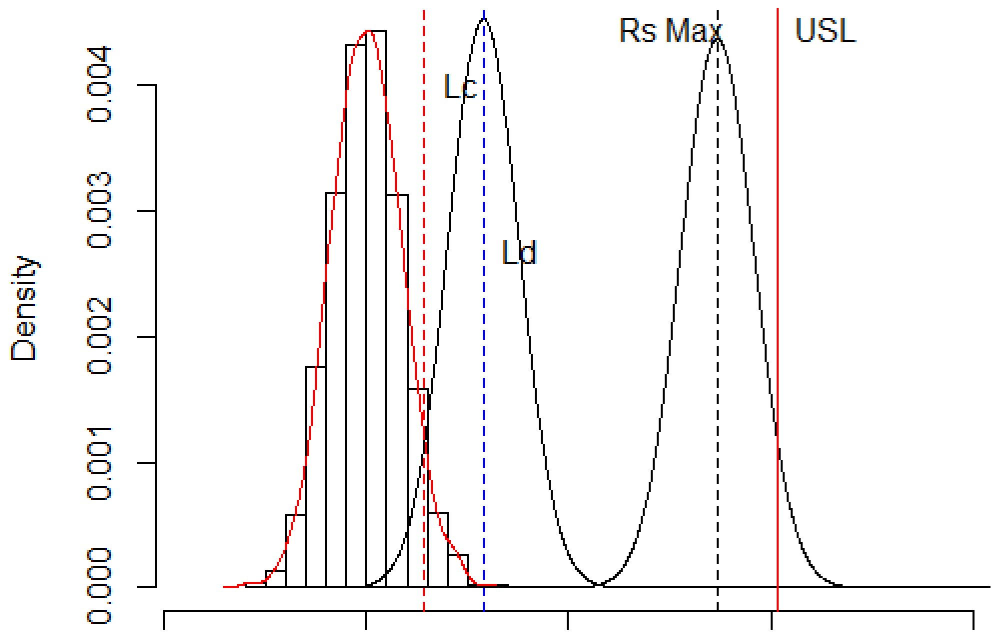

Figure 2: Relationship between signal and the USL

From figure 2, the USL is given by:

$$USL = (R_s^{Max} - R_o) + k_\delta \sigma_S$$

Where the first term is the highest net count rate achievable, and σ_s is the standard deviation of the signal count rate when $R_s = R_s^{Max}$. (i.e. $R_s^{Max} = C_s'/T_s$) .

Therefore,

$$USL = (R_s^{Max} - R_o) + k_\delta \sqrt{\frac{R_s^{Max}}{T_s} + \frac{R_o}{T_s} + \frac{R_o}{T_o}}$$

Where the USL is specified in terms of a count rate. Then,

$$(USL - (R_s^{Max} - R_o))^2 \ = k_\delta^2 \left(\frac{R_s^{Max}}{T_s} + \frac{R_o}{T_s} + \frac{R_o}{T_o} \right)$$

$$(USL - (R_s^{Max} - R_o))^2 \ = k_\delta^2 \frac{R_s^{Max}}{T_s} + \frac{k_\delta^2}{k_\alpha^2} L_c^2$$

$$(R_s^{Max} - R_o)^2 - \left(2USL + \frac{k_\delta^2}{T_s}\right)(R_s^{Max} - R_o) + USL^2 - \frac{k_\delta^2}{k_\alpha^2} L_c^2 - k_\delta^2 \frac{R_o}{T_s} \ = 0$$

For simplification, we again choose $k_\delta = k_\alpha = k$

$$(R_s^{Max} - R_o)^2 - \left(2USL + \frac{k^2}{T_s}\right)(R_s^{Max} - R_o) + USL^2 - L_c^2 - k^2 \frac{R_o}{T_s} \ = 0$$

Solution to the quadratic equation gives:

$$(R_s^{Max} - R_o) = \left(USL + \frac{k^2}{2T_s}\right) \pm \sqrt{\left(USL + \frac{k^2}{2T_s}\right)^2 - \left(USL^2 - L_c^2 - k^2 \frac{R_o}{T_s}\right)}$$

As we are interested in the maximum allowable activity, we take the lower of the two solutions, therefore only keeping the minus sign in the expression.

$$N^{Max} = R_s^{Max} - R_o = \left(USL + \frac{k^2}{2T_s}\right) - \sqrt{\left(USL + \frac{k^2}{2T_s}\right)^2 - \left(USL^2 - L_c^2 - k^2 \frac{R_o}{T_s}\right)} \qquad \textbf{(5)}$$

In terms of activities, we have:

$$A^{Max} = \frac{\left(USL \cdot 60 \cdot A \cdot \epsilon \cdot f + \frac{k^2}{2T_s}\right) \pm \sqrt{\left(USL \cdot 60 \cdot A \cdot \epsilon \cdot f + \frac{k^2}{2T_s}\right)^2 - \left((USL \cdot 60 \cdot A \cdot \epsilon \cdot f)^2 - L_c^2 - k^2 \frac{R_o}{T_s}\right)}}{60 \cdot A \cdot \epsilon \cdot f}$$

$$A^{Max} = USL\left(\frac{Bq}{cm^2}\right) + \frac{\frac{k^2}{2T_s} \pm \sqrt{\left(USL \cdot 60 \cdot A \cdot \epsilon \cdot f + \frac{k^2}{2T_s}\right)^2 - \left((USL \cdot 60 \cdot A \cdot \epsilon \cdot f)^2 - L_c^2 - k^2 \frac{R_o}{T_s}\right)}}{60 \cdot A \cdot \epsilon \cdot f}$$

As an example, consider the following parameters

Ro=	800 cpm
Ts=	1 minute
To=	1 Minute
ε=	0.25
A=	100 cm^2
f=	0.495
USL	0.4 Bq/cm^2

The critical level (or decision level) MDA, and maximum count rate to meet the specifications are:

$$L_c = 1.645\sqrt{\frac{800cpm}{1min}\left(1+\frac{1min}{1min}\right)} = 66\ cpm$$

$$L_c\left(\frac{Bq}{cm^2}\right) = \frac{66\ cpm}{60\cdot 0.25\cdot 100cm^2\cdot 0.495} = 0.09\ Bq/cm^2$$

$$MDA\ (\frac{Bq}{cm^2}) = \frac{2.71 + 3.29\sqrt{800\ cpm\cdot 1min\ \left(1+\frac{1min}{1min}\right)}}{60\cdot 0.25\cdot 100cm^2\cdot 1min\cdot 0.495}$$

$$MDA\ \left(\frac{Bq}{cm^2}\right) = \ 0.181\ Bq/cm^2$$

$$A^{Max} = 0.4Bq/cm^2 + \frac{\frac{2.71}{2\cdot 1min} - \sqrt{\left(0.4Bq/cm^2\cdot 60\cdot 100cm^2\cdot 0.25\cdot 0.495 + \frac{2.71}{2\cdot 1min}\right)^2 - \left((0.4Bq/cm^2\cdot 60\cdot 100cm^2\cdot 0.25\cdot 0.495)^2 - 66cpm^2 - 2.71\frac{800cpm}{1min}\right)}}{60\cdot 100cm^2\cdot 0.25\cdot 0.495}$$

$$A^{Max} = 0.29\ Bq/cm^2$$

The interpretation of this result is that given the specific geometry and detector efficiencies, we can say that if a sample has a measurement of 0.29 Bq/cm^2 there is 95% confidence that the true result is less than the USL. Intuitively then, in order for the method to be capable of meeting the specifications, the minimum detectable activity **MUST** be less than A^{Max}.

Solutions for NUREG 1757

According to NUREG-1757, the method for determining the net count rate is given by equation O-2 in Appendix O

$$N = \left(R_{u,su} - R_{s,su}\right) - R_{rm}$$

Where,

N =	Net counts
$R_{u,su}$ =	Unshielded survey unit count rate
$R_{s,su}$ =	Shielded survey unit count rate
R_{rm} =	Reference material background count rate

$$R_{rm} = R_u - R_s$$

R_u =	Unshielded gross on a background reference material
R_s =	Shielded, background count rate

The count rates in the above expressions can be defined as:

$$R_{u,su} = \frac{C_{u,su}}{T_{u,su}}$$

$$R_{s,su} = \frac{C_{s,su}}{T_{s,su}}$$

$$R_u = \frac{C_u}{T_u}$$

$$R_s = \frac{C_s}{T_s}$$

Where the C and T terms are the corresponding counts and count times for the measurements respectively.

Following the same procedure as in the previous section, we can determine the variance in the net count rate as:

$$\delta N^2 = \left(\delta R_{u,su}\right)^2 + \left(\delta R_{s,su}\right)^2 + (\delta R_{rm})^2$$

$$\delta N^2 = \left(\frac{\sqrt{C_{u,su}}}{T_{u,su}}\right)^2 + \left(\frac{\sqrt{C_{s,su}}}{T_{s,su}}\right)^2 + \left(\frac{\sqrt{C_s}}{T_s}\right)^2 + \left(\frac{\sqrt{C_u}}{T_u}\right)^2$$

The number of counts in the source signal, $C_{u,su}$, is given by the sum of the number of counts due to the source activity plus the counts from gamma background during period $T_{u,su}$, as well as the background from the reference material during the period.

$$C_{u,su} = C'_{u,su} + R_{s,su}T_{u,su} + R_{rm}\,T_{u,su}$$

$$C_{u,su} = C'_{u,su} + (R_{s,su} + R_{rm})\,T_{u,su}$$

where $C'_{u,su}$ is the total signal counts coming from source activity.

Using the same definitions for the critical level as before (i.e. $C'_{u,su} = 0$), and the same methodology, one can arrive at:

$$L_c = k_\alpha \sqrt{\frac{(R_{s,su} + R_u - R_s)}{T_{u,su}} + \frac{R_{s,su}}{T_{s,su}} + \frac{R_u}{T_u} + \frac{R_s}{T_s}} \qquad \textbf{(6)}$$

The detection limit (using $k_\alpha = k_\beta = k$) is given by:

$$L_d = \frac{k^2}{T_{u,su}} + 2k\sqrt{\frac{(R_{s,su} + R_u - R_s)}{T_{u,su}} + \frac{R_{s,su}}{T_{s,su}} + \frac{R_u}{T_u} + \frac{R_s}{T_s}} \qquad \textbf{(7)}$$

And finally, the maximum measureable count rate:

$$N^{Max} = \left(\frac{k^2}{2T_{u,su}} + USL\right) \pm \sqrt{\left(\frac{k^2}{2T_{u,su}} + USL\right)^2 - \left(USL^2 - L_c^2 - \frac{k^2}{T_{u,su}}\left(R_{s,su} + R_{rm}\right)\right)} \qquad \textbf{(8)}$$

Consider the following example from NUREG 1757,

$R_{u,su}$ =	1,000 cpm
$R_{s,su}$ =	500 cpm
R_{rm} =	100 cpm
R_u =	400 cpm
R_s =	300 cpm
$T_{u,su}$ =	1 min
$T_{s,su}$ =	1 min
T_u =	1 min
T_s =	1 min
ε=	0.20
A=	126 cm^2
f=	1

$$N = (1{,}000cpm - 500cpm) - (400cpm - 300cpm)$$

$$N = 400\ cpm$$

The activity is then determined as:

$$A = \frac{N}{\epsilon \cdot A \cdot f}$$

$$A\left(\frac{dpm}{100cm^2}\right) = \frac{400cpm}{0.2 \cdot 1.26 \cdot 1} \cong 1{,}600\ dpm/100cm^2$$

The critical level for the measurement is:

$$L_c = 1.645\sqrt{\frac{(500\ cpm + 400cpm - 300cpm)}{1min} + \frac{500\ cpm}{1min} + \frac{400\ cpm}{1min} + \frac{300cpm}{1min}}$$

$$L_c = 70\ cpm$$

Or in terms of activity,

$$L_c\left(\frac{dpm}{100cm^2}\right) = \frac{70\ cpm}{0.2 \cdot 1.26 \cdot 1} = 277\ dpm/100cm^2$$

The detection limit is:

$$L_d = \frac{2.71}{1min} + 3.29\sqrt{\frac{(500\ cpm + 400cpm - 300cpm)}{1min} + \frac{500\ cpm}{1min} + \frac{400\ cpm}{1min} + \frac{300cpm}{1min}}$$

$$L_d = 142\ cpm$$

Or in terms of activity, the MDA is

$$MDA(\frac{dpm}{100cm^2}) = \frac{142\ cpm}{0.2 \cdot 1.26 \cdot 1} = 565\frac{dpm}{100cm^2}$$

If we assume that the USL is 2,400 dpm/100cm^2 $\left(0.4\frac{Bq}{cm^2} \cdot 60 \cdot 100\right)$. The USL, expressed in terms of a count rate is:

$$UCL\ (cpm) = USL \cdot \epsilon \cdot A \cdot f$$

$$USL\ (cpm) = 2{,}400\ dpm/100cm^2 \cdot 0.2 \cdot 1.26 \cdot 1$$

$$USL\ (cpm) = 605\ cpm$$

Therefore, the maximum net count rate can be determined as:

$$N^{Max} = \left(\frac{2.71}{2 \cdot 1min} + 605cpm\right) - \sqrt{\left(\frac{2.71}{2 \cdot 1min} + 605cpm\right)^2 - \left(605cpm^2 - 70cpm^2 - \frac{2.71}{1min}(500cpm + 100cpm)\right)}$$

$$N^{Max} = 516\ cpm$$

In terms of activity,

$$A^{Max} = \frac{N^{Max}}{\epsilon \cdot A \cdot f} = \frac{516cpm}{0.2 \cdot 1.26 \cdot 1} = 2{,}048\ dpm/100cm^2$$

Therefore if a signal is detected at 2,048 dpm/100 cm^2 there is a 95% chance that the activity is less than the USL of 2,400 dpm/100cm^2.

Scenarios

We want to run scenarios to determine which circumstances would lead to the case in which the Maximum allowable activity is less than the MDA. At minimum, the MDA must be less than the USL. If the MDA > USL, it is not possible to conduct a measurement which would be able to detect anything less than the USL.

Assume for a Ludlum PR4389 beta probe,

ε=	0.1251
A=	100 cm^2
f=	1
USL =	750 dpm/100cm^2
R_u =	0 cpm
R_s =	0 cpm
$T_{u,su}$ =	5 min
T_u = T_s =	1 min

We can plot the MDA for these parameters as a function of the shielded background count rate, $R_{s,su}$, for a variety of count times $T_{s,su}$. This is shown in Figure 3. When these curves cross above the USL, the method is not capable of detecting activities within the specifications. For example, at a count time of $T_{s,su} = 1\ min$, the maximum allowable background $R_{s,su}$ is 670 cpm, otherwise the MDA will cross above the USL.

Ts,su (cpm)	Rs,su (cpm)
1	670
5	2,000
20	3,200

These maximum count rates can be plotted, as shown in Figure 4. This figure provides the maximum allowable Rssu in order for the MDA < USL. As the background count rate Rm increases, the maximum allowable background Rs,su must come down significantly.

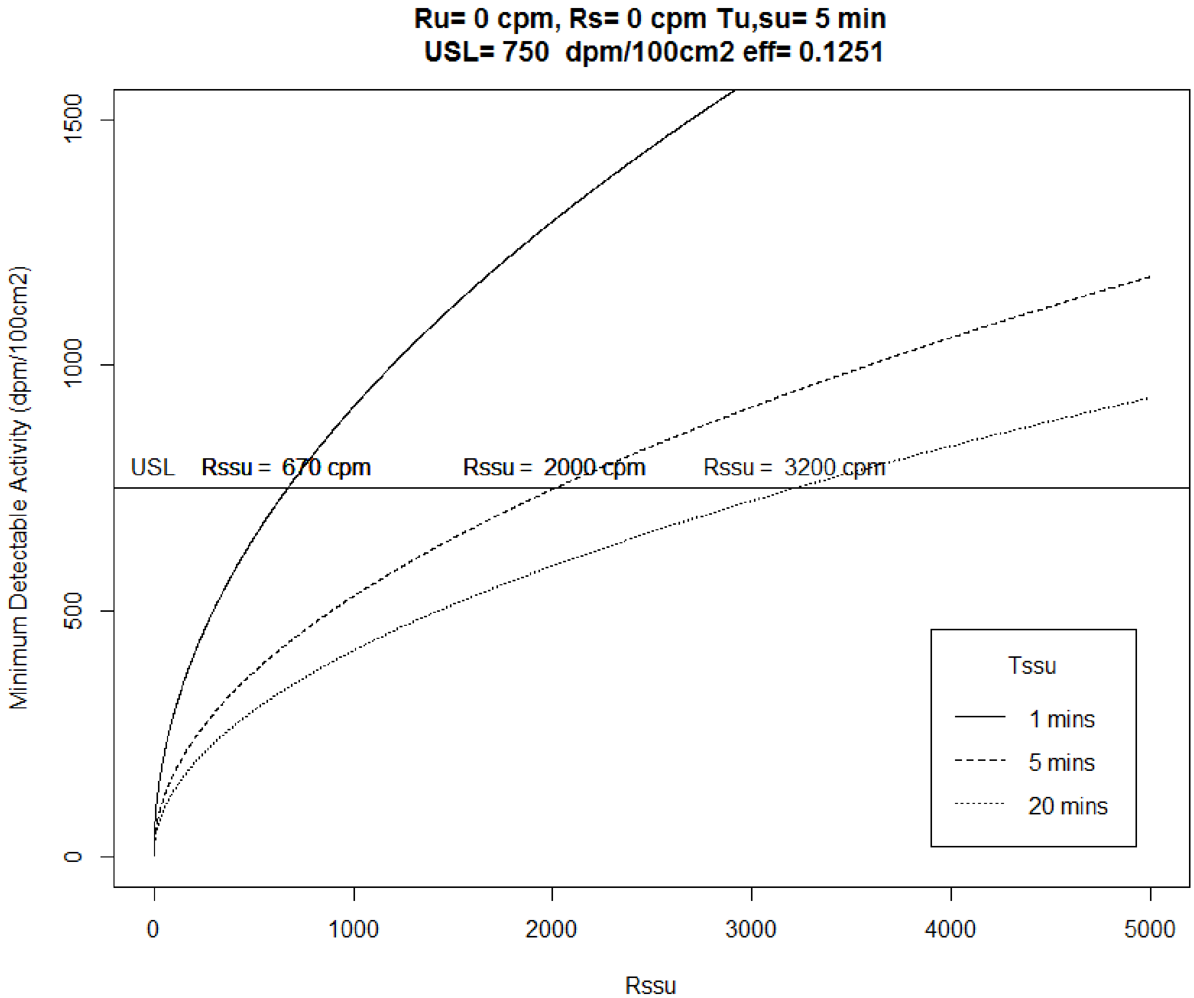

Figure 3: MDA as a function of Rssu

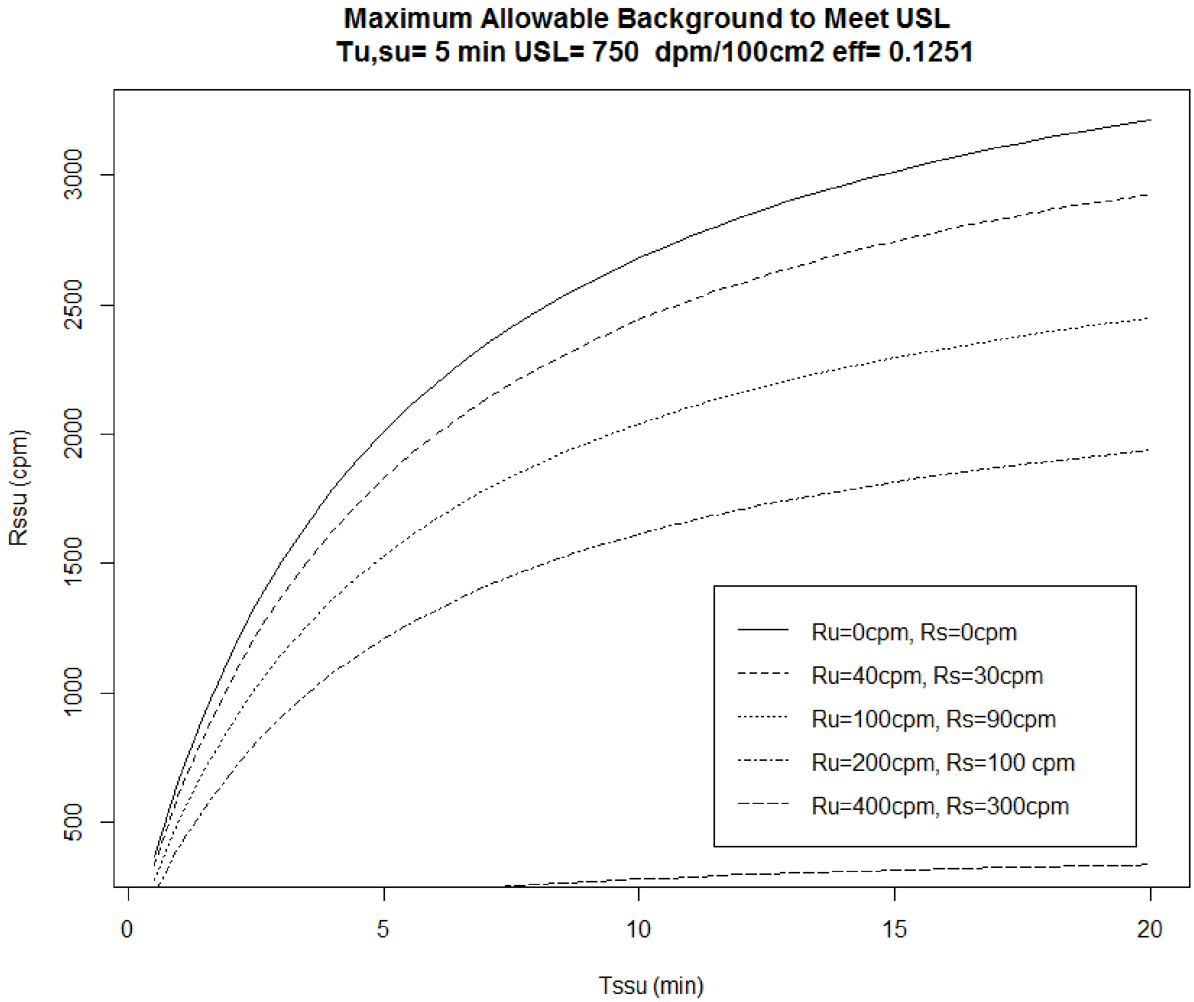

Figure 4: Maximum Allowable Rssu for MDA < USL