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CORRESPONDENCE CONTROL TICKET

Date Printed: Aug 28, 2007 12:16

PAPER NUMBER:	LTR-07-0565	**LOGGING DATE:**	08/27/2007
ACTION OFFICE:	EDO		
AUTHOR:	Po Kee Wong		
AFFILIATION:	MD		
ADDRESSEE:	Dale Klein		
SUBJECT:	TAMCE 2008 symposium by Fw: submission of five papers for ESIA9 to choose for presentation		
ACTION:	Appropriate		
DISTRIBUTION:	Encls to: EDO		
LETTER DATE:	08/25/2007		
ACKNOWLEDGED	No		
SPECIAL HANDLING:	Made publicly available in ADAMS via EDO/DPC		
NOTES:			
FILE LOCATION:	ADAMS		
DATE DUE:		**DATE SIGNED:**	

From: "Po Kee Wong" <pokwong@verizon.net>
To: <richard@mathforum.org>, <Domrosa@snet.net>, "Susan McReynolds" <mooresu@sonic.net>, <anisohedral@yahoo.com>, "Mita Desai" <mita.9955@gmail.com>, <cc0026@tsplescopycenter.com>, <Michelle.Rhee@dc.gov>, <michael.sohlman@nobel.se>, <rsas@kva.se>, <Mayor@dc.gov>, <Jerry_D_Weast@mcpsmd.org>, <Chairman@nrc.gov>, <abement@nsf.gov>
Date: 8/25/2007 2:40:23 PM
Subject: Fw: FW: TAMCE 2008 Symposium by Fw: Submission of five papers for ESIA9 to choose for presentations

Dear Referee at Mathforum ET AL:

You are cordially invited to review and evaluate the contents of the 16 attachments in this E-mail pertinent to the subject matter of " THE IMPACTS OF NEW SOLUTIONS OF OLD PROBLEMS IN MATHEMATICAL AND EXPERIMENTAL SCIENCES " to be presented at TAMCE 2008 in Turkey and at ESIA9 in Beijing for open discussions.

Your time and efforts spent for the subject matter with criticism and open discussions will be appreciated.

Very truly yours,

Wong, Po Kee
□ □ □
2413 Spencer |Road, Silver Spring, Maryland 20910-2344 USA
Tel: 301-585-3453
pokwong@verizon.net

----- Original Message -----
From: Po Kee Wong
To: Michelle.Rhee@dc.gov ; Mayor@dc.gov
Cc: Po Kee Wong ; Adam Wong
Sent: Wednesday, August 01, 2007 12:13 PM
Subject: Fw: FW: TAMCE 2008 Symposium by Fw: Submission of five papers for ESIA9 to choose for presentations

Dear Chancellor Rhee and Mayor Fenty:

With reference to my offer to you to solve your immediate needs of most updated text books in Mathemaitcs and Sciences in my last E-mail having been sent to you, the current E-mail with 16 attachments are submitted to you for your consideration:

----- Original Message -----
From: Po Kee Wong
To: Yonhua Tzeng
Cc: □□□ ; ChihHongChen@aol.com ; Po Kee Wong
Sent: Monday, April 16, 2007 9:28 PM
Subject: Re: FW: TAMCE 2008 Symposium by Fw: Submission of five papers for ESIA9 to choose for presentations

Dear Professor Yonhua Tzeng and professor dshieh:

Thank you for your message

----- Original Message -----
From: Yonhua Tzeng
To: pokwong@verizon.net
Cc: □□□
Sent: Monday, April 16, 2007 7:27 PM
Subject: Re: FW: TAMCE 2008 Symposium by Fw: Submission of five papers for ESIA9 to choose for presentations

Dear Dr. Wong, Po Kee (□□□):

Would you please send me your updated and detailed CV so that I can forward it to departments at NCKU that fit your background and expertise?

Thank you very much for your willingness and enthusiasm in helping with NCKU! We do need a lot of helps from NCKU alumni!

Sincerely,

Yonhua Tzeng
Professor of EE
VP for Research and Development
National Cheng Kung University
Tainan, Taiwan, R.O.C.

Wong, Po Kee
pokwong@verizon.net

On 4/16/07, □□□ <dshieh@mail.ncku.edu.tw> wrote:
Dear Prof. Tzeng:
Got an email from Alumni Center as following. Any suggestion?

DB

□□□□□□□:

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----- Original Message -----
From: Po Kee Wong <mailto:pokwong@verizon.net >
To: info@tmce-symposium.org ; default@ConfMaster.net ; Z.Rusak@tudelft.nl
Cc: Po Kee Wong <mailto:pokwong@verizon.net > ; Wong, Adam <mailto:Adam.Wong@fcps.edu> ; Simon Tam < mailto:dr.tamsimon@gmail.com>
Sent: Saturday, April 14, 2007 1:55 PM
Subject: TAMCE 2008 Symposium by Fw: Submission of five papers for ESIA9 to choose for presentations

Dear TAMCE 2008 Symposium.org ET AL:

The attachments contained in the communications with various organizations worldwide are submitted to you for your consideration for open review and evaluation and to pick the topics that you want to be presented at TAMCE 2008 Symposium:

Your time and effort spent on the subject matter will be appreciated.

Very truly yours,

Po Kee Wong, Ph.D.
Tel:+301-585-3453
Wong, Po Kee □□□□□

Wong, Po Kee
pokwong@verizon.net

----- Original Message -----
From: Po Kee Wong < mailto:pokwong@verizon.net>
To: darchambault@wilmington.co.uk ; Volaniran@wilmington.co.uk
Cc: Po Kee Wong < mailto:pokwong@verizon.net>
Sent: Tuesday, March 20, 2007 9:19 AM
Subject: Fw: Submission of five papers for ESIA9 to choose for presentations
The is the first one of five consecutive 5 E-mails of communications being forwarded to you:

----- Original Message -----
From: Po Kee Wong <mailto:pokwong@verizon.net>
To: ??? <mailto:esia9@buaa.edu.cn> ; fesi@fesi.org.uk ; Dong <mailto:dongjl@tsinghua.edu.cn >
Cc: Po Kee Wong <mailto:pokwong@verizon.net> ; 'simon tam' < mailto:simonfctam@yahoo.com.hk> ; Wong, Adam < mailto:Adam.Wong@fcps.edu>
Sent: Sunday, January 28, 2007 11:39 AM
Subject: Fw: Submission of five papers for ESIA9 to choose for presentations

Dear Professors Wu;Gosney and Dong:

This to inform you that,by the instruction of Professor Wu, the paper ABSTRACT entitled "IMPACTS OF NEW SOLUTIONS OF OLD PROBLEMS IN MATHEMATICAL AND EXPERIMENTAL SCIENCES " has been successfully uploaded to your website and given a paper code:171.

However, the paper ABSTRACT entitled "FORMULATION AND SOLUTION OF A SYSTEM OF NON-COPLANAR BEAMS IN FINITE DEFORMATIONS " has NOT been successfully uploaded to your website and NOT given a paper code number. This paper ABSTRACT has been SUCCESSFULLY forwarded to all of you in our previous communications. One of the reasons may be the Figure in the submitted ABSTRACT occupies too much spaces for transmission during the submission. Please help to input the submitted ABSTRACT and send a paper code number of that submission to me.

Very truly yours,

Wong, Po Kee □□□□□

Wong, Po Kee
pokwong@verizon.net

----- Original Message -----
From: Po Kee Wong < mailto:pokwong@verizon.net>
To: ??? <mailto:esia9@buaa.edu.cn > ; fesi@fesi.org.uk ; Dong < mailto:dongjl@tsinghua.edu.cn>
Cc: Po Kee Wong < mailto:pokwong@verizon.net>
Sent: Saturday, January 27, 2007 4:43 PM
Subject: Fw: Submission of five papers for ESIA9 to choose for presentations

Dear Professors Wu; Gosney and Dong:

The abstract of the submitted 5 papers is attached
together with the 5 papers and other relevant documents for your review and evaluation.

Yours truly,

Wong, Po Kee □□□□□

Wong, Po Kee
2413 Spencer Road, Silver Spring, Maryland 20910-2344 USA
Tel:+301-585-3453
pokwong@verizon.net

----- Original Message -----
From: Po Kee Wong <mailto:pokwong@verizon.net >
To: ??? <mailto:esia9@buaa.edu.cn> ; fesi@fesi.org.uk ; Dong < mailto:dongjl@tsinghua.edu.cn>
Cc: Po Kee Wong <mailto:pokwong@verizon.net >
Sent: Saturday, January 27, 2007 4:18 PM
Subject: Fw: Submission of five papers for ESIA9 to choose for presentations

Dear Professors Wu;Gosney and Dong:

When you open the Fw_mathforrum_org Po Kee Wong Angles-Google Search... you should be able to see 26 topics of mathematical discussions. However, some commercial companies intentionally block some of the websites for their own advertisements without my permission. If you click the topics that you want to see for two or more times you can open the website that you want to see and read !!! None of all those illegal pop-ups belong to me!!!. Both you and I can take them to the Courts in USA; UK and China for their illegally blocking legitimate communications !!!

Very truly yours,

Wong, Po Kee □□□□□

Po Kee Wong
2413 Spencer Road,Silver Spring, Maryland 20910-2344 USA
Tel:+301-585-3453
pokwong@verizon.net

----- Original Message -----
From: Po Kee Wong <mailto:pokwong@verizon.net>
To: ??? < mailto:esia9@buaa.edu.cn> ; fesi@fesi.org.uk ; Dong <mailto:dongjl@tsinghua.edu.cn >
Cc: Po Kee Wong <mailto:pokwong@verizon.net>
Sent: Saturday, January 27, 2007 12:46 PM
Subject: Submission of five papers for ESIA9 to choose for presentations

Dear Professors Wu;Gosney and Dong:

Being forwarded to you in the attachments are 5 papers and one abstract for your selection of the appropriate one or more than one of the papers for the presentations at ESIA9 according to your needs.

Your time and effort spent on the review and evaluation of the submitted papers for your needs will be appreciated. I look forward to hearing from you about your decision will be appreciated.

Sincerely yours,

WONG,PO KEE □□□

Po Kee Wong, Ph.D.
pokwong@verizon.net
------ Forwarded Message
From: Po Kee Wong < pokwong@verizon.net>
Date: Sun, 15 Apr 2007 22:05:02 +0800
To: <ChihHongChen@aol.com >
Cc: Po Kee Wong <pokwong@verizon.net>
Subject: Fw: TAMCE 2008 Symposium by Fw: Submission of five papers for ESIA9 to choose for presentations

------ End of Forwarded Message

CC: "Po Kee Wong" <pokwong@verizon.net>

Mail Envelope Properties (46D0777B.ACE : 10 : 6862)

Subject: Fw: FW: TAMCE 2008 Symposium by Fw: Submission of five papers for ESIA9 to choose for presentations
Creation Date 8/25/2007 2:27:58 PM
From: "Po Kee Wong" <pokwong@verizon.net>

Created By: pokwong@verizon.net

Recipients
nrc.gov
OWGWPO02.HQGWDO01
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mcpsmd.org
Jerry_D_Weast

dc.gov
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Michelle.Rhee

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mooresu (Susan McReynolds)

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Domrosa

mathforum.org
richard

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	snet.net
	mathforum.org

Files	**Size**	**Date & Time**
MESSAGE	10474	8/25/2007 2:27:58 PM
TEXT.htm	26394	
Po Kee Wong 1p resume.doc	28160	
Dr Wong photo.doc	33792	
ECQ Po Wong.doc	31744	
image002.JPG	115945	
image004.JPG	174355	
Fw_ mathforum_org Po Kee Wong Angles - Google Search(http___www_google_com_search_.htm	1227	
ICONE13-50509.pdf	216582	
COVER of COLLECTION OF TRUTH ARTICLES.pdf	2085161	
IMPACTS FROM NEW SOLUTIONS OF OLD PROBLEMS IN MATHEMATICAL AND EXPERIMENTAL SCIENCES.doc	23552	
IMECE 2001.doc	65536	
IMECE2003-43540 (2).doc	51712	
IMECE2003-43586.doc	51200	
IMECE2003-43536.doc	320512	
COMPARISON OF TRAJECTORY SOLID ANGLE WITH GEOMETRIC SOLID ANGLE IN SCATTERING THEORY(FULL PAPER).doc	53760	
Stanford Thesis.pdf	1174515	
FW 11 websites where you can obtain my patents.htm; international conference papers and relevant engineering and scientifi	17684	
Mime.822	6124411	

Options

Expiration Date:	None

Priority:	Standard
ReplyRequested:	No
Return Notification:	None
Concealed Subject:	No
Security:	Standard

Dr. Po Kee Wong
2413 Spencer Road
Silver Spring, MD 20910
Tel. 301-585-3453
Email: pokwong@verizon.net

Work History

Systems Research Company, Silver Spring, MD 1976-Present

Owner and founder of SRC specializing in science and technology. Responsible for the planning, marketing, managing, engineering and producing proposals, reports, technical papers and data for national and international meetings. Presented more than 20 papers. Attended more than 100 conferences. USPTO has granted 3 patents, 1 is pending.

Boston Public Schools, Boston, MA 1979-2001

High school and Middle School teacher. Taught regular and Chinese Bilingual math and science courses. Licensed in the state of Massachusetts. My students have received the top awards in math from BPS. Had taught all math courses using the TI calculators.

Engineer 1971-1975

Worked for Stone & Webster, MA and General Electric, CA performing review and evaluation of computer codes, industrial and governmental technical reports. Used in-house codes to check and to sign out specific areas in structural mechanics.

Education

1970 PHD Aeronautics and Astronautics	STANFORD
1966 ENG Applied Mechanics	CALTECH
1961 MS Mechanical Engineering	UTAH
1956 BS Mechanical Engineering	CHENG-KUNG U.

Professional Organizations:
ASME, AIAA, IAF, NYAS, and AMS

Languages:
Mandarin, Cantonese

Certification:
National Board for Professional Teacher
Massachusetts Department of Education, Professional Development Provider

Dr. Po Kee Wong
2413 Spencer Road
Silver Spring, Maryland 20910-2344 USA
Tel. and Fax.:301-585-3453
E-mail: pokwong@verizon.net

EXECUTIVE CORE QUALIFICATIONS:

(1) Leading Change: Established the sole proprietary SYSTEMS RESEARCH COMPANY (SRC) in 1976. The company has been cleared and recognized by the US Federal Government Small Business Administration (SBA) and the Department of Defense (DOD), Defense Logistic Agency (DLA) which granted SRC a Commercial And Government Enterprise (CAGE) Code: 5R583 for Mechanization Of Contract Administration Services (MOCAS) since February 20, 1979.
He plans, directs, and implements all operations of SYSTEMS RESEARCH COMPANY from 1976 to present.

(2) Leading People: Recruits, selects, supervises and develops professional and support staff involved in diverse endeavors having been evidenced by the contents of his technical proposals written and produced solely by himself and submitted to various federal governmental organizations that include DOD,DOE,NRC,NSF,DOT and NASA. Evidences of this ability can be seen from the following proposals:

A. SRC-DOE unsolicited proposal No. P7900450 entitled " INITIATION OF THE DEFINITION OF TRAJECTORY SOLID ANGLE AND ITS INFLUENCE ON CLASSICAL, QUANTUM AND STATISTICAL MECHNICS " January 17, 1979.
B. SRC-NSF SCREAMS proposal No. 98-12001 entitled " IMPACTS OF HIGH POWER FUNCTIONS; TRAJECTORY SOLID ANGLE; THE WONG'S ANGLES AND THE PHYSICAL ECONOMIC THEORY FOR SCIENTIFIC COMPUTING RESEARCH ENVIROMENTS " February 25, 1998.
C. SRC-NSF SBIR Proposal No. DMI-9760362 entitled " ESSENCE OF HIGH SCHOOL MATHEMATICS: AP CALCULUS &AP STATISTICS " June 10, 1997
D. SRC- BMDO/TOI proposal entitled " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLES TO SOLVE PROBLEMS OF FAILURE OF (THAAD)-Theater High Altitude Area Defense Missile System " November 1, 1997.
E. SRC-NASA proposal No. NRA-96-HEDS-03-076 entitled " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLES TO SOLVE FUNDAMENTAL PROBLEMS IN PHYSICS " March 21, 1997.
F. SRC-NASA proposal No. TRIANA-0003-0006 entitled " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLES FOR TRIANA " July 22, 1998.
G. SRC-NASA PROPOSAL No. AIST-0042-0006 entitled " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLES TO SUPPORT ESTO PROGRAMS: AIST;ATI;IIP AND HPCC/ESS." January 22, 2000.

(3) Result Driven: Obtained the first Federal Governmental Contract for SRC from DOT-TSC under Order No. TS-15054 in 1978 responding to the solicitation of a program in collisions of structures and as a test case of SRC' s unsolicited proposal N0. TSC-UP-77-27.
Established stature in the profession as members of ASME, AIAA, MAA,AMS,AAAS, New York Academy of Sciences; having been cited in 12 published " who's who" biographies.
Presents and publishes high qualitative technical papers in AIAA, ASME, MAA ..etc. professional conferences and meetings at regional, national and international meetings.
Contributes in the invention of several granted and pending US Basic Patents with international impacts in physics, mathematics, engineering and high technologies.

(4) Business Acumen: The ability of Po Kee Wong to acquire and administer human, financial, material

and information resources can be seen from the contents of the SRC-US Governmental various agencies proposals from A,B,C........to G. In all these proposals that provide new technology to the U.S. Government to enhance decision making, they include the key characteristics as being prescribed in this job requirements for candidates.

(5) Building Coalitions/Communications: The ability of Po Kee Wong to explain, advocate, and express facts and ideas in a convincing manner, and negotiate with individuals and groups internally and externally has been repeatedly demonstrated in his U.S. Patents, presentations and publications of technical papers at regional, national and international conferences as well as inside the contents of the above proposals from A,B,C, to G.

(6) TECHNICAL QUALIFICATIONS.

PART I.
TECHINCAL PAPERS RELEVANT TO BOTH US PATENTS HAVING BEEN REVIEWED, PRESENTED AND PUBLISHED AT INTERNATIONAL CONFERENCES: 68pages.

I.1." BASIC NEEDS OF HUMAN BEINGS AS THE PURPOSES AND FOUNDATIONS FOR THE EXISTENCE OF GOVERNING INSTITUTIONS AND THE ADVANCEMENT OF SCIENCE AND TECHNOLOGY ".A paper presented at 2001 ASME International Mechanical Engineering Congress & Exposition on Tuesday, November 13, 2001, 9:30AM at Technical Session # E&TM-11 at Hilton New York/Sheraton New York, New York City, NY.USA.8 pages.

I.2. " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE (TSA) AND THE WONG'S ANGLES (WA) TO SOLVE PROBLEMS OF THAAD FOR BMDO AND FOR FUTURE MISSIONS OF NASA" US Copyright Registration Number TX5-375-549, April 19, 2001, presented at the Proceedings of the Fifteenth SSI/Princeton Conference on Space Manufacturing, May 7-9, 2001.14 pages.

I.3. " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE (TSA) AND THE WONG'S ANGLES (WA) TO SOLVE PROBLEMS OF THAAD FOR BMDO AND FOR FURTURE MISSIONS OF NASA" Excerpts of document No. I.2 published in " SPACE MANUFACTURING 13 SETTLING CIRCUMSOLAR SPACE" Proceeding of the Fifteenth SSI/Princeton Conference on Space Manufacturing May 7-9,2001. Page 98 to page 101. 4pages.

I.4. " APPLICATIONS OF TRAJECTORY SOLID ANGLE (TSA) AND WONG'S ANGLE (WA) TO SOLVE FUNDAMENTAL PROBLEMS IN PHYSICS AND ASTRONOMY " IAF-00-J.1.10 paper presented and published at 51st. International Astronautical Congress, 2-6 Oct 2000/Rio de Janeiro, Brazil. 5pages.

I.5. " APPLICATIONS OF TRAJECTORY SOLID ANGLE (TSA) AND WONG'S ANGLES (WA) FOR LAUNCHING OF SPACE VEHICLES " IAF-00-S.6.03 paper presented and published at the 51st. International Astronautical Congress, 2-6 Oct 2000/Rio de Janeiro, Brazil. 4 pages

I.6. " NUMERICAL DATA FOR SATELLITE ALTITUDE CONTROL BY MEANS OF WONG'S ANGLES " AIAA-96-1047-CP paper presented and published at the 16th. International Communications Satellite Systems Conference, February 25-29,1996 Washington DC. page 517 to page 523, 7pages.

I.7. " ON THE FORMULATION AND SOLUTION OF A CLASS OF MAGNETO-VISCOELASTO-DYNAMICS (MVD) GOVERNING EQUATIONS OF MOTION " presented and published at the 1995 ASME Design Engineering Technical Conferences-The 15th. Biennial Conference on Mechanical Vibration and Noise, September 17-20, 1995 Boston, Massachusetts, DE-Vol. 84-2 Volume 3- Part B page 1451 to 1456. 7pages.

I.8. " ON THE IRROTATIONAL-FLOW VELOCITY POTENTIAL FUNCTION AND A NEW STREAM FUNCTION OF FLUID MECHANICS " paper No. 80-C2/Aero-3 presented and published at the ASME Century 2 Aerospace Conference, San Francisco, California, August 13-15, 1980. 10pages.

I.9. " ON THE UNIFIED GENERAL SOLUTIONS OF LINEAR WAVE MOTIONS OF THERMOELASTODYNAMICS AND HYDRODYNAMICS WITH PRACTICAL EXAMPLES " paper No. 67-APM-32 presented and published at the ASME Applied Mechanics Conference, Pasadena, California, June 26-28, 1967. 9pages.

PART II.
3 (SRC) SYSTEMS RESEARCH COMPANY'S TECHNICAL PROPOSALS HAVING BEEN SUBMITTED TO NASA FOR SUPPORT: 113pages.

II. 1. SRC-NASA proposal No.NRA-96-HEDS-03-076 entitled " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLES TO SOLVE FUNDAMENTAL PROBLEMS IN PHYSICS " submitted on March 21, 1997. 26pages.

II. 2. SRC-NASA proposal No. TRIANA-0003-0006 entitled " APPLIACTIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLE FOR TRIANA " submitted on July 22, 1998. 36 pages

II. 3. SRC-NASA proposal No.AIST-0042-0006 entitled " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLE TO SUPPORT ESTO PROGRAMS:AIST;ATI;IIP AND HPCC/ESS." Submitted on January 22, 2000. 51pages.

At US$295.00 or £147.50 Sterling each PLUS US$20.00 or £10.00 Sterling postage and packaging per copy. This represents a massive saving on the actual publication price of US$390.00 or £195.00 Sterling each PLUS postage and packaging at US$20.00 or £10.00 Sterling per copy.. In addition, I would like to offer you the option of obtaining a specially commissioned Outstanding Intellectual Diploma or Medal to commemorate your inclusion in this title. The Outstanding Diploma or Medal can be acquired either singly or as a set and they both pay tribute to your inclusion in this prestigious project. Further details are given on the enclosed Priority Order Form.

I look forward to hearing from you and hope that we may include your updated biography in **2000 Outstanding Intellectuals of the 21st Century**.

Yours sincerely

Sara Rains
Editor in Chief

Please correct your entry shown overleaf using the headings provided. Further corrections may be written on a separate sheet of paper. If there are no alterations, write 'No Change' against the entry. Include your name, address and signature and return this form by the date shown.

EDUCATION

CAREER

PUBLICATIONS

HONOURS AND AWARDS

MEMBERSHIPS OF ASSOCIATIONS, INSTITUTES, ETC

OTHER WHO'S WHOS IN WHICH YOU ARE LISTED

NAME

FULL ADDRESS (For Publication)

I certify that the above information is correct and may be published.

SIGNATURE: ______________________ DATE: __________

TO ASSIST OUR ADMINISTRATION STAFF, PLEASE RETURN THIS FORM BY: 29th January 2007

INTERNATIONAL BIOGRAPHICAL CENTRE, CAMBRIDGE, ENGLAND

IBC

15th December 2006

Mr Po Kee Wong
2413 Spencer Road
Silver Spring
MD 20910-2344
USA

Ref. TINT4-5/clp

Dear Mr Wong,

2000 OUTSTANDING INTELLECTUALS OF THE 21ST CENTURY
PUBLICATION: 2008

YOUR CHANCE TO UPDATE YOUR BIOGRAPHICAL ENTRY

I have great pleasure in attaching to this letter a clipping of your entry as it appeared in the Fourth Edition of **2000 Outstanding Intellectuals of the 21st Century.** I am personally delighted with the end result and already it is receiving critical acclaim from librarians and researchers alike.

I am now writing to ask if we may include the same entry or an amended version in the Fifth Edition of **2000 Outstanding Intellectuals of the 21st Century** which is due for publication across the world in early 2008. I would ask you to check the clipping opposite and make any relevant amendments and updates before returning it to me at this office.

If you do wish to amend or add to your entry, please include corrections in the relevant sections overleaf. Please sign the form and return the complete letter to me by the date shown. We will then send you a typescript for approval prior to publication. If, however, you have no alterations please return this complete letter on which you need only write 'No Change' along with your signature in the space provided.

As I am sure you are aware there has never been a charge for biographical inclusion but we will make available a limited number of case bound copies to biographees at special reduced rates.

Over....

WONG, Po Kee, b. 5 May 1934, Canton City, China. Educator. m. Ruby Ching Wong, 1 son, 1 daughter. Education: BS, Mechanical Engineering, Taiwan National Cheng Kung University, 1956; MS, Mechanical Engineering, University of Utah, 1961; Engineer's Degree, Mechanical Engineering, Caltech, 1966; PhD, Aeronautical & Astronautical Engineering, Stanford University, 1970. Appointments: Various teaching positions, University of Santa Clara, Caltech, University of Utah, Taiwan National Cheng Kung University, 1958-73; Owner and Chief Executive Officer, Sonetic Research Company, 1976; Teacher, Boston Public Schools, City of Boston, Massachusetts, USA, 1978-2001; Retired from teaching, 2001; Industrial positions: Lockheed Missile & Space Company; General Electric [illegible] Department; Stone & Webster Eng. Inc. Publications: Owns US basic patents in aerospace and hydrospace objects detecting, tracking and targeting; nuclear power plant safety code licensing; curriculum development in mathematics and physics for education of learners of all levels. Honours: Graduate School Scholarships: Utah, Caltech and Stanford University; Lockheed-Stanford Fellowship; China Foundation in America [illegible] Fellowship. Memberships: ASME Life Member; AIAA Associate Fellow. Address: 2413 Spencer Road, Silver Spring, Maryland 20910-2344, USA.

IBC Correspondence [illegible] International Biographical Centre, St Thomas' Place,
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Telephone +44 1353 646600
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From: pokwong <pokwong@rcn.com>
To: pokwong@verizon.net <pokwong@verizon.net>; pokwong <pokwong@rcn.com>
Date: Wednesday, September 06, 2006 5:32 AM
Subject: mathforum.org Po Kee Wong Angles - Google Search (http://www.google.com/search?

http://www.google.com/search?q=+site:mathforum.org+Po+Kee+Wong+Angles&hl=en&lr=&ie=UTF-8&filter=0

13th International Conference on Nuclear Engineering
Beijing, China, May 16-20, 2005
ICONE13-50509

Fundamental Challenging Problems for Developing New Nuclear Safety Standard Computer Codes

Po Kee Wong	Adam E. Wong	Anita Wong
Systems Research Company (SRC)	Systems Research Company (SRC)	Systems Research Company (SRC)
Silver Spring, Maryland	Silver Spring, Maryland	Silver Spring, Maryland
USA	USA	USA
Phone: 301-585-3453	Phone: 301-585-3453	Phone: 301-585-3453
Fax: 301-585-3453	Fax: 301-585-3453	Fax: 301-585-3453
Email: pokwong@rcn.com	Email: pokwong@rcn.com	Email: pokwong@rcn.com

KEYWORDS: TRAJECTORY SOLID ANGLE, GEOMETRIC SOLID ANGLE, WONG'S ANGLE

ABSTRACT

Based on the claims of the US Basic patents number 5,084,232; 5,848,377 and 6,430,516 that can be obtained from typing the Patent Numbers into the Box of the Website
http://164.195.1 00.11/netahtml/srchnum.ht m
and their associated published technical papers having been presented and published at International Conferences in the last three years and that all these had been sent into US-NRC by E-mail on March 26, 2003 at 2:46 PM., three fundamental challenging problems for developing new nuclear safety standard computer codes had been presented at the US-NRC RIC2003 Session W4. 2:15-3:15 PM. at the Washington D.C. Capital Hilton Hotel, Presidential Ballroom on April 16, 2003 in front of more than 800 nuclear professionals from many countries worldwide. The objective and scope of this paper is to invite all nuclear professionals to examine and evaluate all the current computer codes being used in their own countries by means of comparison of numerical data from these three specific openly challenging fundamental problems in order to set up a global safety standard for all nuclear power plants in the world.

INTRODUCTION

Problem Number (1) provides impacts enumerated in the followings:

1. Provides a new Statistical Mechanics in Physics.
2. The calculation of collision cross-sections in particle physics, based on by utilization of Geometric Solid Angle (as that was used in the well known Sir Rutherford's Alpha Scattering), must be re-examined again in comparison with that by utilization of Trajectory Solid Angle (TSA) (US Patent 5,084,232). Please also read the IAC-02-J.P.02 paper.

3. A new Hydrogen Model is being proposed by substituting the trajectory equations of the complete Two-Body solution by Max Born into the definition of Trajectory Solid Angle (US Patent 5,084,232).
4. Application of TSA in a cold-atom atomic fountain clock as described by Bigelow (Ref 10 of paper IAF-00-J.1.10) This is the JPL project trying to improve the accuracy of the Norman Ramsey' Atomic Clock.
5. All the proposed tasks in the 1979 Unsolicited Department of Energy Proposal number P7900450 should be re-examined again.

Problem Number (2) Wong's Angles (WA) Patent provides:
1. A precise method to measure and determine the real trajectories of object s under the actions of many-force fields in various environment s and
to guide the using of appropriate instrument s for measurement s in experiment s (for examples: How to set up stations around the Nuclear Power Plant to shoot down in-coming missiles to hit the Nuclear Power Plant; How to set up instrument s around a nuclear reactor core to track the internal flow conditions inside the nuclear reac tor core... etc.)
2. The (WA) provides a unique and precise method to design the Digital Sensing Processor (DSP). The (DSP) can be used directly in all instrument s or to be integrated into the Central Processor Unit (CPU) in all calculators and computers attached to regulate and control all scientific instrument s including but not limiting to: Digital Cameras; Digital Telescopes; Digital Microscopes; Digital Theodolites; High Definition TV; CAD-CAM System Design in Mechanical Engineering; Surveying System Design in Civil and Architectural Engineering; Aircraf t and Airport Landing System Design and all other relevant instrument s for measuring; tracking and controlling of object s.
Problem number (3) provides a unique correction of the calculating procedures that have been prevailingly used in all computers and calculators for several decades. This correction must be made, because of the errors in computers and calculators have been extensively used in various fields of sciences, engineering, technologies and mathematics in education. The corrections that involve in functions of complex variables are very import ant to aerosp ace re-entry vehicles and nuclear reactor cores both include the coupling motions of dynamics and heat transfer. Therefore, the corrections must be made.

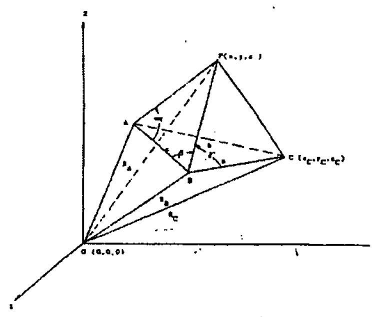
SPECIFIC EXAMPLES OF PROBLEMS

Initial ejection velocity of a particle
V sub.zero = 9.8 m/sec at point O
The gravitation in the direction perpendicular to the plane XOY g = 9.8 m/sec squared a = square of v sub. zero / g = 9.8 m = Max. range of the particle can hit r1 = 9 m r2 = 10 m r2 - a = 10 - 9.8 = 0.2 m will not be hit sector angle =pi/6 shaded targeted area = 4.974 square m. Find the cumulative probability for the particle ejected from point O randomly to hit the prescribed targeted area.

Given

Problem No. (2):

Given all reliable sensing instrument s used for detection of particles or object s moving in various environment s, how to arrange and set them up in stations to track; to measur e; the position, velocity and acceleration vectors of the moving particle or object with the least amount of parameters to be experiment ally measured to obt ain the accurate dat a?



Specific example: three ground stations identified as stations A, B, C. Stations A is at the West of Station B. Stations C is at the north of Station B. Segment AB which is underneath the earth surface =0.4R(R=Radius of the earth at sea level=OR, where O is the center of the earth). Segment BC which is also underneath the earth surface = 0.3 R. A high altitude object is observed with two sets of Wong's Angles from Stations A, and B as Alpha 1 = 30 degrees; Beta 1 = 60 degrees; Gamma 1 = 60 degrees at time t = t1 at point P1. Alpha 2 = 60 degrees; Beta 2 = 30 degrees; Gamma 2 = 90 degrees at time t = t2 at point P2
Find: The position, velocity and acceleration vectors of the object moving from point P1 to point P2.

Problem No. (3):
Given:
Z1=x1 +I y1, Z2 = x2 + I y2, where x1, y1, x2, y2 are unknown real numbers to be determined from solving the two following simultaneous equations involved in High Power Functions of Complex Numbers. where i = (- 1) ^ (1/2) is the unit imaginary number.
Find:
The Principal Solution of Z1 and Z2 from the following two simultaneous equations (1) and (2):

Arc Sin (Z1 + Z2)=
(3^(1/2)-i)^((1 +i3^(1/2))^(-1+ i)) Eq.(1)

Arc Sin (Z1 - Z2)=
((3^(1/2)-i)^(1+ i3^(1/2)))^(-1 + i) ....Eq.(2)

The above specific problem is considered the simplest problem in comparison with other problems involved in functions of complex variables of many Elementary Transcendental Functions in the general solutions of a set governing equations of Thermo-Visco-Elastodynamics appeared in many references shown in the U.S. Patent No.: 6,640,516 that have been used for LOCA, Fuel Pin Design, and Thermal Hydraulic Transient Analysis in Nuclear Power Plants and Aerospace Industries for years since 1968.

SUMMARY

1. The invention of TRAJECTORY SOLID ANGLE provides the most precise definition to solve the problem for the first time in October 1974. Comparing the (TSA) method with all other methods at that time, all other methods became approximate. For examples: the Monte Carol Methods; the Geometric Solid Angle (GSA) method are all conditionally accurate in some given ranges of parameters but not precise in all ranges of the given parameters.

2. The definition of (TSA) is explicitly defined with all parameters implicitly contained within the definition while all the other methods do not.

3. Due to the precise definition of (TSA), it is applicable for macroscopic bodies as well as for microscopic particles of mathematically defined infinitely small size under the actions of any force and moment fields between and among the bodies and particles. Therefore, the (TSA) provides great impacts to the entire range of physics; from the calculation of the collision cross sections of sub-nucleus particles in high-energy physics and to that of galaxies in astronomy. The applications of all other methods are relatively limited. force fields (which include: the hydrogen model; Alpha scattering; moon-earth model; Comet

4. The Geometric Solid Angle (GSA) of any targeted area, being finite or infinitesimally small, is unchanged with respect to the location of the source where the particle is ejected. The (GSA) is not related to the parameters of ejection of the particle at all. It is a pure mathematical quantity. The TRAJECTORY SOLID ANGLE (TSA) is a term containing all the parameters of generating the particle and the targeted area to be hit. Thus the (GSA) of any targeted area is always finite and unchanged while that of (TSA) can be zero. This explains why the (TSA) can be and should be used to solve the P.sub.2 targeting problem for particles and bodies under the action of any force and moment fields and that the (GSA) cannot and should not be considered as the correct solution for the P.sub.2 problem. There will be errors comparing the use of (TSA) between the uses of (GSA) to solve the same problem. The errors will range from 0% to more than 100%.

5. Since the collision cross sections of many problems in central Halley scatters around the solar system etc.) have been based on the use of (GSA) for calculation and have been published in textbooks around the world, the future assertion the truth of (TSA) will provide a great impact to all those results in the past.

6. The (TSA) concept and its definition not only confirms the well known Heisenberg's principle of uncertainty in physics, but also provides the precise definition and procedures to calculate the uncertainty in term of numbers as precise as we want.

7. The most important concept of (TSA) is that the definition can be applied to discover new laws and new particles by comparisons and matches of the unknown results with the already confirmed and proved results. If there are new laws of physics that describe the particle motions other than those of Newton's classical mechanics and Einstein's narrow and general relativity, the present (TSA) concept is still applicable to obtain the precise P.sub.2 function for the problem.

8. Four examples are selected to illustrate how to obtain the probability distribution functions by means of (TSA). They are: Alpha scattering; A particle in uniform, isotropic linear motion; A particle under assumed constant-gravity pull from a plane surface; A particle in a medium where the resistance force is ' linearly proportional to the velocity of the particle and under a uniform gravitational field. These examples are selected on the basis that they are well known and can be found from the open literatures. They were selected with the intention to show that even with such simple well-known examples, the correct probability functions and cumulative distribution functions of these problems have never been obtained before. Whether exact solutions can be obtained from the equations of motion that govern other problems will not be the issue because the equation of motions can always be solved by means of numerical analysis together with computer programming. The key issue is that through the definition of (TSA), the P.sub.2 functions can be precisely defined and obtained. The (TSA) can be applied to solve the most fundamental problems in physics that include all the subjects listed as cited references in this application.

NOMENCLATURE

BRIEF DESCRIPTION OF THE DRAWINGS
FIGURE No.1:
FIGURE No. 1 shows a finite surface area ABCDA that can be described with respect to a fixed coordinate system designated to be hit by a particle generated and ejected from the origin of the coordinate system. The finite area ABCDA to be hit can be subdivided into almost infinite number of infinitesimal surface areas, thus the total surface area in vector form can be expressed as

$$P_s = \frac{\vec{T} \cdot \Delta \vec{A}_s}{4\pi R_s^2}$$

$$\vec{A} = \vec{A}_{ABCDA} = \sum_{s=1}^{s=\infty} \Delta \vec{A}_s$$

$$P_2 = \sum_{s=1}^{s=\infty} P_s = \sum_{s=1}^{s=\infty} \frac{\vec{T}_s \cdot \Delta \vec{A}_s}{4\pi R_s^2} = \int_A \frac{\vec{T} \cdot d\vec{A}}{4\pi R^2} = \int_A \frac{\cos\delta\, d\vec{A}}{4\pi R^2} = \frac{\Omega}{4\pi}$$

Each infinitesimal surface area.delt a.A sub.s is connected by the position vector R.sub.s that defines the equations of a surface. The spherical surface formed by the position vector passing through a particular infinitesimal surface area (which is as small as almost like a point) is 4.pi.R.sub.s. sup.2, where 0<R.sub.s< Infinity.

The probability for the particle to hit a infinitesimal surface area .delta.A.sub.s depends on the particle's unit tangent vector T.

If T is parallel (direct common sense, since the surface area vector has a unit vector perpendicular to the surface area, thus mathematically it should have been said perpendicular) to the surface area .delta.A sub.s, the particle will miss the surface.

If T is perpendicular (it should have been said parallel mathematically) to.delta.A.sub. s the particle will hit the surface at a right angle.
The probability for the particle to hit on.delt a.A sub.s is **P.** sub. s which can be expressed in the following.

The probability for the particle to hit the entire surface area ABCDA is therefore **P.** sub.2 that equals to the summation of **P.** sub.s, where s is summed from s=1 to s= to infinity.

These complete the proof that the (TSA) can be used to solve the P.sub.2 targeting problem. (The unit tangent vector T contains all the parameters of generation and ejection of the particle and satisfies the governing equations of the laws of physics).

FIGURE No. 2 shows that particles under assumed constant gravitational pull on a plane are generated and ejected from the origin of the coordinates. The particles can be electrons, ions or unchanged particles. The gravitation g can also be simulated by an electric and/or magnetic field for the charged particles. The plane is the target surface to collect the particles. The figure represent s the schematic diagram of a mass spectrometer . Given the initial velocity v. sub .o of an ejected particle, find the probab ility of the particle that would hit the predetermined area bound by r.sub.1 less than or equal to .r that is also less than or equal to r.sub.2; phi. sub.1 less than or equal to phi that is also less than or equal to phi.sub.2.

FIGURE No. 3 shows the probability density function of the particle hitting on the plane surface. The cumulative probability to hit on any area on the plane surface can be calculated by carrying out the integration precisely. For example:
Given v sub.o =9.8m/sec , g=9.8m/sec^2

a= (v su.o)^2/g =9.8m = Max. Range r sub.1 = 9 m
r sub.2 =10 m r sub.2-a =10-9.8 = 0.2 m will not be hit
phi sub.2 -phi sub.1 = pi/6 =0.5235987756 radian

The targeted area =4.974 m^2

The cumulative probability for the particle to hit the target area=.OMEGA./(4. pi.)=0.0035846.

FIG. No. 4 shows the classification of regions that can be reached only by the high; only by the low, and by both high and low trajectories of the particles.

Region H bounded by OPBT.sub.h AO can be reached by high trajectory only.

Region L bounded by DBCD can be reached by low trajectory only.

Region HL bounded by OPBDIO can be reached by both high and low trajectories.

Region outside of OAT.sub.h BCIO cannot be reached by either high or low trajectories.

The TRAJECTORY SOLID ANGLE (TSA), Probability Density Function (pdf), and Cumulative Distribution Function (cdf) are all different in each region. They are all zero outside of the region bounded by OAT.sub.hBCIO.

FIGURE No. 1 is applicable for the general targeting problems in mass spectrometers particle accelerators, super-colliders; actual missiles and rockets targeting problems scattering and collision of astronomical bodies, chaos of classical dynamics and quantum mechanics, fluid dynamics and the weather prediction . . . etc.

FIGS. No. 2, 3, 4 are demonstrated in great details how to apply the invention of (TSA) to solve a specific well-known-simple problem but its Probability Density Function (pdf) and Cumulative Distribution Function (cdf) have never been precisely obtained by all other methods before the invention of (TSA).

The procedures to find the distribution function P.sub.2 for a particle striking a predetermined area, given all its parameters of generation and ejection can be systematically summarized in the following steps:

1. Solve the set of governi ng equations that govern the trajectory of the particle and obtain the position vector, the velocity vector and the trajectory equation in terms of initial conditions and all other parameters in the governing equations.

2. Find the unit tangent vector from the velocity vector or from the trajectory equation.

3. Find the unit normal vector and the differential surface area from the governing equation of the surface to be struck.

4. Find the intersection of the trajectory on the surface and set the intersection coordinates in terms of the two independent variables that define the surface.

5. The incident angle of the particle on the trajectory striking at the surface and be defined from the inner product of the unit tangent vector to the unit normal surface vector expressed in terms of the two independent variables at the intersection.

6. The trajectory solid angle for the problem can be obtained from integration over the cosine of the incident angle multiplying the differential surface area divided by the square of the position vector of the surface.

7. The probability distribution function can be defined as the ratio of the trajectory solid angles (TSA).

There are 11 separate tasks proposed tasks to be done in the SYSTEMS RESEARCH COMPANY'S 143 pages technical proposal DOE No. P7900450 that was sent for supports in 1979 to the High Energy Physics Division of the US Department of Energy.

There are also at least 3 technical proposals having been submitted to NASA for funding and support.

The values of the invention depend on whether the solution of the P.sub.2 targeting problem by means of the (TSA) is TRUE. If it is, it will provide all the impacts to practitioners, public and private decision makers, and the general public especially involved in education:

It will affect many previous Nobel Laureates' work in scattering and collision crossections of particles; in Statistical Mechanics and Quantum Mechanics. Specific work of interests include: Rutherford's Alpha Scattering; Hofstadter's electron scattering; Yang's p-p collision and its scattering and the geometric picture; Fermi-Dirac, Bose-Einstein, Maxwell-Boltzmann statistics; quantum mechanics based on Schrodinger's equation; Schwinger and Feynman's quantum electrodynamics and Heisenberg's uncertainty principle... etc.

All these topics are in the current text books of physics for graduate and undergraduate levels in all universities in the world. It follows that will influence the selection of materials for the secondary curriculum planning and development according to the impacts.

ACKNOWLEDGEMENT

This paper is an original excerpt from the U.S. Patent No. 5,084,232 entitled "Trajectory Solid Angle's Impacts to Physics and High Technologies "Published by the U.S. Patent Office on January 28, 1992. The previous work contributed by many others leading to the invention of Trajectory Solid Angle (TSA) to solve the P sub. 2 targeting problem and that also become the foundation of the new statistical mechanics is gratefully appreciated. The information by many others can be read from the REFERENCES CITED and the BACKGROUND OF THE INVENTION from the patent specification, which can be obtained from:

http://164.195.100.11/netahtml/srchnum.htm

The U.S. Patent 5,848,377 entitled "Wong's Angles (WA) to Determine Trajectories of Objects " published by the U.S. Patent Office on December 8, 1998 can also be obtained from the above site.

Technical papers recently produced by means of the claims of both the (TSA) and the (WA) patents having been reviewed, presented and published at international Conferences are provided in the following to our colleagues for review and evaluation of this submitted paper for ICONE10:

REFERENCES

PART I.
TECHINCAL PAPERS RELEVANT TO BOTH US PATENTS HAVING BEEN REVIEWED, PRESENTED AND PUBLISHED AT INTERNATIONAL CONFERENCES: 68pages.

I.1."BASIC NEEDS OF HUMAN BEINGS AS THE PURPOSES AND FOUNDATIONS FOR THE EXISTENCE OF GOVERNING INSTITUTIONS AND THE ADVANCEMENT OF SCIENCE AND TECHNOLOGY ". A paper presented at 2001 ASME International Mechanical Engineering Congress & Exposition on Tuesday, November 13, 2001, 9:30AM at Technical Session # E&TM-1 1 at Hilton New York/Sheraton New York, New York City, NY.USA.8 pages.

I.2. " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE (TSA) AND THE WONG'S ANGLES (WA) TO SOLVE PROBLEMS OF THAAD FOR BMDO AND FOR FUTURE MISSIONS OF NASA" US Copyright Registration Number TX5-375-549, April 19, 2001, presented at the Proceedings of the Fifteenth SSI/Princeton Conference on Space Manufacturing, May 7-9, 2001.14 pages.

I.3. "APPLICATIONS OF THE TRAJECTORY SOLID ANGLE (TSA) AND THE WONG'S ANGLES (WA) TO SOLVE PROBLEMS OF THAAD FOR BMDO AND FOR FURTURE MISSIONS OF NASA" Excerpts of document No. I.2 published in "SPACE MANUFACTURING 13 SETTLING CIRCUMSOLAR SPACE" Proceeding of the Fifteenth SSI/Princeton Conference on Space Manufacturing May 7-9, 2001.Page 98 to page 101. 4pages.

I.4. "APPLICATIONS OF TRAJECTORY SOLID ANGLE (TSA) AND WONG'S ANGLE (WA) TO SOLVE FUNDAMENTAL PROBLEMS IN PHYSICS AND ASTRONOMY " IAF-00-J.1.10 paper presented and published at 51st. International Astronautical Congress, 2-6 Oct 2000/Rio de Janeiro, Brazil. 5pages.

I.5. "APPLICATIONS OF TRAJECTORY SOLID ANGLE (TSA) AND WONG'S ANGLES (WA) FOR LAUNCHING OF SPACE VEHICLES " IAF-00-S.6.03 paper presented and published at the 51st. International Astronautical Congress, 2-6 Oct 2000/Rio de Janeiro, Brazil. 4 pages

I.6. "NUMERICAL DATA FOR SATELLITE ALTITUDE CONTROL BY MEANS OF WONG'S ANGLES" AIAA-96-1047-CP paper presented and published at the 16th. International Communications Satellite Systems Conference, February 25-29, 1996 Washington DC. page 517 to page 523, 7 pages.

I.7. "ON THE FORMULATION AND SOLUTION OF A CLASS OF MAGNETO-VISCOELASTO-DYNAMICS (MVD) GOVERNING EQUATIONS OF MOTION" presented and published at the 1995 ASME Design Engineering Technical Conferences-The 15th. Biennial Conference on Mechanical Vibration and Noise, September 17-20, 1995 Boston, Massachusetts, DE-Vol. 84-2 Volume 3- Part B page 1451 to 1456. 7pages.

I.8. "ON THE IRROTATIONAL-FLOW VELOCITY POTENTIAL FUNCTION AND A NEW STREAM FUNCTION OF FLUID MECHANICS" paper No. 80-C2/Aero-3 presented and published at the ASME Century 2 Aerosp ace Conference, San Francisco, California, August 13-15, 1980. 10p ages.

I.9. "ON THE UNIFIED GENERAL SOLUTIONS OF LINEAR WAVE MOTIONS OFTHERMOELAST ODYNAMICS AND HYDRODYNAMICS WITH PRACTICAL EXAMPLES "paper No. 67-APM-32 presented and published at the ASME Applied Mechanics Conference, Pasadena, California, June 26-28, 1967. 9pages.

PART II.
THREE (SRC) SYSTEMS RESEARCH COMPANY'S TECHNICAL PROPOSALS HAVING BEEN SUBMITTED TO NASA FOR SUPPOR T: 113p ages.

II. 1. SRC-NASA proposal No.NRA-96-HEDS-0 3-076 entitled "APPLICA TIONS OF THE TRAJ ECTORY SOLID ANGLE AND THE WONG'S ANGLES TO SOLVE FUNDAMENT AL PROBLEMS IN PHYSICS " submitted on March 21,1997.26 pages.

II. 2.SRC-NASA proposal No. TRIANA-0003-0006 entitled "APPLIACTIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLE FOR TRIANA" submitted on July 22, 1998. 36 p ages

II. 3. SRC-NASA proposal No.AIST-0042-0006 entitled " APPLIACTIONS OF THE TRAJ ECTORY SOLID ANGLE AND THE WONG'S ANGLES TO SUPPOR T ESTO PROGRAMS: AIST; ATI; IIP AND HPCC/ESS." Submitted on January 22, 2000. 51 pages.

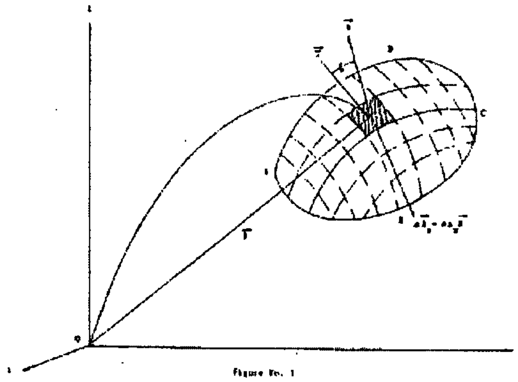


Figure No. 1

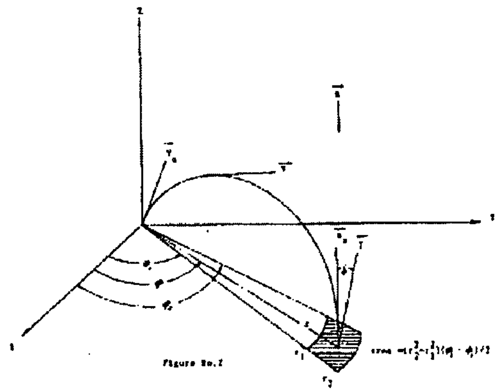


Figure No. 2

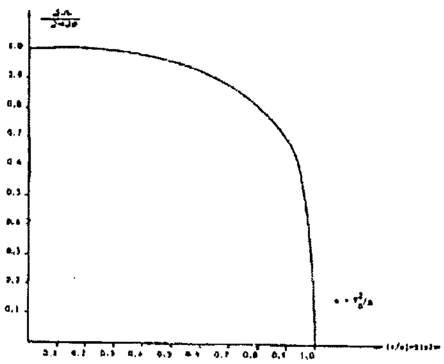


Figure No. 3

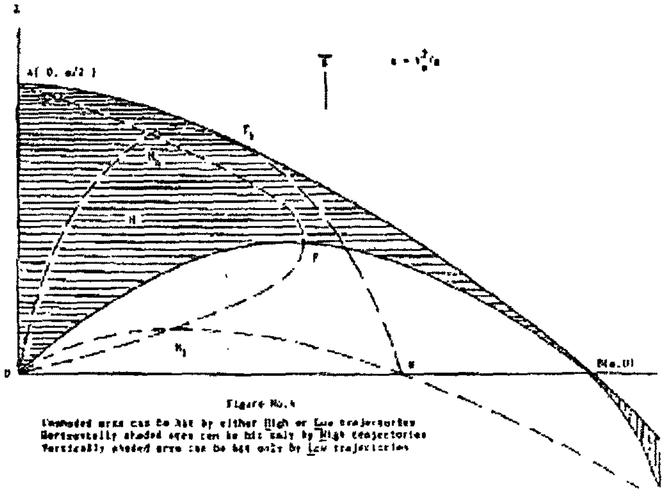
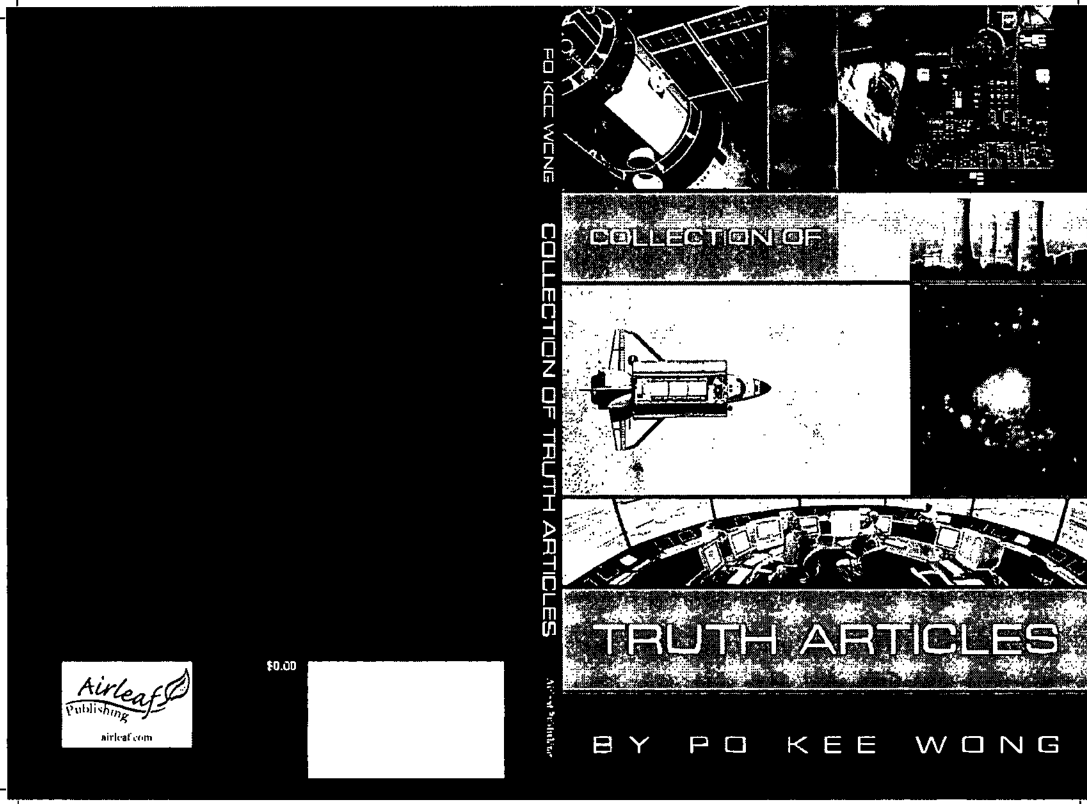


Figure No.4

Unshaded area can be hit by either High or Low trajectories
Horizontally shaded area can be hit only by High trajectories
Vertically shaded area can be hit only by Low trajectories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IMPACTS FROM NEW SOLUTIONS OF OLD PROBLEMS IN MATHEMATICAL AND EXPERIMENTAL SCIENCES

ABSTRACT

Submitted to

NINTH INTERNATIONAL CONFERENCE ON ENGINEERING STRUCTURAL INTEGRITY ASSESSMENT

ESIA9
"Engineering Structural Integrity ~ Research, Development and Application"
15-19 October 2007, Beijing, China

By
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In response to the **Second Call-for-Papers** from **ESIA9**, five papers are being submitted to all participants of our colleagues worldwide for open review and evaluation and to assess their impacts and values in mathematical and experimental sciences with their applications in **Engineering Structural Integrity**:

(1) IMECE 2001/T&S-23408 paper, 7 pages with partial section translation in Chinese.
(2) IMECE 2003-43540 paper, 3 pages.
(3) IMECE 2003-43536 paper, 5 pages.
(4) IAC-02-J.P.02 paper , 7 pages
(5) IMECE 2003-43586 paper, 3 pages.

The above 5 papers together with this abstract are being submitted electronically to Professor Sujun Wu (□□□)

2003 ASME INTERNATIONAL MECHANICAL ENGINEERING CONGRESS AND R&D EXPO Proceedings
IMECE2003
November 15-21, 2003, Washington DC USA

IMECE2003-43540

the uniquely-corrected method to compute high power functions

Po Kee Wong/SYSTEMS RESEARCH COMPANY (SRC) U.S. Federal Supply Code: 5R583

Adam Wong/SYSTEMS RESEARCH COMPANY(SRC) U.S. Federal Supply Code: 5R583

Anita Wong/SYSTEMS RESEARCH COMPANY (SRC) U.S. Federal Supply Code : 5R583

Abstract

The present paper provides a unique correction of the calculating procedures that have been prevailingly used in all computers and calculators for several decades. This correction must be made, based on the impacts of the computers and calculators have been extensively used in various fields of sciences, engineering, technologies and mathematics in education.

introduction

In recent years, several large companies that produce hand-held calculators have been competing continuously with one and other to break into the market for educators of all levels to learn and to use their calculators for teaching in the classrooms. During the summer months of June-August, 1995, many seminars were conducted in the metropolitan Boston Areas in the State of Massachusetts. CASIO offered workshops for the use of CFX-9800G; Texas Instruments offered workshops for the use of TI-82; Hewlett Packard offered for the use of HP 38G. Educators from the Boston Public Schools of the City of Boston, Massachusetts were invited and assigned to attend the workshops offered by CASIO and TI. Educators from everywhere were invited to attend the HP 38G workshops. It was undoubtedly that all the participants in these workshops were benefited from utilizing the calculators to implement their mathematics and science curricula in one way or the other. In particular, educators from the State of Massachusetts came for the workshops enthusiastically because they were given Professional Development Points (PDP) to fulfill the requirements for their re-certification to teach in the State of Massachusetts. Others came to seek for the choices of the appropriate calculator in order to implement their curricula effectively in their own classes. In addition to the above reasons, the first author of this paper also participated all the workshops in order to select the appropriate calculator for the Advanced Placement Calculus that was being offered at the Charlestown High School of the Boston Public Schools under the grant funded by the EAGLE program of the Boston Plan For Excellence in The Public Schools in academic year 1995-96. The author learned a lot from attending those workshops and also fed back his opinions that could and should be updated and to be built-in into the calculators for wider applications not only for teaching but also for research in Physics and Mathematics. For examples: special functions like circular cylindrical and spherical Bessel's Functions; Legendre Functions and Error Function should be built-in into the calculator to solve many problems in Physics and in Engineering; likewise the Largrange Interpolation Formula should also be built-in for curve fitting...etc. After finishing the participation of all four workshops in July-August 1995, the first author was asked by Mr. Richard Stutman, a BPS mathematics teacher and colleague working in the Boston Teacher's Union (BTU), to solve a fun-and- game problem that was involved in high power functions of infinite orders. Responding to his request, the author sought to solve the problem by means of the CFX-9800G; TI-82 and HP-38G. As a result of this effort, a major error in the procedures of calculating the high power functions was found simultaneously in all three calculators CFX-9800G; TI-82 and HP-38G. The major error had been corrected and filed for examination with the U.S. Patent Office in order to clear the legal liability problems from the companies.

SUMMARY OF THE CORRECTION

Mathematical procedures of calculation of a mathematical function in symbolic form can be defined in many ways almost at our own wills. However, there are examples that procedures and the symbolic expression of the mathematical functions will not be unique if one changes its standard calculating procedures. The power functions are some of these examples. The errors to calculate the high power functions contribute from CFX-9800G; TI-82; and HP-38G are that they all start from the base upward to the higher exponential power, while the correct way should be started downward from the top exponential power to the base. These can be cleared from the following examples A and B:

A. Errors in Numerical Computations

$$4^{3^{2}} = 64^{2} = 4096 \text{ is not correct}$$

$$3^{2^{3}} = 9^{3} = 729 \text{ is not correct}$$

$$2^{3^{2}} = 8^{2} = 64 \text{ is not correct}$$

B. Errors in Symbolic Representation uniquely Involved in Solving Equations of High Power Functions.

$x^{x^{x}} - 2 = 0$ means $x^{x^{2}} - 2 = 0$ leads to wrong answer x = 1.1414213562

$x^{x^{x^{x}}} - 2 = 0$ means $x^{x^{3}} - 2 = 0$ leads to wrong answer x= 1.336709735

$x^{(2x)^{(3x)}} -2 = 0$ means $x^{(6x^{2})} -2 = 0$ leads to wrong answer x = 1.100152079

DETAIL DESCRIPTION OF THE CORRECTION

The above examples A and B in errors can be corrected as the followings:

A. Correct Numerical Computation.

$$4^{3^{2}} = 4^{9} = 262{,}144$$

$$3^{2^{3}} = 3^{8} = 6{,}561$$

$$2^{3^{2}} = 2^{9} = 512$$

B. Correct Symbolic Representation Uniquely Involved in Solving Equations of High Power Functions.

$x^{x^{x}} - 2 = 0 \qquad x = 1.476684337$

$x^{x^{x^{x}}} - 2 = 0 \qquad x= 1.446601432$

$x^{(2x)^{(3x)}} - 2 = 0 \qquad x = 1.064146805$

$(3x)^{(2x)^{x}} - 2 = 0 \qquad x= .6140723908$

C. Examples of Correct Solutions of more Complicated Equations of High Power Functions.

2 2

$$x^{2\,(x\ \ )} - 5\,x^{(x\ \ )} + 6 = 0$$

x= 1.41421356 and x= 1.565552276

$$(x)^{2(x)^{(x)}} - 5\,(x)^{(x)^{(x)}} + 6 = 0$$

x = 1.476684337 and x = 1.635078475

Denote $y1(x) = x$; $y2(x) = x^{x}$; $y3(x) = x^{x^{x}}$;.....etc.

The solutions of the following equations of High Power Functions can be obtained:

$y2(x) - 2 = 0$ x = 1.559610469

.$y3(x) - 2 = 0$ x = 1.476684337

$y4(x) - 2 = 0$ x = 1.446601432

$y5(x) - 2 = 0$ x = 1.432694806

$y6(x) - 2 = 0$ x = 1.425385621

$y7(x) - 2 = 0$ x = 1.421227912

$y8(x) - 2 = 0$ x = 1.418734462

$y9(x) - 2 = 0$ x = 1.417182504

$y10(x) - 2 = 0$ x = 1.416190183

$y15(x) - 2 = 0$ x = 1.414502086

$y20(x) - 2 = 0$ x = 1.414258764

$y30(x) - 2 = 0$ x = 1.414214713

$y40(x) - 2 = 0$ x = 1.414213592

$y40(x) - 3 = 0$ x = 1.447839583

$y40(x) - 4 = 0$ x = 1.449395757

$y40(x) - 5 = 0$ x = 1.44979292

$y40(x) - 6 = 0$ x = 1.449978187

$y40(x) - 7 = 0$ x = 1.450087526

$y40(x) - 8 = 0$ x = 1.4501607

$y40(x) - 9 = 0$ x = 1.450213659

$y40(x) - 10 = 0$ x = 1.450254088

CONCLUSION

What is claimed is:

1. A unique method of calculating and solving equations involved with High Power Functions has been made for all current and future computers and calculators that are built-in with the wrong procedures to calculate the High Power Functions.

Acknowledgments

The first author of this paper thanks to his friend and colleague Mr. Richard Stutman of Boston Public Schools for the fun-and-game problem which led to the discovery of the errors for the calculation of High Power Functions in computers and calculators back to 1995.

References

All documents that have been filed with the U.S. Patent Applications No. 08/980,657 by Po Kee Wong since 1995.

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AN ALTERNATE COMPLETE SOLUTION OF THE "COCONUTS" PROBLEM IN NUMBER THEORY

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Abstract

The " Coconuts " problem was described in 1926 by Ben Ames Williams in a short story that appeared in The Saturday Evening Post. Solutions have been given by Dirac and by Whitehead, among others. Their solutions involve six indeterminate equations that have been reduced to a single Diophantine equation with two unknowns. We give a new alternate solution obtained by means of a geometric series.

introduction

As can be seen from the article in reference No. 1 by Gardner, the history of the "Coconuts" problem had been described in the October 9, 1926 issue of the Saturday Evening Post appeared as a short story by Ben Ames Williams. The proposed solutions of the problem had been previously involved with Noble Laureate Physicist Professor P. A. M. Dirac of the Cambridge University and by Mathematician Professor J. H. C. Whitehead of the University of Oxford in England. The older version of the problem can be expressed in six indeterminate equations that can further be reduced into a single Diophantine equation with two unknowns. Up to the present, this Diophantine equation can be solved by means of computer programming. The modern version of the older version of the Williams' "Coconuts" problem was re-stated by Herda in reference No. 2 as following:

" A pile of coconuts was collected by four woman, assisted by a monkey. During the night, one of the woman arose while the others were asleep and divided the nuts into four equal shares, with one nut left over which she gave to the monkey. She hid one share, put the rest of the nuts into a single pile and went back to sleep. In turn, each of the other three women went through the same procedure, and in each case there was always one nut to give to the monkey! In the morning, the four women divided the remaining nuts into four equal shares and again had one nut left over for the monkey. What is the smallest number of nuts could have been in the original pile? "

The problem was re-stated by Professor Herda of the University of Massachusetts at Boston to his students as a midterm take-home examination question and to write a computer program to solve the problem. The authors, without previous knowledge in Number Theory and knowing the paper by Gardner, provided an independent complete solution of the problem. The method of obtaining the complete closed-form solution is by means of a Geometric Series that appears in the formulation.

Nomenclature

Assuming that there are N nuts in the pile and that there are P women in the game. Denoting:

W1---the amount of nuts hidden by the first woman with R1 nuts left over.

W2---the amount of nuts hidden by the second woman with R2 nuts left over.

W3---the amount of nuts hidden by the third woman with R3 nuts left over.

Then

Wr---the amount of nuts hidden by the r th. Woman with Rr nuts left over.

Wp---the amount of nuts hidden by the p th. Woman with

Rp nuts left over.

Where r are positive integers between 1 and p.

Nuts hidden by each woman can be expressed in general Wr = (R sub(r-1) - 1) / P for r equal to and greater than 2 while

W1= (N –1) / P

Nuts left over in the pile Rr after each woman took her share can be expressed in general

Rr = (P-1) Wr

These two general expressions can be understood better by the followings:

W1 = (N-1)/P

W2 = (R1-1)/P =(N-1)(P-1) / P^2 – 1/P

W3 = (R2-1)/P= (N-1) (P-1)^2/P^3 –(P-1)/P^2 –1/P

And

R1 = (P-1) W1=(N-1)(p-1)/P

R2 =(P-1) W2=(N-1)(P-1)^2/P^2 – (P-1)/P

R3 =(p-1) W3=(N-1)(P-1)^3/P^3 – (P-1)^2/P^2 - (P-1)/P

Based on the above results, both Wr and Rr can be expressed, in general as:

Wr=(N-1)(P-1)^(r-1)/P^r-(P-1)^(r-2)/P^(r-1)…(P-1)/P^2-1/P.

Rr=(P-1)Wr=(N-1)(P-1)^r/P^r – ((P-1)P^r –P(P-1)^r)/P^r

And when r=P

Rp=(N-1)(P-1)^P/P^P- ((P-1)P^P-P(P-1)^P)/P^P
=(P-1) Wp ....Eq. (1)

Equation (1) represents the nuts left over in the next morning.

The P th. Woman hid the amount of nuts Wp is represented by Equation (2) expressed in the following:

Wp=(N-1)(P-1)^(P-1)/P^P–(P^P–P(P-1)^(P-1))/P^P.....(2)

Again in the next morning, the left over amount of nuts Rp in Equation (1) is to be divided by the P th. Woman and gave one to the monkey, then

(Rp – 1)/ P = I ....(3)

The net gain of nuts by the r th. woman is denoted by Tr.

Tr = Wr + I =Wr +(Rp-1)/P= Wr +((P-1) Wp-1)/P......(4)

Substituting Rp from Equation (1) into Equation (3) with further simplification and obtaining

(N+P– 1)(P – 1)^P = (1 + I) P^ (P+1) ....(5)

The problem is now reduced to find N and I as integers that satisfy Equation (5) for any given P as total number of women in the game.

It is when (N+P – 1) = P^ (P+1) and (1+I) =(P – 1)^P the left-hand side and the right-hand side of Equation (5) becomes identical to each other for any given P values, thus

N=P^(P+1) –P + 1 .... (6)

I= (P – 1) ^ P - 1...(7)

Are obtained as the general solution of the problem.

The solution represented by equations (6) and (7) is the smallest solution because the numbers on both sides of Equation (5) are divisible by (P – 1)^P x P ^(P+1)

The complete general solution of the problem can now be summarized in the following:

P = Total number of women in the game

The total number of nuts in the pile is N

N=P^(P+1) –P +1

The amount of nuts that the r th. woman hid is Wr

Wr=(P-1)^(r-1) P^(P+1)/P^r – 1

The amount of nuts that the P th. woman hid is Wp

Wp = P (P – 1)^(P – 1) – 1

The total number of nuts that the r th. woman got is Tr

Tr = Wr +I =(P-1)^(r-1) P^(P+1) / P^r +(P-1)^P – 2

The total number of nuts that the P th. woman got is Tp

Tp = Wp + I =P (P-1)^(P-1) + (P-1)^P - 2

The last pile of nuts that each woman got is I

I = (P – 1) ^P – 1

The monkey got (P + 1) coconuts

All these results can also be counter-checked from the total number of nuts in the pile that should be equal to the sum of all nuts that each woman got plus the nuts that the monkey got, that is

N = Summation of Tr (from r=1 to r=P) + (P + 1)
=Summation of Wr (from r=1 to r=P) + PI + (P+1)
=Summation of Wr (from r=1 to r=P) + P (P-1)^P+1
=Summation of ((P-1)^(r-1) P^(P-r+1) –1) (from r=1 to r=P) + P (P-1)^P +1
= P^(P+1) – P +1

The above general and complete solution of the problem can be used to make numerical check for the older version of the Williams' " Coconuts" problem as indicated by the paper by Gardner by setting P = 5 sailors then N = 15,621 nuts. When P becomes very large, there will be as many indeterminate equations to be set up according to the paper by Gardner. The reduction of these indeterminate equations into the single Diophantine equation and then seeking the solution of the Diophantine equation is extremely tedious and time consuming even with the help of a powerful computer.

Based on the presentation of this paper, the advantages of the complete closed-form solution over the previous solution by means of the Diophantine equation is therefore distinctive.

CONCLUSIONS

1. The current complete solution is in closed –form without solving the simultaneous linear equations of congruence.
2. Its advantages over all other previous solutions by means of the reduced Diophantine equations are:

a. Saving a great deal of computing time to obtain the solutions especially when the number of women in the game is a very large number.
b. Providing the complete information of the solution of the problem that has never been obtained before : such as Wr, Wp, Tr, and Tp terms as described in this paper.

3. Equation No. (6) N = P^(P+1) – P + 1 can be integrated into the current computer programs worldwide to search for the largest prime numbers continuously.

Acknowledgments
The authors would like to thank Professor Herda for his comments about the authors' independent complete solution of the "coconuts" problem and his reference to search Gardner's paper. The authors are also thankful to Adele Premice of Scientific America for providing information to obtain Gardner's papers. The authors appreciate Professor Robin Brooks and Professor Richard Sampson of Bates College to have revised the abstract of the paper and making it concise and precise to have aroused the attentions from all the mathematicians in the world. The authors also thank Mr. L. Rizzo of Bates College for producing the video record of the presentation of this paper at Bates College on August 9, 1995.

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APPLICATION OF THE WONG'S FORMULA FOR GEAR DESIGN AND ANALYSIS

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Abstract

Even though the famous ancient Greek Appollonius problem had been solved by many others by means of the Coordinate Geometry (Analytical Geometry) as indicated, for example , from page 56 to page 59 in the book " CRC Concise Encyclopedia of Mathematics" by Eric W. Weisstein in (Reference No. 1), the complete closed-form solution of the problem by means of Trigonometry and Plane Geometry that is independent from the coordinate systems has never been obtained until it was first derived, obtained and filed for a U.S. Patent Application by the first author of this paper on October 31, 1994 (Reference No. 2).

The closed-form formula can be applied for CAD/CAM packaging of circular cylindrical objects. They are simple to be used, cost effective and that they also provide the highest precision and accuracy in closed-form equations containing only six parameters. With all advantages over all other and previous methods, the Wong's Formula are highly competitive in worldwide market for automobile, ship and airplane manufacturing industries.

Introduction

The Wong's Formulas are simple to be used almost by anyone who understands basic mathematics and/or algebra I. The inventor of the Wong's Formulas had already submitted the specification of the invention containing the Formulas in a sealed envelope to the U.S. Patent Office because of the simplicity of the Wong's Formulas can be easily infringed. The submitted Wong's Formulas should remain to be sealed before and even after the application is to be granted until when the U.S. Patent Law fails to protect this patent.

The procedures of using the Wong's Formulas can best be described in the following with reference to the notations shown under each figure:

(1) Measure or design to set the values of a, b, c, r sub.A, r sub.B, r subC. by lengths that fulfill the conditions for non-penetrating gears.

(2) Substitute these values into the Wong's Formulas r=r(a,b,c,rsub.A,rsub.B,rsub.C), then r can be calculated precisely and accurately.

(3) The center of the constructed fourth gear W sub.I can be determined from points A, B, C, with:

(4) AWsub.I=Rsub.A=r+rsub.A
Bwsub.I=Rsub.B=r+rsub.B
Cwsub.I=Rsub.C=r+rsub.C where rsub.A,rsub.B,rsub.C are positive values when the fourth gear is tangented externally to the three fixed given gears A,B,C respectively as being shown by Figure No.1. They should be set as negative values when the fourth gear is tengented internally to the three fixed given gears A,B,C respectively as being shown by Figure No.2. Figures No.3 to No.8 show clearly the meaning of the positive and negative values of rsub.A, rsub.B, rsub.C relevant to the configurations of the arrangement of the designed gears.

Nomenclature

Figure No. 1 to No. 8 provide a specific numerical example of how to use the Wong's Formulas to determine the locations of the centers of the fourth gears and their radii tangented to the other three given gears. As can be seen from these 8 figures, there are 8 different solutions for the problem.

Figures No.9 to No. 10 provide degenerated cases of the problem represented by Figures No.1 to No. 8. The three given gears are initially tengented to one and others. No movements can be transmitted from one to the other. Two solutions as indicated by the Figures are possible. These two Figures show how to apply the Wong's Formulas for static packaging of objects of circular cylindrical shapes. The Wong's Formulas can be repeatedly applied to calculate the centers and new radii of other circular cylindrical objects to fill in the emty spaces bound by the three initially tangented circles.

Figure No. 11 provides a specific example of how to use the Wong's Formulas to find the center of the fourth gear such that the sum of its distances to the centers of the other three given gears is a minimum. That is to find (CWsub.S + AWsub.S + BWsub.S) = 6.7664362 = minimum

Figure No. 12 is the general figure used to derive the Wong's Formulas that should be described in the followings:

A, B, C, are the center of rotation of each gear A, B, C respectively.

rsub.A, rsub.B, rsub.C are the radii of each gear A, B, C respectively.

Segment BC=a, Segment CA=b, Segment AB=c are the distances separated the rotation axis of gears A, B, C respectively.

The radius of the fourth gear to be constructed and tangented to the three given gears is denoted as r.

Rsub.A, Rsub.B, Rsub.C are to be used to locate the centers Wsub.I of the fourth gears. They are measured from centers A, B, C respectively with the calculated values of r that can be obtained from the Wong's Formulas as r=r(a,b,c,rsub.A,rsub.B,rsub.C).

The Wong's Formulas holds true for the following conditions for non-penetrating gears:

(a+b) is greater than or equal to c, (b+c) is greater than or equal to a, (c+a) is greater than or equal to a and that a is greater than or equal to (rsub.B +rsub.C) , b is greater than or equal to (rsub.C + rsub.A), c is greater than or equal to (rsub.A + rsub.B).

Acknowledgments

The author Po Kee Wong appreciates the administrators and examiners of the U.S. Patent Office for their time and their efforts to have examined all the Patents that have been submitted to them for a fair examination according to the U.S. Patent Law .

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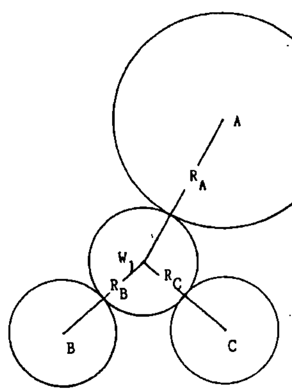
2. Po Kee Wong, " WONG'S FORMULA FOR CAD/CAM PACKAGING OF OBJECTS " U.S.Patent Application Number 08/331,980 filed on 10/31/94

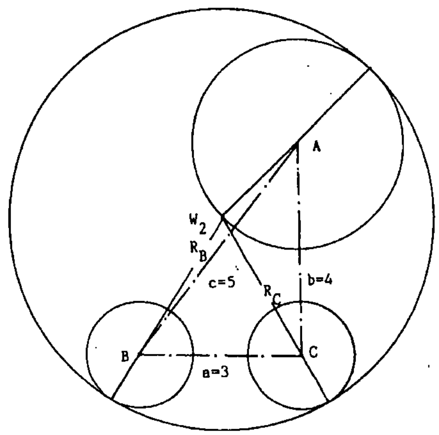
Figure No.1



a=3 b=4 c=5 ; r_A=2 r_B=1 r_C=1 ; radius r =1.0298221

R_A= AW_1=r+r_A=3.0298221 R_B=2.0298221 =R_C centered at W_1

Figure No.2

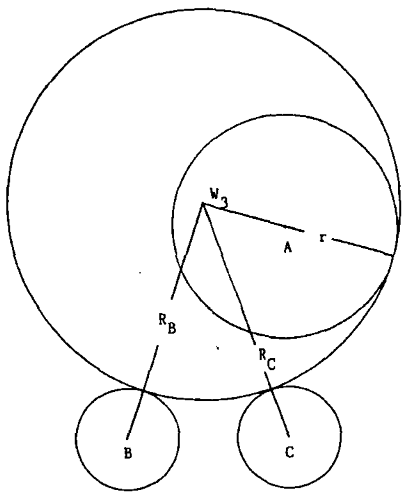


a=3 b=4 c=5 ; r_A=-2 r_B=-1 r_C=-1 ; radius r =4.0298221

R_A=AW_2=2.0298221 R_B=BW_2=3.0298221 R_C=CW_2=3.0298221

centered at W_2

Figure No.3

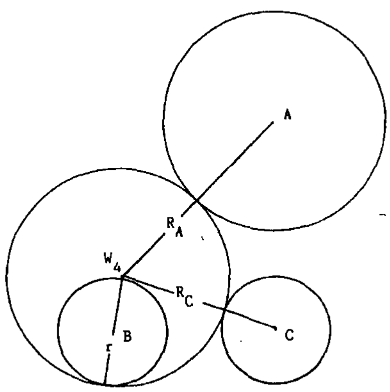


$a=3$ $b=4$ $c=5$; $r_A=-2$ $r_B=1$ $r_C=1$; radius $r=3.5237158$

$R_A=AW_3=1.5237158$ $R_B=BW_3=4.5237158$ $R_C=CW_3=4.5237158$

centered at W_3

Figure No.4

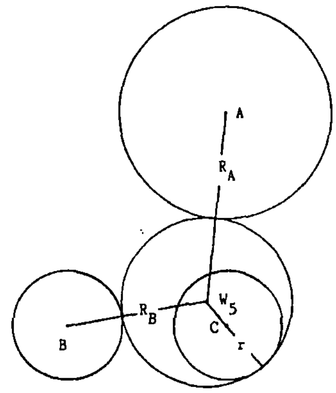


$a=3$ $b=4$ $c=5$; $r_A=2$ $r_B=-1$ $r_C=1$; radius $r=2.1034662$

$R_A=AW_4=4.1034662$ $R_B=BW_4=1.1034662$ $R_C=CW_4=3.1034662$

centered at W_4

Figure No. 5

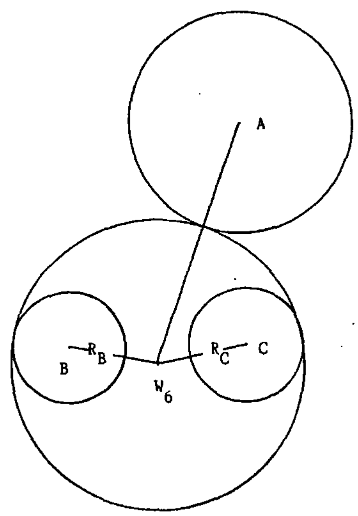


$a=3$ $b=4$ $c=5$; $r_A=2$ $r_B=1$ $r_c=-1$; radius $r=1.603479$

$R_A=AW_5=3.603479$ $R_B=BW_5=2.603479$ $R_C=CW_5=0.603479$

centered at W_5

Figure No. 6

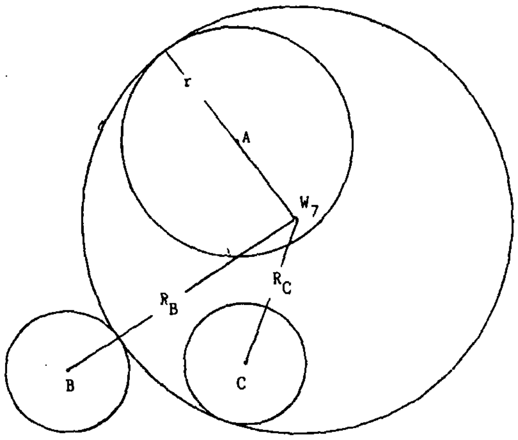


$a=3$ $b=4$ $c=5$; $r_A=2$ $r_B=-1$ $r_C=-1$; radius $r=2.5237158$

$R_A=AW_6=4.523715$ $R_B=BW_6=1.523715$ $R_C=CW_6=1.5237158$

centered at W_6

Figure No.7

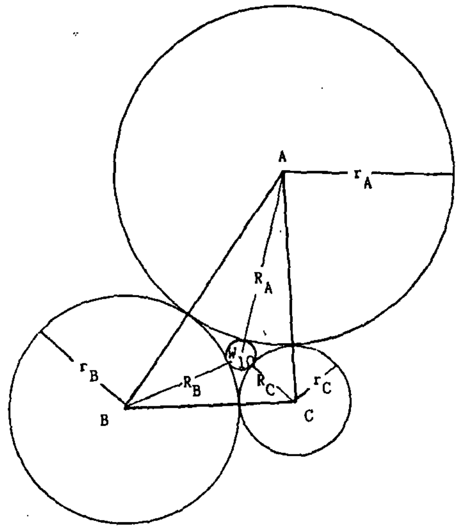


a=3 b=4 c=5 ; r_A=-2 r_B=1 r_C=-1 ; radius r =3.7513535

R_A=AW_7=1.7513535 r_B=BW_7=4.7513535 R_C=CW_7=2.7513535

centered at W_7

Figure No.10

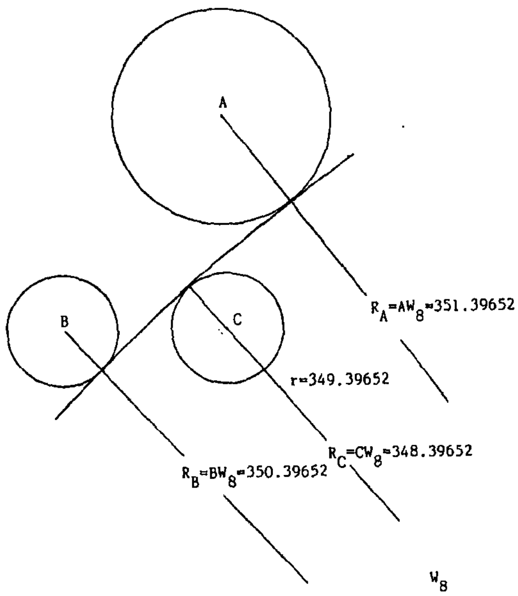


a=3 b=4 c=5 ; r_A=3 r_B=2 r_C=1 ; radius r = 6/23

R_A= AW_{10}= $3\,\frac{6}{23}$ R_B= BW_{10}= $2\,\frac{6}{23}$ R_C= CW_{10}= $1\,\frac{6}{23}$

centered at W_{10}

Figure No.8

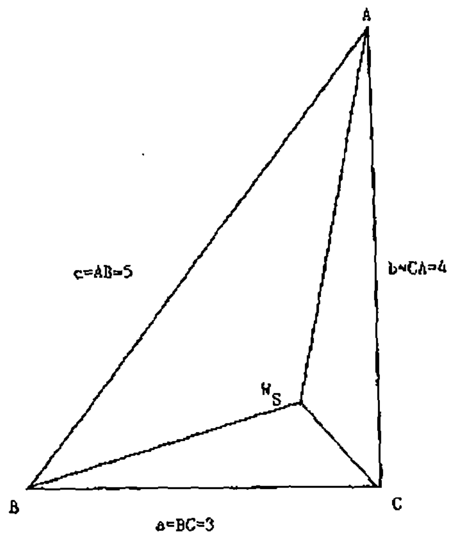


a=3 b=4 c=5 ; r_A=-2 r_B=-1 r_C=1 ; radius r =-349.39652

R_A=AW_8= -351.39652 R_B=BW_8= -350.39652 R_C=CW_8= -348.39652

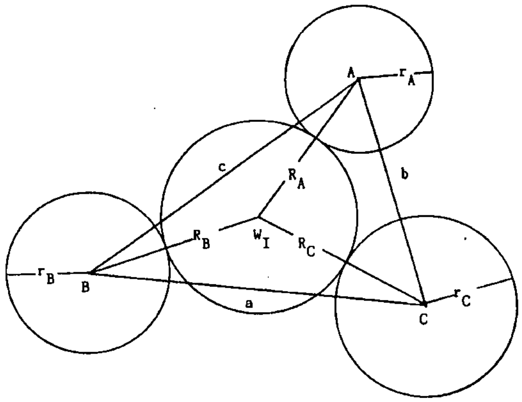
centered at W_8

Figure No. 11



CW_S= 1.0239078 AW_S= 3.3885249 BW_S=2.3540035

Figure No. 12



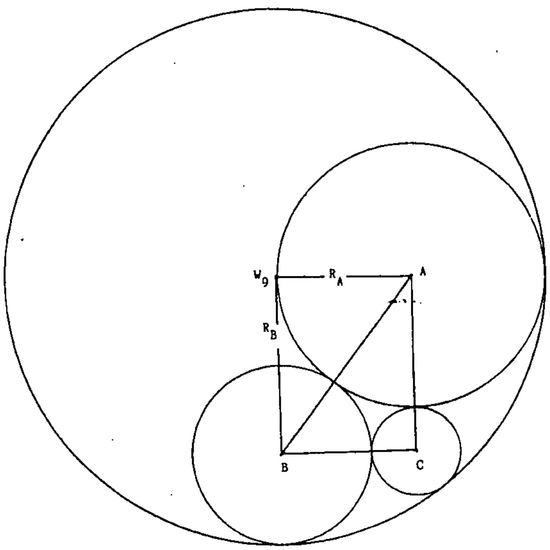
a=BC b=CA c=AB ; r_A, r_B, and r_C each is the radius of circle A,B,C respectively

r is the radius of the circle to be constructed and tangented with the given circles A,B,C respectively

$R_A = AW_I = r + r_A$ $R_B = BW_I = r + r_B$ $R_C = CW_I = r + r_C$

W_I is the center of the circle to be constructed

Figure No. 9



a=3 b=4 c=5 ; $r_A = -3$ $r_B = -2$ $r_C = -1$; radius r=6

$R_A = AW_9 = r + r_A = 3$ $R_B = BW_9 = r + r_B = 4$ $R_C = CW_9 = r + r_C = 5$ centered at W_9

IAC-02-J.P.02

COMPARISON OF TRAJECTORY SOLID ANGLE WITH GEOMETRIC SOLID ANGLE IN SCATTERING THEORY

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IAC-02-J.P.02

COMPARISON OF TRAJECTORY SOLID ANGLE WITH GEOMETRIC SOLID ANGLE IN SCATTERING THEORY

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ABSTRACT

The objective of the paper is openly to invite all physicists, mathematicians and engineers in the world to re-examine and to confirm the ultimate truth and the worldwide impacts of two U.S. Basic Patents No. 5,084,232 and No. 5,848,377 which can be obtained from: http://164.195.100.11/netahtml/srchnum.htm
The application of Trajectory Solid Angle (TSA) to obtain the correct collision cross-sections in Nuclear Physics and in Astronomy by the example of obtaining the correct scattering cross-section of the well-known Alpha Scattering was shown in a paper IAF-00-J.1.10. entitled " Applications of Trajectory Solid Angle (TSA) and Wong's Angles (WA) Solving Fundamental Problems in Physics and Astronomy " presented and published at the 51st. International Astronautical Congress, 2-6 Oct 2000/Rio de Janeiro, Brazil.

The Alpha Scattering was done in theory and in experiment by Sir Rutherford. The differential scattering cross section derived from using the geometric solid angle can be seen from all the textbooks of physics in the world. However, the differential scattering cross section derived from using the TSA has not been known by most of our colleagues in the world and it is different from the previous results. The present and the previous theoretical results converge to be the same only when the Alpha particle is far away from the stationary heavy nucleus. That was where Sir Rutherford made his measurement and therefore the old theory and the experiment were confirmed. The Alpha Scattering is really similar to the scattering of the Comet Halley by our solar system even though they are under the actions of different force fields. In 1976-79, the senior author of this paper

communicated with JPL of NASA and urged JPL to conduct an experiment to confirm the curvature effects of the trajectory of the Comet Halley coming closer to our solar system in those years. It is unfortunate that the communications have never been answered even up to now. Without repeating the analysis, the trajectory equation can be expressed by means of the spherical coordinate system that was shown in the paper IAF-00-J.1.10 Ref. 2 Figure No.1 and in Ref.3 Figure No. 1 for axially symmetric motion. The scattered particle is restricted in a plane surface which is perpendicular to the XOY plane; the particle is coming from the negative Z axis going upward as the scattering angle @ that is measured from the XOY plane as zero degree. After going through all the process as indicated in the above example A. of the paper IAF-00-J.1.10, the differential trajectory solid angle can be obtained as

(2 pi (e)(Sin @)^2 d @)/((1+eCos @)^2 +(e Sin @)^2) ^(1/2) (1)

that is obviously different from the usual differential geometric solid angle obtained as 2pi Sin @ d@ (2)

where e is the eccentricity of the orbit and pi = 3.14159.......

This paper will be concentrated in the presentation and publication of the graphical results of numerical data obtained from the above two differential scattering cross-sections in greater detail in order to distinguish the differences by Comparison of the Trajectory Solid Angle (TSA) with the Geometric Solid Angle (GSA) in Scattering Theory for the Central Force Fields.

INTRODCUTION

As mentioned in the above Abstract, in order to fulfill the objectives of this paper, the following direct technical communications with the 2001 Nobel Laureate Professor Ketterle of MIT are directly quoted without excerptions and modifications in the following for open invitation of other physicists; mathematicians; engineers and other relevant scientists to join our open reviews and evaluations either to confirm or to deny the truth of this paper in Scattering Theory for the Central Force Fields in Physics:

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Dear Professor Ketterle:
Thank you for your E-mail replying with reference to the subject matters and giving me your web site to search for your technical papers contributing your success and achievements in BEC.
As a result of searching for your technical papers, I found that two of your papers are highly educational to me with possibility of creating our mutual technical interests. They are:
(1) " Collective enhancement and suppression in Bose-Einstein condensates " by Wolfgang Ketterle and Shin Inouye, January 23, 2001, Lecture notes of the Cargese Summer School 2000. 38 pages.
(2) " Does Matter Wave Amplification Work for Fermions?" by Wolfgang Ketterle and Shin Inouye, PHYSICAL REVIEW LETTERS Volume 86, Number 19, 7 May 2001. page 4203 to 4206. 4 pages.
The particular technical areas in these two papers (1) and (2) that we may have mutual interests to discuss and share are that:
In your paper No.(1) page 2 under the title of I. SCATTERING OF LIGHT AND MASSIVE PARTICLES Equation (1), the Hamiltonian contains QUOTED " The strength of the coupling is parametrized by the coefficient C (which in general may depend on the momentum transfer) " UNQUOTED . The term C appears

again in Equations (11), (12) on page 5; Equation (16) on page 6; Equations (19), (21) of page 8; Equations (26), (27) on pages 9 and 10.
In particular, Equation (26) C = (4pi *h^2/MV)a where "a" is now the scattering length for collisions between condensate and impurity atoms. 4pi a^2 = collision cross-section.
In your paper No. (2), page 4203 under the title of "Scattering theory.---"
"Scattering Theory" relevant to the calculation of " Collision Cross-section" have been repeatedly demonstrated in my patents and the IAF -00--J.1.10 paper entitled " APPLICATIONS OF TRAJECTORY SOLID ANGLE (TSA) AND THE WONG'S ANGLES (WA) SOLVING FUNDAMENTAL PROBLEMS IN PHYSICS AND ASTRONOMY "
It is my humble opinion that the example B. Application of TSA to obtain the correct collision cross-sections in Nuclear Physics and in Astronomy by the example of obtaining the correct scattering cross-section of the well-known Alpha Scattering in the paper IAF-00-J.1.10 should be related to our mutual interests.
Based on the above in-depth technical discussions, I hope that you are willing to provide your expertise to examine a copy of my SRC-NASA proposal No. NRA-96-HEDS-03-076 entitled " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE AND THE WONG'S ANGLES TO SOLVE FUNDAMENTAL PROBLEMS IN PHYSICS " March 21, 1997 which is also the reference No.6 in the IAF-00-J.1.10 paper.
Your are cordially invited to participate many of the proposed tasks listed in the SRC-NASA proposal NRA-96-HEDS-03-076 of your choices. The proposal (29 pages) is being sent to you by fax to you at No. 1-617-253-4876 (your fax. number).
With my best regards, I am,
Sincerely yours,
Po Kee Wong, Ph.D. & P.ME.
CEO, SYSTEMS RESEARCH COMPANY (SRC)
U.S. Federal Supply Code: 5R583 for Mechanization Of Contract Administration Services (MOCAS)
50 Bradley Street, Somerville, Mass. 02145-2930 USA
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-----Original Message-----
From: Wolfgang Ketterle <ketterle@MIT.EDU>
To: pokwong <pokwong@massed.net>
Date: Sunday, October 14, 2001 2:45 PM
Subject: Re: (1) Congratulation to you for winning the Nobel Prize (2) Sending you two US basic Patents and 5 international technical papers to assess their truth relevant to the Bose-Einstein Statistics that is the foundation of your BEC

At 11:45 PM 10/13/2001 -0400, you wrote:

> Dear Professor Ketterle:
>
> The original E-mail to you was failed to be executed by the E-mail server.
> Please excuse the separate E-mails being sent to you in piecewise condition.
>
> Your time and effort to help me out will be gratefully appreciated. I look forward to hearing from you about the subject matter No. (2) again.
>
> Very truly yours,
>
> Po Kee Wong

Thanks for your congratulations.
I am not an expert with regard to your other request.
Regards
Wolfgang Ketterle

Wolfgang Ketterle
John D. MacArthur Professor of Physics
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MIT-Harvard Center for Ultracold Atoms,
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NUMERCIAL DATA AND GRAPHICAL PRESENTATION

The most recent data about the Comet Halley can be obtained from:

http://www.ssd.jpl.nasa.gov/horizons_doc.html

Go to Appendices/Examples and Click on Comet data screen then the followings will be shown:

JPL/DASTCOM3 Halley 1997-Apr-02 11:13:57 Rec #: 20181 soln data arc: 1835-1989 # obs: n.a. FK5/J2000.0 osculating elements (AU, DAYS, DEG, period in Julian years): EPOCH= 2446480.5 != 1986-Feb-19.0000000 (TDB) EC= .967276875 QR= .587103582 TP= 2446470.9589491 OM= 58.8601271 W= 111.8656638 IN= 162.2421694 A= 17.94154996 MA=.1237401 ADIST= 35.295996338 PER= 75.9973 N= .012969228 ANGMOM= .018487217 DAN= 1.80537 DDN= .84911 L= 191.5461888 B= -56.6792985 TP= 1986-Feb-09.4589491 Physical & non-grav parameters (KM, SEC; A1 & A2 in AU/d^2): GM= n.a. RAD= 5.6 A1= 3.88D-10 A2= 1.55D-10 M1= 5.5 M2= 13. k1= 8. k2= 5. PHCOF= .030 COMET comments 1: soln ref.= IHW 61, radius ref. is Belton,M (1991) 2: k1=8, k2=5, phase coef.=0.03; ref. for magnitude laws is ICQ 1994 Handbook
****
****
Select ... [E]phemeris, [F]tp, [K]ermit, [M]ail, [R]edisplay, ?, :

==========================

Based on the above data, the orbital equations can be described in the followings:

GIVEN: (1)R(MIN)=0.587103582 AU ; (2) ANGLE OF INCLINATION OF COMET HALLEY'S OBITS TO THE ECLITIC PLANE = 18 DEGREES; (3) THE PERIOD OF THE COMET HALLEY REVOLVING AROUND THE SOLAR SYSTEM = T = 75.9973 YEARS ; (4) THE AVERAGE DISTANCE FROM THE CENTER OF THE SUN TO THE CENTER OF THE EARTH IS DEFINED AS ONE ASTRONAUTICAL UNIT (AU)

FIND: ALL OTHER OBITAL ELEMENTS AND RELEVANT PHYSICAL QUANTITIES OF THE COMET HALLEY.

RESULTS: R(MAX)=35.295996338 AU ; SEMI-MAJOR AXIS = a =17.94154996 AU ;SEMI-MINOR AXIS=b=4.552186934 AU; DISTANCE BETWEEN THE CENTER OF THER ELLIPTICAL ORBIT OF THE COMET HALLEY TO THE FOCUS (CENTER OF THE SUN) = c =17.35423961 AU; ECCENTRICITY OF THE ORBIT e=0.967276875 ; DISTANCE FROM THE POINT OF INTERSECTION OF THE ORBIT WITH THE ECLITIC PLANE TO THE CENTER OF THE SUN = 1.154967099 AU

Then, the orbital equation of the Comet Halley in plane motion about the Sun is

r = a (1- e^2)/(1 +eCos@) (3)

Based on Equation (3), the following Equations (4), (5) and (6) can be derived from using the Trajectory Solid Angle Patent No. 5,084,232 .
The differential Trajectory Solid Angle with respect to the differential Scattering angle d@ and divided by 2pi is

r1= e(Sin @) ^ 2 / ((1+eCos @) ^2 + (e Sin @)^2)^(.5) (4)

The differential Geometric Solid Angle with respect to the differential scattering angle d@ and divided by 2pi is

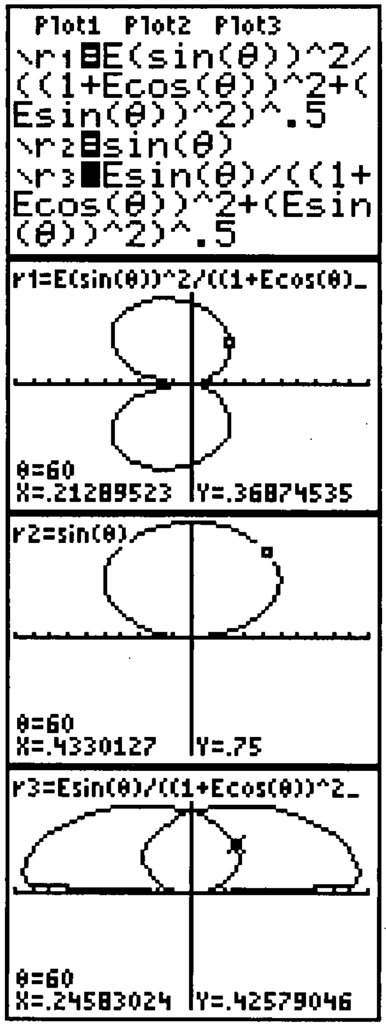
r2 = Sin @ . (5)

The RATIO of the Differential Trajectory Solid Angle (TSA) to the Differential Geometric Solid Angle (GSA) is

r3= r1/r2 = e Sin @ / ((1 + e Cos @) ^ 2 + (e Sin @) ^2)^(.5) (6)

Equations (4), (5) and (6) were first shown on page No. 12 in the 1979 SRC-DOE proposal No.P7900450 that is the reference No.7 in the U.S. Patent No. 5,084,232.

Equations (4), (5) and (6) can be typed into a TI-83 calculator. Each equation can be graphed as a function of the scattering angle @.
where E=e=0.967276875. in the TI-83 Equations (4), (5) and (6) is the Eccentricity of the orbit of Comet Halley about the Sun.



CONCLUSIONS

The claims and impacts of the Basic Patents No. 5,084,232 and No. 5,848,377 can be seen and traced from the Website given in the abstract. The claims are also summarized in the

IMECE 2001 paper recently presented in New York City on November 13, 2001.

However, it is very amusing to provide the old conclusions written more than 23 years ago in the SRC-DOE 1979 proposal No. P7900450 on page No. 22:

1. The probability distribution function for a particle striking a designated area, given the parameters of its generation and ejection is presented. It is being initiated and defined as the ratio of the trajectory solid angle, being subtended to the point from where the particle is released, over 4pi. The trajectory solid angle deviates but includes the geometric solid angle as a special case. It is defined from the definition of the incident angle of a trajectory that intersects the surface to be struck by the particle. The trajectory of the particle is governed by the fundamental laws in classical and modern physics.

2. The present result, in contrast with the technique presented by Maxwell-Boltzmann statistics, is derived from the basic principles of classical mechanics and differential geometry. If the presentation is reviewed and confirmed to be correct, the subject may provide an impact on the theory of statistical mechanics and quantum mechanics of modern physics thus affecting the theory and its interpretation of experimental results.

3. .Important applications of the subject are widely open in numerous areas from problems in physics to astronomy, from aerospace engineering, nuclear safety to precision design of electronic instruments.

REFERENCES

1. Wolfgang Ketterle and Shin Inouye, "Collective Enhancement and Suppression in Bose-Einstein Condensates " January 23, 2001, Lecture notes of the Cargese Summer School 2000. 38 pages.
2. Wolfgang Ketterle and Shin Inouye, " Does Matter Wave Ampliflication Work for Fermions ? " PHYSICAL REVIEW LETTERS Volume 86, Number 19, 7 May 2001. Page 4203 to 4206. 4 pages.

3." Horizons User Manual " from: http://www.ssd.jpl.nasa.gov/horizons_doc.html

4. " BASIC NEEDS OF HUMAN BEINGS AS THE PURPOSES AND FOUNDATIONS FOR THE EXISTENCE OF GOVERNING INSTITIONS AND THE ADVANCEDMENT OF SCIENCE AND TECHNOLOGY " A paper presented at 2001 ASME International Mechanical Engineering Congress & Exposition on Tuesday, November 13, 2001 , New York City, NY. USA. 8 pages.

5. " APPLICATIONS OF TRAJECTORY SOLID ANGLE (TSA) AND THE WONG'S ANGLES (WA) TO SOLVE FUNDAMENTAL PROBLEMS IN PHYSICS AND ASTRONOMY" IAF-00-J.1.10 paper presented and published at 51st. International Astronautical Congress, 2-6 Oct

2000/Rio de Janeiro, Brazil.

6. " APPLICATIONS OF THE TRAJECTORY SOLID ANGLE (TSA) AND THE WONG'S ANGLES (WA) TO SOLVE PROBLEMS OF THAAD FOR BMDO AND FOR FUTURE MISSIONS OF NASA " published in " SPACE MANUFACTURING 13 SETTLING CIRCUMSOLAR SPACE " Proceeding of the 5th. SSI/Princeton Conference on Space Manufacturing, May 7-9,2001. Page 98 to page 101. 4 pages.

Waves in Viscous Fluids, Elastic Solids, and Viscoelastic Materials

A Dissertation
Submitted to the Department of Aeronautics and Astronautics and the Committee on the Graduate Division of Stanford University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

By
Po Kee Wong

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ISBN: 1-60002-009-7

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Approved for the University Committee on the Graduate Division:

Dean of the Graduate Division

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professor Max Anliker for the opportunity to participate in the Stanford biomechanics research program. With his encouragement ample time and freedom were provided to develop the present theory as a basis for further research.

The author is also indebted to Professors I-Dee Chang and C. Chao for their critical comments and for reading the manuscript.

Mrs. Jane Fajardo and Miss Kathy Hunt have contributed their help in editing and typing the manuscript, and the author would like to express his most sincere thanks to them.

Financial support from the United States Office of Naval Research under Contract No. N000 14-67-A-0112-0007 is gratefully acknowledged.

LIST OF SYMBOLS

a_i radii of the concentric spherical shells with $i = 1,2,3,4,$ ...identifying the number of the layer

$\vec{A}_m$ vector potential function with $m = 1,2,3,4,...$ identifying the kind of material

$\left.\begin{array}{l} A_{1m} \\ A_{2m} \\ B_{1m} \\ B_{2m} \end{array}\right\}$ arbitrary constants

b geometrical length

c_o sound speed of the viscous fluid

c_{dm} dilatational wave speed in an infinite viscoelastic medium of material m

c_{sm} shear wave speed in an infinite viscoelastic medium of material m

D partial differential operator with respect to time

e base of natural log $e = 2.732$ ...

$\in$ onedimensional strain, small number of a ratio of two characteristic lengths, perturbation of wave number, or small radius

E Young's modulus of the elastic material

f frequency of vibration

h thickness of a spherical shell

i subscript i =1,2,3,4,...

j subscript j =1,2,3,4,...identifying the order of the vibrational mode

ϑ_n spherical Bessel function

k_{dm} dilatational wave number

k_{sm} shear wave number

$c_m = 4k^2_{dm} / k^2_{sm}$

k bulk modulus of an elastic solid

m subscript identifying the kind of material

n degree of Legendre's function Pn (cos θ), or the meridianal wave number of the spheres or spherical shells, or the order of Bessel's functions ϑ_n and Y_n

P instantaneous pressure

P_o equilibrium pressure

P_e excess pressure (perturbation pressure) $P_e = P - P_o$

r spherical radial coordinate

T period of vibration

t time as an independent variable in the governing equations

t_d attenuation coefficient (time constant) of overdamped or damped motion

$\vec{u}$ displacement vector

$\dot{\vec{u}}$ velocity vector

$\vec{u}_m$ displacement vector of a continuum of material m

$\vec{u}_{rm}$ radial component of the displacement vector of a continuum of material m

Y_n spherical Bessel's function

Z mechanical resistance or impedance of a viscoelastic material model; Z is equivalent to the reciprocal of Young's modulus $\frac{1}{E}$ in the case of an elastic solid, otherwise it is a linear operator

Z_d dilatational mechanical resistance of a viscoelastic material model; Z_d is equivalent to the reciprocal of Lamé's constant $\frac{1}{\lambda}$ in the case of an elastic solid, otherwise it is a linear operator

Z_s shear mechanical resistance of a viscoelastic material model; Z_s is equivalent to the reciprocal of Lamé's constant $\frac{1}{\mu}$ in the case of an elastic solid, otherwise it is a linear operator

$$\left.\begin{array}{l} P \\ P' \\ P'' \\ Q \\ Q' \\ Q'' \end{array}\right\}$$ Flügge's[41)] viscoelastic linear operators

Y_μ / Y_s Bland's[42)] viscoelastic moduli

μ / λ Lamé's constants in elastic solids

$$\lambda_m = \frac{1}{Z_d}$$

$\mu_m = \frac{1}{Z_s}$

η' first shear viscosity of a viscous fluid

η second shear viscosity of a viscous fluid

υ' first kinematic viscosity of a viscous fluid

υ second kinematic viscosity of a viscous fluid

$\dot{\Omega}$ rotational vector of a continuum

ω angular frequency of vibration

α_{jn} roots of characteristi c equations, where subscript j identifies the number of modes in the spherical radial direction, while n identifies the number of modes in the meridianal direction

ρ_o density of the continuum in equilibrium

$\left.\begin{matrix}\varphi_m \\ \psi_m \\ \chi_m\end{matrix}\right\}$ scalar potential functions of a continuum of material m

θ spherical coordinate in the circumferential direction

ϕ spherical coordinate in the meridianal direction

CHAPTER 1

INTRODUCTION

The work presented here deals with the formulation of a theory that can be effectively applied to study the interactions of fluid, solid, and viscoelastic materials. On the basis of the linearized Navier-Stokes equation of a viscous fluid and the displacement equation of motion of an elastic solid, the governing equations of motion for a class of three-dimensional homogeneous, isotropic linearly viscoelastic continua are systematically constructed. The general solutions of these equations of motion are obtained by means of Lamé-Helmholtz-Stokes potentials. Each equation of motion is then transformed into a scalar and a vector potential equation. The scalar potential equation is separable in eleven coordinate systems, whereas the vector potential equation is separable in only six coordinate systems. Choosing the spherical coordinate system for illustration and introducing the Debye potentials to resolve the vector potential for the purpose of obtaining a separable solution, we can reduce each equation of motion into three independent scalar potential equations which are all separable in spherical coordinates. The field quantities are obtained in terms of these three scalar potentials from which the transient or harmonic solutions can be sought for mixed or nonmixed boundary value problems. Finally, applications of the general solutions for physical problems in spherically symmetric, axially symmetric, torsional and nontorsional motion are given in thirteen examples. These examples can be used for the modeling studies of aero and hydro space vehicles, geological wave problems and macroscopic biomechanics.

Recent developments in this field were edited by Greenspon[1)] in a symposium volume of eleven articles from four fields, namely, air blast loading and response, acoustic interaction, aeroelasticity and hydroelasticity. However, another field which also involves fluid-solid interactions and is rapidly expanding is biomechanics. It is actually an old subject which was dormant for many years and has only recently regained its popularity. Typical examples of fluid-solid interaction in biomechanics can be seen in a series of articles by Anliker et al.[2-6] on the theoretical and experimental developments of noninvasive methods to determine the elastic and inelastic properties of individual arteries, veins, eyes and hearts. Other examples can be found from articles edited and reviewed by Fung[7, 8]. In view of the importance of this field to medical technology and the need for systematic organization and efficient application of engineering and scientific principles to medicine and biology, we are presenting here a mathematical analysis of the interaction of viscous fluids, elastic solids and viscoelastic materials.

The theory is formulated starting out from the classical conservation laws-of momentum, mass and energy for continua with different constitutive equations. This approach is conventional from the point of view of continuum mechanics[8]. Texts and articles on this subject were given by Prager[9], Scipio[10], Jaunzemis[11], Eringen[12], Truesdell[13], and many others. It is well known that the governing equations are in general nonlinear and that their analytic solutions are quite formidable. For example, in the case of a viscous fluid the general solution of the Navier-Stokes equation still remains unknown except in a few particular cases as indicated by Schlichtig[14]. In view of the difficulties in solving the simultaneous nonlinear partial differential equations and the fact that the linearized solutions of these equations still represent a wide class of practical engineering and scientific problems, we confine our self to first seeking the linearized solutions of these equations and then applying the successive iteration method to find the nonlinear solutions. In the case of viscous fluids this method, known as the perturbation method, is for example described in articles by Eckart[15], Truesdell[16] and Hunt[17]. It had been systematically developed by Kaplun[18], Van Dyke[19], Cole[20], Chang[21] and others.

Having confined our goal, we linearize the equations about the equilibrium state and obtain a set of displacement equations of motion for different materials. For simplicity these sets of equations are derived for isothermal conditions which means that the energy equation will not be coupled with the momentum equations and the equation of continuity. The coupling of these equations with the thermal and electromagnetic fields can undoubtedly be included later. Examples of thermal coupling can be found in articles by Truesdell[16) who provided the harmonic plane wave solution and in publications by Wu[22) who gave the transient plane wave solution for the case of viscous, thermally conducting fluids. Examples of thermally conducting elastic solids are described in articles by Deresiewicz[23) who obtained the harmonic solutions of dilatational plane waves in an unbounded solid, axially symmetric nontorsional (longitudinal) motion of a circular cylinder, and the thickness-stretch motion in an infinite plate. Wong[24) also dealt with problems involving thermal coupling and provided the harmonic solutions of multilayered cylindrical shells in torsional, longitudinal, flexural and circumferential motion and the harmonic solutions of multilayered spherical shells in axially symmetric torsional and nontorsional motion. Examples of electromagnetic or other field couplings were given by Scipio[10).

The displacement equations of motion are linear partial differential equations and may be solved by means of the spectral operator theory systematically presented by Friedman[25). However, there are at least two difficulties encountered. First, the separability of the general solutions of these equations for a given coordinate system is in question; second, the boundary condition is generally in tensorial form.

In order to remove the first difficulty the Helmholtz-Stokes potentials were introduced in the field of fluid mechanics for the velocity vector of the fluid field. The decomposition theorems of the velocity vector are described for example by Lagerstrom et al.[26) and Wu[22). In elastodynamics, the identical potential functions, named after Lamé, were introduced for the displacement vector of the elastic solid. A detailed review, in which Lamé's potentials were used for the integration of the equations of motion of elasticity, was given by Sternberg and Curtin[28). In either case, the velocity vector or the displacement vector is decomposed into two parts. This decomposition of a vector is now generally known as Helmholtz's theorem. It will be shown later in this text that the use of Lamé's potentials is sufficient for the solution of the governing displacement equations of motion for elastic solids and viscous fluids as well as viscoelastic materials. By means of Lamé's potentials, the governing displacement equations of motion of isotropic materials` are now decomposed into two separate partial differential equations: the scalar potential equation and the vector potential equation.

In order to remove the second difficulty, these potential equations should be transformed into the proper orthogonal coordinates describing the boundaries. When this is done, one again faces the first difficulty: the transformed potential equations may not have a separable solution. It should be clarified here that the general solutions of the potential equations do not depend upon the coordinate system, which is arbitrarily chosen to describe the boundary conditions. We require not only the separable solutions of the field equations (i.e., the governing potential equations), but also separable solutions of those field quantities which are actually used to match the boundary conditions. The potential equations can further be reduced into a scalar Helmholtz equation and a vector Helmholtz equation by suppressing the time variable either by Laplace transform methods or by separation of variables. The separable solution of the scalar Helmholtz equation in eleven curvilinear orthogonal coordinate systems was shown by Moon and Spencer[29)]; however, he separable solution of the vector Helmholtz equation is considerably involved. A common technique of achieving the vector solution is to find some scalar potentials which can be satisfied with the separable scalar Helmholtz equation. These potentials, when they are constructed to represent the Lamé vector potential, should then be consistent with the Helmholtz theorem. In a spherical coordinate system these scalar potentials, known as Debye potentials, were discussed by Wilcox[30)]. The necessary conditions for the separable solution of the vector Helmholtz equation in a general curvilinear orthogonal coordinate system were given by Morse and Feshbach[31)]. They found that separation of variables can only be achieved for spherical, conical and general cylindrical coordinates. Seeking the harmonic solution for general cylindrical coordinates in the case of a homogeneous, isotropic elastic solid, they gave the results of the displacement and the stress fields in terms of three scalar potential functions that satisfy the potential equations of motion of elastodynamics. However, Morse and Feshbach[31] assert that these general harmonic solutions are useful only in the case of circular cylindrical coordinates. For other general cylindrical coordinates on a given coordinate surface (such as elliptical or rectangular cylindrical coordinates), the field quantities that should be satisfied with the boundary conditions are not separable on that coordinate surface as they are on the radial surface in the case of circular cylindrical coordinates.

For other than circular cylindrical boundaries the boundary value problems are considerably more complicated, and very limited information is available. For example, the harmonic wave propagation in elastic rods of elliptical cross section with a stress-free surface has been given by Wong et al.[32]. This applies only to an infinite rod. If the rod is of finite length, the stresses at both ends of the rod can not be zero. This fact was also pointed out in Pochammer's[33] and Love's[34] work on a circular cylindrical rod. The impossibility of devising separable solutions for problems with nonmixed boundary conditions can also be demonstrated for rectangular cylindrical coordinates. Mindlin and Fox[35] gave a set of discrete points of the frequency spectrum for an infinite rectangular rod with stress-free boundaries, for special values of the ratio of width to depth. As pointed out by Mindlin[36], the frequency equation in this case can not be expressed in terms of a finite number of known transcendental functions, because of the complexities arising from mode conversion at the two perpendicular boundaries. Mathematically these difficulties could be anticipated because of the presence of two characteristic lengths in the problem which arise whenever the nonmixed conditions should be satisfied. However, recently Miklowitz[37,38] presented a means for solving some problems in this class, focusing on the semi-infinite plate as an example. He pointed out that for problems involving nonmixed end conditions, the direct separation methods fail to yield solutions, and the usual transform methods have been oriented only to the mixed-end conditions. In view of this, to accommodate a wider class of boundary value problems we should seek a general solution of the governing displacement equations of motion that can be applied for harmonic as well as transient and for mixed as well as nonmixed boundary and initial value problems. This has been done by Wong[39,40] only in the case of an elastic solid for the spherical coordinates and the general cylindrical coordinates, and in the case of a viscous fluid for the circular cylindrical coordinates. However, as pointed out by Anliker et al.[2-6] and by Fung[7,8], the materials involved in biomechanics are far more complicated than a linearly elastic solid or a viscous fluid; thus, the research results should be generalized to cover a wider class of materials.

CHAPTER 2

DEVELOPMENT OF POTENTIAL EQUATIONS OF MOTION

If we neglect the effects of thermal conduction, the linearized equations which govern the mechanical behavior of a continuum are reduced to the equation of continuity and three equations expressing equilibrium in the sense of d'Alembert:

$$\frac{\partial \rho}{\partial t} + \rho_o (\nabla \cdot \dot{\vec{u}}) = 0 \tag{2.1}$$

$$\rho_o \frac{\partial \dot{\vec{u}}}{\partial t} - \nabla \cdot \varsigma = 0 \tag{2.2}$$

where ρ and ρ_o are the local densities of the continuum in the perturbed and unperturbed states, $\dot{\vec{u}}$ is the velocity vector and ς is the stress tensor. For homogeneous, isotropic, linearly elastic solids, the components of the stress tensor are related to the displacement vector $\vec{u}$ through Hooke's law

$$\varsigma_{ij} = \lambda \ (\nabla \cdot \vec{u})\delta_{ij} + \mu \, [\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\vec{u}] \tag{2.3}$$

where λ and μ are Lamé's constants; ∇ is the "gradient" operator; $\vec{e}_i$ and $\vec{e}_j$ are unit vectors in the direction of coordinates χ_i and χ_j; and δ_{ij} is the Kronecker delta. If we are dealing with a homogeneous, isotropic, and linearly viscous fluid, the components of the stress tensor are related to the velocity vector $\dot{\vec{u}}$ through the Newtonian law

$$\varsigma_{ij} = [-P + \eta' \, (\nabla \cdot \dot{\vec{u}})]\delta_{ij} + \eta \, [\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\dot{\vec{u}} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\dot{\vec{u}}] \tag{2.4}$$

In this equation η' and η are the dilatational and shear viscosities; and P is the instantaneous pressure defined by the linearized equation of state which for isothermal conditions is given by

$$P - P_o = c_o^2(\rho - \rho_o) = \frac{P_o}{\rho_o}(\rho - \rho_o) \tag{2.5}$$

where P_o is the pressure in the unperturbed fluid $c_o^2 = \left(\frac{\partial P}{\partial \rho}\right)$.

Substituting (2.3) into (2.2), one obtains the displacement equations of motion for a homogeneous, isotropic, and linearly elastic solid:

$$\rho_o \frac{\partial^2 \vec{u}}{\partial t^2} - (\lambda + 2\mu)\nabla(\nabla \cdot \vec{u}) + \mu \nabla \times (\nabla \times \vec{u}) = 0 \tag{2.6}$$

By combining (2.2), (2.4), and (2.5), we arrive at the equations of motion for a homogeneous, isotropic, and linearly viscous fluid:

$$\rho_o \frac{\partial^2 \vec{u}}{\partial t^2} - (\eta' + 2\eta)\frac{\partial}{\partial t}\nabla(\nabla \cdot \vec{u}) + \eta \frac{\partial}{\partial t}\nabla \times (\nabla \times \vec{u}) + c_o^2 \nabla(\rho - \rho_o) = 0 \tag{2.7}$$

Taking the divergence of (2.7) and using (2.1) and (2.5) to eliminate $\vec{u}$ one obtains

$$\left[\left(c_o^2 + (\upsilon' + 2\upsilon)\frac{\partial}{\partial t}\right)\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\left(\begin{matrix}\dfrac{\rho - \rho_o}{\rho_o} \\ \dfrac{P - P}{P_o}\end{matrix}\right) = 0 \tag{2.8}$$

where c_o is the speed of sound in the viscous fluid, $\upsilon' = \eta' \rho_o =$ dilatational kinematic viscosity $\upsilon = \eta / \rho_o =$ shear kinematic viscosity, $\frac{\rho - \rho_o}{\rho_o} = \frac{\rho_e}{\rho_o} =$ excess density ratio, and $\frac{P - P_o}{P_o} = \frac{P_e}{P_o} =$ excess pressure ratio.

In terms of the dilatation Δ and the rotation vector $2\vec{\Omega}$ which are defined by

$$\Delta = \nabla \cdot \vec{u} \tag{2.9}$$

$$2\vec{\Omega} = \nabla \, x \vec{u} \tag{2.10}$$

equations (2.6) and (2.7) can be written respectively as

$$\rho_o \frac{\partial^2 \vec{u}}{\partial t^2} - (\lambda + 2\mu)\, \nabla \Delta + \mu \, \nabla \, x \, 2\vec{\Omega} = 0 \tag{2.11}$$

$$\rho_o \frac{\partial^2 \vec{u}}{\partial t^2} - (\eta' + 2\eta) \frac{\partial}{\partial t} \nabla \Delta + \eta \frac{\partial}{\partial t} \nabla \, x \, 2\vec{\Omega} + c_o^2 \nabla (\rho - \rho_o) = 0 \tag{2.12}$$

By taking the divergence and the curl of (2.11) and (2.12) and noting from (2.10) that $\nabla \cdot \vec{\Omega} \equiv 0$, we find

$$\rho_o \frac{\partial^2 \Delta}{\partial t^2} - (\lambda + 2\mu) \nabla^2 \Delta = 0 \tag{2.13}$$

$$\rho_o \frac{\partial^2 2\vec{\Omega}}{\partial t^2} - \mu \nabla^2 2\vec{\Omega} = 0 \tag{2.14}$$

$$\rho_o \frac{\partial^2 \Delta}{\partial t^2} - (\eta' + 2\eta) \frac{\partial}{\partial t} \nabla^2 \Delta + c_o^2 \nabla^2 (\rho - \rho_o) = 0 \tag{2.15}$$

$$\rho_o \frac{\partial^2 (2\vec{\Omega})}{\partial t^2} - \eta \frac{\partial}{\partial t} \nabla^2 (2\vec{\Omega}) = 0 \qquad (2.16)$$

The displacement vector $\vec{u}$ can be written in terms of the Lamé's potentials[27,31] φ_m and $\vec{A}_m$ (see also Appendix B)

$$\vec{u}_m = \nabla \varphi_m + \nabla \times \vec{A}_m \qquad (2.17)$$

where the subscript m is introduced to identify the constitutive laws of the various materials which will be considered. The left-hand side of this expression involves three unknowns, whereas the right-hand side contains four, namely, φ_m and the three components of $\vec{A}_m$. Thus another condition must be imposed on $\vec{A}_m$, but this condition must be such that the field quantity $\vec{u}$ is not affected, i.e., the field quantities must be gauge invariant[31]. The forms of this condition will be discussed later.

The substitution of (2.17) into (2.9) and (2.10) gives

$$\Delta_m = \nabla^2 \varphi_m \qquad (2.18)$$

$$2\vec{\Omega}_m = \nabla \times \nabla \times \vec{A}_m = \nabla (\nabla \cdot \vec{A}_m) - \nabla^2 \vec{A}_m \qquad (2.19)$$

Choosing $m = 1$ to represent the viscous fluid and $m = 2$ the elastic solid, we obtain by substituting (2.18) and (2.19) into (2.13), (2.14), (2.15), and (2.16)

$$\nabla^2 [\rho_o \frac{\partial^2 \varphi_2}{\partial t^2} - (\lambda + 2\mu) \nabla^2 \varphi_2] = 0 \qquad (2.20)$$

$$\nabla \times \nabla \times \left[\rho_o \frac{\partial^2 \vec{A}_2}{\partial t^2} - \mu \nabla^2 \vec{A}_2 \right] = 0 \qquad (2.21)$$

$$\nabla^2\left[\rho_o \frac{\partial^2\varphi_1}{\partial t^2} - (\eta'+2\eta)\frac{\partial}{\partial t}\nabla^2\varphi_1 + c_o^2(\rho-\rho_o)\right] = 0 \tag{2.22}$$

$$\nabla\times\nabla\times\left[\rho_o\frac{\partial^2\vec{A}_1}{\partial t^2} - \eta\frac{\partial}{\partial t}\nabla^2\vec{A}_1\right] = 0 \tag{2.23}$$

A sufficient condition for the satisfaction of these equations is that

$$\left(\nabla^2 - \frac{1}{c_d^2}\frac{\partial^2}{\partial t^2}\right)\varphi_2 = 0 \tag{2.24}$$

$$\left(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}_2 = 0 \tag{2.25}$$

$$(\upsilon'+2\upsilon)\frac{\partial}{\partial t}\nabla^2\varphi_1 - \frac{\partial^2\varphi_1}{\partial t_2} - c_o^2\frac{(\rho-\rho_o)}{\rho_o} = 0 \tag{2.26}$$

$$\left(\nabla^2 - \frac{1}{\upsilon}\frac{\partial}{\partial t}\right)\vec{A}_1 = 0 \tag{2.27}$$

where $c_d^2 = (\lambda + 2\mu)/\rho_o$ is the square of the dilatational wave speed and $c_s^2 = \mu/\rho_o$ that of the shear wave speed for the elastic solid.

For $m = 1$ we can eliminate $(\rho - \rho_o)/\rho_o$ with the aid of (2.1) and (2.17) and therefore write equation (2.26) also in the form

$$\left[\left(c_o^2 + (\upsilon'+2\upsilon)\frac{\partial}{\partial t}\right)\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\rho_1 = 0 \tag{2.28}$$

Equations (2.24) and (2.25) are the governing potential equations for a linearly elastic solid whereas (2.27) and (2.28) are the governing potential equations for a viscous fluid. The solutions of these equations are contingent on the gauge invariance just mentioned.

If we take the divergence of (2.19) and make use of (2.10) and (2.19) we find

$$\nabla\cdot(2\vec{\Omega}_m)\equiv 0 = \nabla^2(\nabla\cdot\vec{A}_m) - \nabla\cdot(\nabla^2\vec{A}_m)$$

which shows that the operators ∇^2 and ∇° are commutative. Hence the divergence of (2.25) and (2.27) leads to

$$\left[\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right](\nabla\cdot\vec{A}_2) = 0 \tag{2.29}$$

$$\left[\nabla^2 - \frac{1}{\upsilon}\frac{\partial}{\partial t}\right](\nabla\cdot\vec{A}_1) = 0 \tag{2.30}$$

Thus it is seen that (2.17) is automatically satisfied if we impose the additional conditions:

$$\nabla\cdot\vec{A}_2 = 0 \tag{2.31}$$

$$\nabla^2(\nabla\cdot\vec{A}_2) - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}(\nabla\cdot\vec{A}_2) = 0 \tag{2.32}$$

with $\vec{u}_2 = \nabla'\varphi_2 + \nabla\times\vec{A}_2$ in the case of a linearly elastic solid, and

$$\nabla\cdot\vec{A}_1 = 0 \tag{2.33}$$

$$\nabla^2(\nabla\cdot\vec{A}_1) - \frac{1}{\upsilon}\frac{\partial}{\partial t}(\upsilon\cdot\vec{A}_1) = 0 \tag{2.34}$$

with $\vec{u}_1 = \nabla \varphi_1 + \nabla \times \vec{A}_1$ in the case of a viscous fluid.

Since $\vec{A}_2$ must satisfy either (2.31) or (2.32) and $\vec{A}_1$ must satisfy either (2.33) or (2.34), there are in effect only two independent components of the vector potentials $\vec{A}_2$ and $\vec{A}_1$. In spherical and general cylindrical coordinates it is possible to choose these components such that they satisfy the scalar diffusion equations for a viscous fluid and satisfy the scalar shear wave equation for a linearly elastic solid.[39,40)

It is shown in Appendix A that in spherical coordinates the vector $\vec{A}_1$, written as

$$\vec{A}_1 = \nabla \times (\vec{e}_r r \psi_1) + \nabla \times \nabla \times (\vec{e}_r r \chi_1) \tag{2.35}$$

with $\vec{e}_r$ representing the unit vector in the radial direction, satisfies the vector diffusion equation (2.27), provided the scalar functions ψ_1 and χ_1 satisfy the scalar diffusion equations

$$\nabla^2 \psi_1 = \frac{1}{\upsilon} \frac{\partial \psi_1}{\partial t} \tag{2.36}$$

$$\nabla^2 \chi_1 = \frac{1}{\upsilon} \frac{\partial \chi_1}{\partial t} \tag{2.37}$$

Similarly the vector

$$\vec{A}_2 = \nabla \times (\vec{e}_r r \psi_2) + \nabla \times \nabla \times (\vec{e}_r r \chi_2) \tag{2.38}$$

satisfies the vector wave equation (2.25), if ψ_2 and χ_2 are solutions to the scalar shear wave equations

$$\nabla^2\psi_2 = \frac{1}{c_s^2}\frac{\partial^2\psi_2}{\partial t^2} \tag{2.39}$$

$$\nabla^2\chi_2 = \frac{1}{c_s^2}\frac{\partial^2\chi_2}{\partial t^2} \tag{2.40}$$

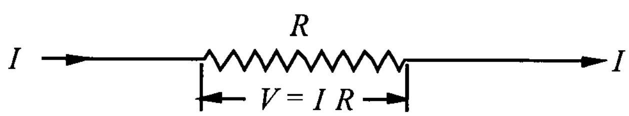
So far we have obtained the linearized Navier-Stokes equations for a viscous fluid and the equations of motion for a linearly elastic solid. We have written these equations in terms of the corresponding relations which govern the associated Lamé and Helmholtz (Stokes) potentials. It will be shown that these equations can be used to construct the equations of motion for a class of linear viscoelastic continua.

The basic idea and method of approach can best be explained through electrical circuit analogy. We begin with the simplest model in one dimension which we then generalize to allow for more complex constitutive laws and three dimensions.

According to Ohm's law we have

$$I = \frac{V}{R} \tag{2.41}$$

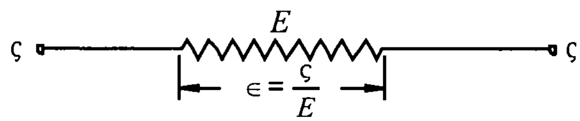
where I is the current flowing through an electric resistance R and V is the difference in voltage across R.



The corresponding mechanical model of Ohm's law is the one-dimensional Hooke's law

$$\varsigma = E\epsilon = \frac{\epsilon}{\left(\frac{1}{E}\right)} = \frac{\epsilon}{Z} \tag{2.42}$$

where ς is the stress, ϵ the strain, E the Young's modulus, and Z the mechanical resistance or impedance.

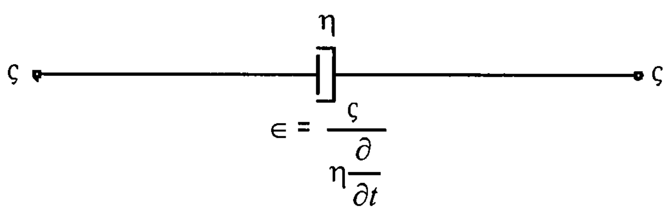


By taking advantage of elementary circuit theory we can readily find the total mechanical resistance Z for continua with more complicated constitutive laws.

For a viscous fluid the Newtonian law specifies

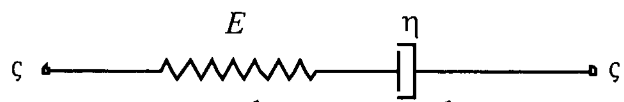
$$\varsigma = \eta\frac{\partial \epsilon}{\partial t} = \frac{\epsilon}{\frac{1}{\eta\frac{\partial}{\partial t}}} = \frac{\epsilon}{Z} \tag{2.43}$$

where ς is the shearing stress, ϵ the strain, and η the viscosity coefficient. $\frac{\partial}{\partial t}$ is treated as an operator.



Comparing (2.43) with (2.42) we immediately identify the mechanical resistance of a dashpot as $z = \dfrac{1}{\eta \frac{\partial}{\partial t}}$.

In the case of the so-called Maxwell model we have a spring and a dashpot in series:



$$z_1 = \frac{1}{E} \qquad z_2 = \frac{1}{\eta \frac{\partial}{\partial t}}$$

The total mechanical resistance is simply

$$z = z_1 + z_2 = \frac{1}{E} + \frac{1}{\eta \frac{\partial}{\partial t}} \tag{2.44}$$

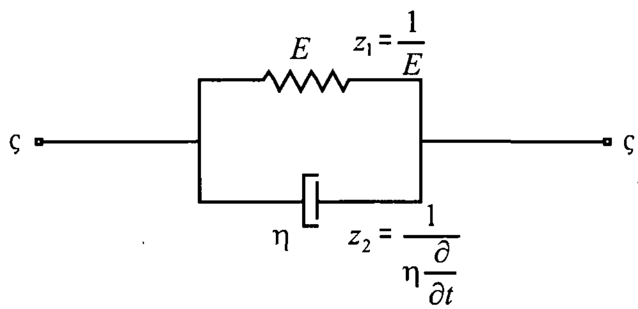
in analogy with the electric resistance. Substituting (2.44) into

$$\varsigma = \frac{\in}{z} = \frac{\in}{\frac{1}{E} + \frac{1}{\eta \frac{\partial}{\partial t}}} \tag{2.45}$$

we obtain the constitutive relation defined by the Maxwell model:

$$\left(\frac{1}{\eta} + \frac{1}{E}\frac{\partial}{\partial t} \right) \varsigma = \frac{\partial \in}{\partial t} \tag{2.46}$$

The Kelvin solid is obtained by arranging a spring and a dashpot in parallel:



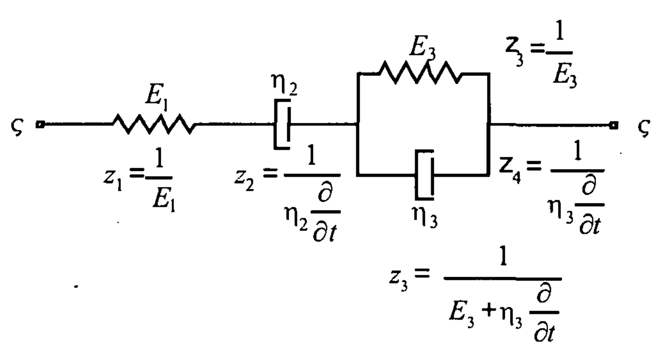
the total mechanical resistance is now

$$z = \frac{1}{\dfrac{1}{z_1} + \dfrac{1}{z_2}} = \frac{1}{E + \eta \dfrac{\partial}{\partial E}} \tag{2.47}$$

and the corresponding constitutive law is

$$\varsigma = (E + \eta \frac{\partial}{\partial t}) \in \tag{2.48}$$

In the ease of a four parameter fluid defined by the diagram below



the total mechanical resistance is given by

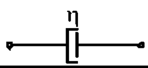
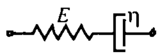
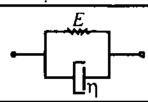
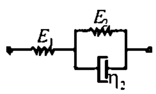
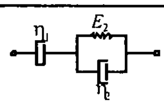
$$z = z_1 + z_2 + z_3 = \frac{1}{E_1} + \frac{1}{\eta_2 \frac{\partial}{\partial t}} + \frac{1}{E_3 + \eta_2 \frac{\partial}{\partial t}} \tag{2.49}$$

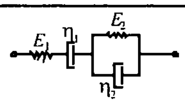
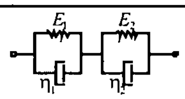
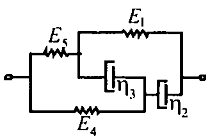
and the constitutive equation can be written as

$$\left[\frac{1}{E_1}\frac{\partial^2}{\partial t^2} + \left(\frac{E_3}{E_1\eta_3} + \frac{1}{\eta_2} + \frac{1}{\eta_3}\right)\frac{\partial}{\partial t} + \frac{E_3}{\eta_2\eta_3}\right]\varsigma = \left[\frac{E_3}{\eta_3}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\right]\epsilon \tag{2.50}$$

For one-dimensional problems the constitutive equation of a viscoelastic material can be systematicall y constructed on the basis of circuit analogy. A representativ e set of examples are given in Table 1.

TABLE 1

Model	Name	Mechanical Resistance	Constitutive Equation
	elastic solid	$Z = \frac{1}{E}$	$\varsigma = \frac{\in}{Z}$
	viscous fluid	$Z = \frac{1}{\eta \frac{\partial}{\partial t}}$	$\varsigma = \frac{\in}{Z}$
	Maxwell fluid	$Z = \frac{1}{E} + \frac{1}{\eta \frac{\partial}{\partial t}}$	$\varsigma = \frac{\in}{Z}$
	Kelvin solid	$Z = \frac{1}{E + \eta \frac{\partial}{\partial t}}$	$\varsigma = \frac{\in}{Z}$
	3-parameter solid	$Z = \frac{1}{E_1} + \frac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}}$	$\varsigma = \frac{\in}{Z}$
	3-parameter fluid	$Z = \frac{1}{\eta_1 \frac{\partial}{\partial t}} + \frac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}}$	$\varsigma = \frac{\in}{Z}$

Model	Name	Mechanical Resistance	Constitutive Equation
	4-parameter fluid	$Z = \frac{1}{E_1} + \frac{1}{\eta_1 \frac{\partial}{\partial t}} + \frac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}}$	$\varsigma = \frac{\epsilon}{Z}$
	4-parameter solid	$Z = \frac{1}{E_1 + \eta_1 \frac{\partial}{\partial t}} + \frac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}}$	$\varsigma = \frac{\epsilon}{Z}$
	Flügge[41)]	$Z = \dfrac{\left(\frac{1}{E_1} + \frac{1}{\eta_1 \frac{\partial}{\partial t}}\right)\left[\frac{1}{E_5}\left(\frac{\eta_2 \frac{\partial}{\partial t}}{E_1} + \frac{E_4}{\eta_3 \frac{\partial}{\partial t}} + \frac{\eta_3}{\eta_3} + 1\right) + \frac{1}{E_1}\left(\frac{E_4}{E_5} + \frac{E_4}{\eta_3 \frac{\partial}{\partial t}} + \frac{\eta_2}{\eta_3} + 1\right)\right]}{\left(\frac{1}{E_1} + \frac{1}{\eta_2 \frac{\partial}{\partial t}}\right)\left(\frac{E_4}{E_5} + \frac{E_4}{\eta_3 \frac{\partial}{\partial t}} + \frac{\eta_2}{\eta_3} + 1\right) + \left(\frac{1}{E_5} + \frac{1}{E_4}\right)\left(\frac{\eta_2 \frac{\partial}{\partial t}}{E_1} + \frac{E_4}{\eta_3 \frac{\partial}{\partial t}} + \frac{\eta_2}{\eta_3} + 1\right)}$	$\varsigma = \frac{\epsilon}{Z}$

The mechanical resistance or impedance Z given in Table 1 can be related to the creep compliance $J(t)$ and the relaxation modulus $\gamma(t)$ introduced by Lee,[44,45] Flugge,[41] and Bland.[42] This relationship is given by

$$\in(t) = Z\left(\frac{\partial}{\partial t}\right)\varsigma(t) = \varsigma_o J(t) \tag{2.51}$$

and

$$\varsigma(t) = \frac{\in(t)}{Z\left(\frac{\partial}{\partial t}\right)} = \in_o Y(t) \tag{2.52}$$

where the mechanical resistance is written as a differential operator $Z\left(\frac{\partial}{\partial t}\right)$

As indicated earlier, the electrical analogy offers several convenient features:

a) Extremely complicated material models can be dealt with in a simple manner by utilizing electrical network analysis.

b) As will be shown, we can extend this analogy to three dimensions and if the continuum is isotropic, homogeneous, and linearly viscoelastic, its constitutive equations can be systematicall y constructed.

c) Without suppressing the time variable the governing field equations are main-rained in th eir general form such that a wider class of boundary value problems can be accommodated.

We are now deriving the linearized Navier-Stokes equations for a viscous fluid and the equations of motion for an elastic solid and the equivalent governing potential for the materials given in Table 1.

In the one-dimensional case the linear operator which interrelates the stress and the strain varies with the constitutive law of the material. Likewise do the corresponding two linear operators in the three-dimensional case where one of the operators governs the dilatational motions of the material, the other the shearing motions. The constitutive law for a class of isotropic, homogeneous, linearly viscoelastic materials is proposed in the following form:

$$\begin{aligned}\varsigma_{ijm} &= \left[-P_m + \lambda_m \left(\nabla \cdot \vec{u}\right)\right] \delta_{ij} + \mu_m \left[\vec{e}_i \cdot \left(\vec{e}_j \cdot \nabla\right)\vec{u} + \vec{e}_j \cdot \left(\vec{e}_i \cdot \nabla\right)\vec{u}\right] \\ &= \left[P_m + \frac{1}{Z_d}\left(\nabla \cdot \vec{u}\right)\right]\delta_{ij} + \frac{1}{Z_s}\left[\vec{e}_i \cdot \left(\vec{e}_j \cdot \nabla\right)\vec{u} + \vec{e}_j \cdot \left(\vec{e}_i \cdot \nabla\right)\vec{u}\right]\end{aligned}$$

where $\dfrac{1}{Z_d} = \lambda_m$ and $\dfrac{1}{Z_s} = \mu_m$ are linear operators. In particular, for the case of a viscous fluid $(m = 1)$, the constitutive law is given by

$$\varsigma_{ij1} = \left[P + \eta' \frac{\partial}{\partial t}\left(\nabla \cdot \vec{u}\right)\right]\delta_{ij} + \eta \frac{\partial}{\partial t}\left[\vec{e}_i \cdot \left(\vec{e}_j \cdot \nabla\right)\vec{u} + \vec{e}_j \cdot \left(\vec{e}_i \cdot \nabla\right)\vec{u}\right]$$

and according to Table 1 the linear operators are

$$\frac{1}{Z_d} = \lambda_1 = \eta' \frac{\partial}{\partial t} \qquad \frac{1}{Z_s} = \mu_1 = \eta \frac{\partial}{\partial t}$$

For the case of an elastic solid $m = 2$ the constitutive law is given by

$$\varsigma_{ij2} = \lambda \left(\nabla \cdot \vec{u}\right)\delta_{ij} + \mu \left[\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\overline{u} + \vec{e}_j \cdot \left(\vec{e}_i \cdot \nabla\right)\vec{u}\right]$$

and the linear operators $\dfrac{1}{Z_d}$ and $\dfrac{1}{Z_s}$ are the ordinary Lamé's constants

$$\frac{1}{Z_d} = \lambda_2 = \lambda \qquad \frac{1}{Z_s} = \mu_2 = \mu$$

(See Table 1).

With these two examples for $m = 1$ and $m = 2$ we can construct the three dimensional constitutive equations for viscoelastic materials. Considering for example a three dimensional Maxwell fluid $(m = 3)$, we have as the operator connected with the dilatational motion

$$\frac{1}{Z_d} = \lambda_3 = \frac{1}{\frac{1}{\lambda} + \frac{1}{\eta' \frac{\partial}{\partial t}}}$$

and as the operator pertaining to the shearing motion

$$\frac{1}{Z_s} = \mu_3 = \frac{1}{\frac{1}{\mu} + \frac{1}{\eta \frac{\partial}{\partial t}}}$$

The corresponding constitutive equation can therefore be written as

$$\varsigma_{ij} = [-P + \frac{1}{\frac{1}{\lambda} + \frac{1}{\eta' \frac{\partial}{\partial t}}}(\nabla \cdot \vec{u})]\delta_{ij} + \frac{1}{\frac{1}{\mu} + \frac{1}{\eta \frac{\partial}{\partial t}}}[\vec{e}_i(\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\vec{u}]$$

For the three dimensional Kelvin solid the operator associated with the dilatational motion is

$$\frac{1}{Z_d} = \lambda_4 = \lambda + \eta' \frac{\partial}{\partial t}$$

while that connected with the shearing motion is

$$\frac{1}{Z_s} = \mu_4 = \mu + \eta \frac{\partial}{\partial t}$$

Thus the constitutive equation for such a material can be expressed in the form

$$\varsigma_{ij} = [-P + (\lambda + \eta' \frac{\partial}{\partial t})(\nabla \cdot \vec{u})]\delta_{ij} + (u + \eta \frac{\partial}{\partial t})[\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\vec{u}]$$

Proceeding in a similar fashion we can construct the three dimensional constitutive equations for the remaining viscoelastic material models given in Table 1. The results are summarized in Table 2.

TABLE 2

CONSTITUTIVE EQUATIONS OF THREE-DIMENSIONAL HOMOGENEOUS, ISOTROPIC, LINEARLY VISCOELASTIC MATERIALS

Name	Constitutive Equation
	$\varsigma_{ijm} = \left[-P_m + \lambda_m(\nabla\cdot\vec{u})\right]\delta_{ij} + \mu_m\left[\vec{e}_i\cdot(\vec{e}_j\cdot\nabla)\vec{u} + \vec{e}_j\cdot(\vec{e}_i\cdot\nabla)\vec{u}\right]$ $\frac{1}{Z_d} = \lambda_m \qquad \frac{1}{Z_s} = \mu_m$
viscous fluid *m=1*	$\varsigma_{ij1} = \left[-P + \eta'\frac{\partial}{\partial t}(\nabla\cdot\vec{u})\right]\delta_{ij} + \eta\frac{\partial}{\partial t}\left[\vec{e}_i\cdot(\vec{e}_j\cdot\nabla)\vec{u} + \vec{e}_j\cdot(\vec{e}_i\cdot\nabla)\vec{u}\right]$ $\frac{1}{Z_d} = \lambda_1 = \eta'\frac{\partial}{\partial t} \qquad \frac{1}{Z_s} = \mu_1 = \eta\frac{\partial}{\partial t}$
elastic solid *m = 2*	$\varsigma_{ij2} = \lambda(\nabla\cdot\vec{u})\delta_{ij} + \mu\left[\vec{e}_i\cdot(\vec{e}_j\cdot\nabla)\vec{u} + \vec{e}_j\cdot(\vec{e}_i\cdot\nabla)\vec{u}\right]$ $\frac{1}{Z_d} = \lambda_2 = \lambda \qquad \frac{1}{Z_s} = \mu_2 = \mu$
Maxwell fluid *m=3*	$\varsigma_{ij3} = \left[-P + \frac{1}{\frac{1}{\lambda} + \frac{1}{\eta'\frac{\partial}{\partial t}}}(\nabla\cdot\vec{u})\right]\delta_{ij} + \frac{1}{\frac{1}{\mu} + \frac{1}{\eta\frac{\partial}{\partial t}}}\left[\vec{e}_i\cdot(\vec{e}_j\cdot\nabla)\vec{u} + \vec{e}_j\cdot(\vec{e}_i\cdot\nabla)\vec{u}\right]$ $\frac{1}{Z_d} = \lambda_3 = \frac{1}{\frac{1}{\lambda} + \frac{1}{\eta'\frac{\partial}{\partial t}}} \qquad \frac{1}{Z_s} = \mu_3 = \frac{1}{\frac{1}{\mu} + \frac{1}{\eta\frac{\partial}{\partial t}}}$

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Name	Constitutive Equation
Kelvin solid *m=4*	$\varsigma_{ij4} = \left[-P + (\lambda + \eta'\frac{\partial}{\partial t})(\nabla \cdot \vec{u})\right]\delta_{ij}$ $+ (\mu + \eta\frac{\partial}{\partial t})\left[\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\vec{u}\right]$ $\frac{1}{Z_d} = \lambda_4 = \lambda + \eta'\frac{\partial}{\partial t} \qquad \frac{1}{Z_s} = \mu_4 = \mu + \eta\frac{\partial}{\partial t}$
3-parameter solid *m=5*	$\varsigma_{ij5} = [-P + \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2 + \eta'_2\frac{\partial}{\partial t}}}(\nabla \cdot \vec{u})]\delta_{ij}$ $+ \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2 + \eta_2\frac{\partial}{\partial t}}}\left[\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\vec{u}\right]$ $\frac{1}{Z_d} = \lambda_5 = \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2 + \eta'_2\frac{\partial}{\partial t}}} \qquad \frac{1}{Z_s} = \mu_5 = \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2 + \eta_2\frac{\partial}{\partial t}}}$
3-parameter fluid *m=6*	$\varsigma_{ij6} = [-P + \frac{1}{\frac{1}{\eta'\frac{\partial}{\partial t}} + \frac{1}{\lambda_2 + \eta'_2\frac{\partial}{\partial t}}}(\nabla \cdot \vec{u})]\delta_{ij}$ $+ \frac{1}{\frac{1}{\eta_1\frac{\partial}{\partial t}} + \frac{1}{\mu_2 + \eta_2\frac{\partial}{\partial t}}}\left[\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\vec{u}\right]$ $\frac{1}{Z_d} = \lambda_6 = \frac{1}{\frac{1}{\eta'\frac{\partial}{\partial t}} + \frac{1}{\lambda_2 + \eta'_2\frac{\partial}{\partial t}}}$ $\frac{1}{Z_s} = \mu_6 = \frac{1}{\frac{1}{\eta_1\frac{\partial}{\partial t}} + \frac{1}{\mu_2 + \eta_2\frac{\partial}{\partial t}}}$

Name	Constitutive Equation
4-parameter fluid *m=7*	$\varsigma_{ij7} = [-P + \dfrac{1}{\dfrac{1}{\lambda_1} + \dfrac{1}{\eta'_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\lambda_2 + \eta'_2 \dfrac{\partial}{\partial t}}} (\nabla \cdot \vec{u})]\delta_{ij}$ $+ \dfrac{1}{\dfrac{1}{\mu_1} + \dfrac{1}{\eta_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\mu_2 + \eta_2 \dfrac{\partial}{\partial t}}} \left[\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j \cdot (\vec{e}_i \cdot \nabla)\vec{u}\right]$ $\dfrac{1}{Z_d} = \lambda_7 \dfrac{1}{\dfrac{1}{\lambda_1} + \dfrac{1}{\eta'_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\lambda_2 + \eta'_2 \dfrac{\partial}{\partial t}}}$ $\dfrac{1}{Z_s} = \mu_7 \dfrac{1}{\dfrac{1}{\mu_1} + \dfrac{1}{\eta'_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\mu_2 + \eta_2 \dfrac{\partial}{\partial t}}}$
4-parameter solid *m=8*	$\varsigma_{ij8} = [-P + \dfrac{1}{\dfrac{1}{\lambda_1 + \eta'_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\lambda_2 + \eta'_2 \dfrac{\partial}{\partial t}}} (\nabla \cdot \vec{u})]\delta_{ij}$ $+ \dfrac{1}{\dfrac{1}{\mu_1 + \eta_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\mu_2 + \eta_2 \dfrac{\partial}{\partial t}}} \left[\vec{e}_i \cdot (\vec{e}_j \cdot \nabla)\vec{u} + \vec{e}_j(\vec{e}_i \cdot \nabla)\vec{u}\right]$ $\dfrac{1}{Z_d} = \lambda_8 = \dfrac{1}{\dfrac{1}{\lambda_1 + \eta'_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\lambda_2 + \eta'_2 \dfrac{\partial}{\partial t}}}$ $\dfrac{1}{Z_s} = \mu_8 = \dfrac{1}{\dfrac{1}{\mu_1 + \eta'_1 \dfrac{\partial}{\partial t}} + \dfrac{1}{\mu_2 + \eta_2 \dfrac{\partial}{\partial t}}}$

TABLE 3

Elastic Moduli	Viscoelastic Operators		Viscoelastic Moduli (transformed operator)
	Flügge [41]	the present text	Bland [42]
Bulk 3k	$\frac{Q''}{P''}$	$\frac{3}{Z_d} + \frac{2}{Z_s}$	Y_υ
Shear 2μ	$\frac{Q'}{P'}$	$\frac{2}{Z_s} = 2\mu_m$	Y_s
Young's E	$\frac{Q}{P} = 3\frac{Q'Q''}{2P'Q''+P''Q'}$	$\frac{1}{Z} = \frac{3Z_s + 2Z_d}{Z_s(Z_d + Z_s)}$	$3\frac{Y_s Y_\upsilon}{2Y_\upsilon + Y_s}$
Poisson's ration ς	$\frac{P'Q''- P''Q'}{2P'Q''+P''Q'}$	$\frac{Z_s}{2(Z_d + Z_s)}$	$\frac{Y_\upsilon - Y_s}{2Y_\upsilon + Y_s}$
Lamé's constant λ	$\frac{1}{3}\left(\frac{Q''}{P''} - \frac{Q'}{P'}\right)$	$\frac{1}{Z_d} = \lambda_m$	$\frac{1}{3}(Y_\upsilon - Y_s)$

The constitutive equations of three-dimensional homogeneous, isotropic, linearly viscoelastic materials given in Table 2 were obtained by direct analogy from the constitutive laws listed in Table 1 for one-dimensional materials. In contrast to the one-dimensional materials for which we have a single operator, the mechanical resistance Z, we have two operators for the three-dimensional materials, a mechanical resistance Z_d due to dilatation, and a Z_s due to shear. Z, Z_d, and Z_s are differential operators which after linear transformation assume the form of the so-called viscoelastic moduli. For comparison purposes the various operators are given also in Table 3 together with those obtained by Flugge[41)] and Bland[42)] on the basis of the correspondence principle.

Following the formation of Table 2 and Table 3, the generalized linear Navier-Stokes equations and their equivalent governing potential equations are obtained by inspection from Table 3. The results are shown in Table 4.

TABLE 4

GENERALIZED LINEAR NAVIER-STOKES EQUATIONS AND THEIR EQUIVALENT GOVERNING POTENTIAL EQUATIONS

Name	Governing Field Equations
Materials	$\rho_o \frac{\partial^2 \vec{u}_m}{\partial t^2} - (\lambda_m + 2\mu_m)\nabla(\nabla \cdot \vec{u}) + \mu_m \nabla \times (\nabla \times \vec{u}_m) = 0$ $\vec{u}_m = \nabla\varphi_m + \nabla \times \vec{A}_m$ $\left(c_{dm}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right)\varphi_m = 0$ $\left(c_{sm}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right)\vec{A}_m = 0$ $c_{dm}^2 = \frac{\lambda_m + 2\mu_m}{\rho_o} = \frac{1}{\rho_o}\left(\frac{1}{Z_d} + \frac{2}{Z_s}\right)$ $c_{sm}^2 = \frac{\mu_m}{\rho_o} = \frac{1}{\rho_o}\frac{1}{Z_s}$
Viscous fluid $m = 1$	$\rho_o \frac{\partial^2 \vec{u}_l}{\partial t^2} - \left(\rho_o c_o^2 + (\eta' + 2_i \eta)\frac{\partial}{\partial t}\right)\nabla(\nabla \cdot \vec{u}_l) + \eta \frac{\partial}{\partial t}\nabla \times (\nabla \times \vec{u}_l) = 0$ $\vec{u}_l = \nabla\varphi_l + \nabla \times \vec{A}_l$ $L_{\varphi l}\varphi_l = \left(c_{dl}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right)\varphi_l = 0$ $L_{AS}\vec{A}_l = \left(c_{sl}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right)\vec{A}_l = 0$

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	$c_{d1}^2 = c_o^2 + \frac{1}{\rho_o}(\eta' + 2\eta)\frac{\partial}{\partial t} = c_o^2 + \frac{1}{\rho_o Z_d} + \frac{2}{\rho_o Z_s}$ $c_{s1}^2 = \frac{\eta}{\rho_o}\frac{\partial}{\partial t} = \frac{1}{\rho_o Z_s}$
Elastic solid $m = 2$	$\rho_o \frac{\partial^2 \vec{u}_2}{\partial t^2} - (\lambda + 2\mu)\nabla(\nabla \cdot \vec{u}_2) + \mu\nabla \times (\nabla \times \vec{u}_2) = 0$ $u_2 = \nabla\varphi_2 + \nabla \times \vec{A}_2$ $L_{\varphi 2}\varphi_2 = \left[c_{d2}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\varphi_2 = 0$ $L_{A2}\vec{A}_2 = \left[c_{s2}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\vec{A}_2 = 0$ $c_{d2}^2 = \frac{\lambda + 2\mu}{\rho_o} = \frac{1}{\rho_o Z_d} + \frac{2}{\rho_o Z_s}$ $c_{s2}^2 = \frac{\mu}{\rho_o} = \frac{1}{\rho_o Z_s}$

Maxwell fluid $m = 3$	$\rho_o \frac{\partial^2 \vec{u}_3}{\partial t^2} - [\rho_o c_o^2 + \frac{1}{\frac{1}{\lambda}+\frac{1}{\eta'\frac{\partial}{\partial t}}} + \frac{2}{\frac{1}{\mu}+\frac{1}{\eta\frac{\partial}{\partial t}}}]\nabla(\nabla\cdot\vec{u}_3) + \frac{1}{\frac{1}{\mu}+\frac{1}{\eta\frac{\partial}{\partial t}}}\nabla \times (\nabla\times\vec{u}_3) = 0$ $\vec{u}_3 = \nabla\varphi_3 + \nabla\times\nabla\vec{A}_3$ $L_{\varphi 3}\varphi_3 = \left(c_{d3}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\varphi_3 = 0$ $L_{A3}\vec{A}_3 = \left(c_{s3}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\vec{A}_3 = 0$ $c_{d3}^2 = c_o^2 + \frac{1}{\frac{\rho_o}{\lambda}+\frac{\rho_o}{\eta'\frac{\partial}{\partial t}}} + \frac{2}{\frac{\rho_o}{\mu}+\frac{\rho_o}{\eta\frac{\partial}{\partial t}}} = \frac{1}{\rho_o Z_d}+\frac{2}{\rho_o Z_s}+c_o^2$ $c_{s3}^2 = \frac{1}{\frac{\rho_o}{\mu}+\frac{\rho_o}{\eta\frac{\partial}{\partial t}}} = \frac{1}{\rho_o Z_s}$

Kelvin solid $m = 4$	$\rho_o \frac{\partial^2 \vec{u}_4}{\partial t^2} - \left[\rho_o c_o^2 + \left(\lambda + \eta' \frac{\partial}{\partial t}\right) + 2\left(\mu + \eta \frac{\partial}{\partial t}\right)\right] \nabla (\nabla \cdot \vec{u}_4)$ $+ \left(\mu + \eta \frac{\partial}{\partial t}\right) \nabla \times (\nabla \times \vec{u}_4) = 0$ $\vec{u}_4 = \nabla \varphi_4 + \nabla \times \nabla \vec{A}_4$ $L_{\varphi 4} \varphi_4 = \left(c_{d4}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right) \varphi_4 = 0$ $L_{A4} \vec{A}_4 = \left(c_{s4}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right) \vec{A}_4 = 0$ $c_{d4}^2 = c_o^2 + \frac{1}{\rho_o}\left(\lambda + \eta' \frac{\partial}{\partial t}\right) + \frac{2}{\rho_o}\left(\mu + \eta \frac{\partial}{\partial t}\right) = c_o^2 \frac{1}{\rho_o Z_d} + \frac{2}{\rho_o Z_s}$ $c_{s4}^2 = \frac{1}{\rho_o}\left(\mu + \eta \frac{\partial}{\partial t}\right) = \frac{1}{\rho_o Z_s}$

<table>
<tr>
<td>3-parameter solid

$m = 5$</td>
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$$\rho_o \frac{\partial^2 \vec{u}_s}{\partial t^2} - \left[\rho_o^2 c_o^2 + \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2 + \eta'_2 \frac{\partial}{\partial t}}} + \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2 + \eta_2 \frac{\partial}{\partial t}}}\right] \nabla \left(\nabla \cdot \vec{u}_5\right)$$

$$+ \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2 + \eta_2^2 \frac{\partial}{\partial t}}} \nabla \times \left(\nabla \times \vec{u}_5\right) = 0$$

$$\vec{u}_5 = \nabla \varphi_5 + \nabla \times \vec{A}_5$$

$$L_{\varphi 5} \varphi_5 = \left[c_{d5}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right] \varphi_5 = 0$$

$$L_{A5} \vec{A}_5 = \left[c_{s5}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right] \vec{A}_5 = 0$$

$$c_{d5}^2 = c_0^2 + \frac{1}{\frac{\rho_o}{\lambda_1} + \frac{\rho_o}{\lambda'_2 + \eta'_2 \frac{\partial}{\partial t}}} + \frac{2}{\frac{\rho_o}{\mu_1} + \frac{\rho_o}{\mu_2 + \eta_2 \frac{\partial}{\partial t}}} = c_0^2 + \frac{1}{\rho_o Z_d} + \frac{2}{\rho_o Z_s}$$

$$c_{s5}^2 = \frac{1}{\frac{\rho_o}{\mu_1} + \frac{\rho_o}{\mu_2 + \eta_2 \frac{\partial}{\partial t}}} = \frac{1}{\rho_o Z_s}$$

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</table>

3-parameter fluid $m = 6$	$\rho_0 \frac{\partial^2 \vec{u}_6}{\partial t^2} - \left[\rho_0 c_0^2 + \frac{1}{\frac{1}{\eta'_1 \frac{\partial}{\partial t}} + \frac{1}{\lambda_2 + \eta'_2 \frac{\partial}{\partial t}}} + \frac{2}{\frac{1}{\eta_1 \frac{\partial}{\partial t}} + \frac{1}{\mu_2 + \eta_2 \frac{\partial}{\partial t}}} \right] \nabla \left(\nabla \cdot \vec{u}_6 \right)$ $+ \frac{1}{\frac{1}{\eta_1 \frac{\partial}{\partial t}} + \frac{1}{\mu_2 + \eta_2 \frac{\partial}{\partial t}}} \nabla \times \left(\nabla \times \vec{u}_6 \right) = 0$ $\vec{u}_6 = \nabla \varphi_6 + \nabla \times \vec{A}_6$ $L_{\varphi 6} \varphi_6 = \left[c_{d6}^2 \nabla^2 - \frac{\partial^2}{\partial t^2} \right] \varphi_6 = 0$ $L_{A6} \vec{A}_6 = \left[c_{s6}^2 \nabla^2 - \frac{\partial^2}{\partial t^2} \right] \vec{A}_6 = 0$ $c_{d6}^2 = c_0^2 + \frac{1}{\frac{\rho_0}{\eta' \frac{\partial}{\partial t}} + \frac{\rho_0}{\lambda_2 + \eta'_2 \frac{\partial}{\partial t}}} + \frac{2}{\frac{\rho_0}{\eta_1 \frac{\partial}{\partial t}} + \frac{\rho_0}{\mu_2 + \eta_2 \frac{\partial}{\partial t}}} = c_0^2 + \frac{1}{\rho_0 Z_d} + \frac{2}{\rho_0 Z_s}$ $c_{s6}^2 = \frac{1}{\frac{\rho_o}{\eta_1 \frac{\partial}{\partial t}} + \frac{\rho_o}{\mu_2 + \eta_2 \frac{\partial}{\partial t}}} = \frac{1}{\rho_o Z_s}$

4-parameter fluid $m = 7$	$\rho_o \frac{\partial^2 \vec{u}_7}{\partial t^2} - \left[\rho_o c_o^2 + \frac{1}{\frac{1}{\lambda_1}+\frac{1}{\eta'_1\frac{\partial}{\partial t}}+\frac{1}{\lambda_2+\eta'_2\frac{\partial}{\partial t}}} + \frac{2}{\frac{1}{\mu_1}+\frac{1}{\eta_1\frac{\partial}{\partial t}}+\frac{1}{\mu_2+\eta_2\frac{\partial}{\partial t}}}\right]\nabla(\nabla\cdot\vec{u}_7)$ $+\frac{1}{\frac{1}{\mu_1}+\frac{1}{\eta_1\frac{\partial}{\partial t}}+\frac{1}{\mu_2+\eta_2\frac{\partial}{\partial t}}}\nabla\times\left(\nabla\times\vec{u}_7\right)=0$ $\vec{u}_7 = \nabla\varphi_7 + \nabla\times\vec{u}_7$ $L_{\varphi 7}\varphi_7 = \left(c_{d7}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\varphi_7 = 0$ $L_{A7}\vec{A}_7 = \left(c_{s7}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\vec{A}_7 = 0$ $c_{d7}^2 = c_0^2 + \frac{1}{\frac{\rho_0}{\lambda_1}+\frac{\rho_0}{\eta'_1\frac{\partial}{\partial t}}+\frac{1}{\lambda_2+\eta'_2\frac{\partial}{\partial t}}} + \frac{2}{\frac{\rho_0}{\mu_1}+\frac{\rho_0}{\eta_1\frac{\partial}{\partial t}}+\frac{1}{\mu_2+\eta_2\frac{\partial}{\partial t}}} = c_0^2+\frac{1}{\rho_0 Z_d}+\frac{1}{\rho_0 Z_s}$ $c_{s7}^2 = \frac{1}{\frac{\rho_o}{\mu_1}+\frac{\rho_o}{\eta_1\frac{\partial}{\partial t}}+\frac{\rho_o}{\mu_2+\eta_2\frac{\partial}{\partial t}}} = \frac{1}{\rho_o Z_s}$

4-parameter solid $m=8$	$\rho_o \frac{\partial^2 \vec{u}_8}{\partial t^2} - \left(\rho_o c_o^2 + \frac{1}{\frac{1}{\lambda_1+\eta_1'\frac{\partial}{\partial t}}+\frac{1}{\lambda_2+\eta_2'\frac{\partial}{\partial t}}} + \frac{2}{\frac{1}{\mu_1+\eta_1\frac{\partial}{\partial t}}+\frac{1}{\mu_2+\eta_2\frac{\partial}{\partial t}}}\right)\nabla(\nabla\cdot\vec{u}_8)$ $+\frac{1}{\frac{1}{\mu_1+\eta_1\frac{\partial}{\partial t}}+\frac{1}{\mu_2+\eta_2\frac{\partial}{\partial t}}}\nabla\times(\nabla\times\vec{u}_8)=0$ $\vec{u}_8=\nabla\varphi_8+\nabla\times\vec{A}_8$ $L_{\varphi 8}\varphi_8=\left(c_{d8}^2\nabla^2-\frac{\partial^2}{\partial t^2}\right)\varphi_8=0$ $L_{A8}\vec{A}_8=\left(c_{s8}^2\nabla^2-\frac{\partial^2}{\partial t^2}\right)\vec{A}_8=0$ $c_{d8}^2=c_0^2+\frac{1}{\frac{\rho_0}{\lambda_1+\eta'_1\frac{\partial}{\partial t}}+\frac{\rho_0}{\lambda_2+\eta'_2\frac{\partial}{\partial t}}}+\frac{2}{\frac{\rho_0}{\mu_1+\eta_1\frac{\partial}{\partial t}}+\frac{\rho_0}{\mu_2+\eta_2\frac{\partial}{\partial t}}}=c_0^2+\frac{1}{\rho_0 Z_d}+\frac{2}{\rho_0 Z_s}$ $c_{s8}^2=\frac{1}{\frac{\rho_o}{\mu_1+\eta_1\frac{\partial}{\partial t}}+\frac{\rho_o}{\mu_2+\eta_2\frac{\partial}{\partial t}}}=\frac{1}{\rho_o Z_s}$

CHAPTER 3

GENERAL SOLUTION OF THE FIELD QUANTITIES IN TERMS OF THE POTENTIAL FUNCTIONS IN SPHERICAL COORDINATES

In Chapter II it was shown that the differential equations of motion for any isotropic, homogeneous, viscoelastic solid and/or fluid can be systematically obtained. By utilizing Helmholtz's theorem, these differential equations can be transformed into two governing potential equations, a scalar potential equation, and a vector potential equation:

$$L_{\varphi m}\varphi_m = \left(c_{dm}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\varphi_m = 0 \qquad (3.1)$$

$$L_{Am}\vec{A}_m = \left(c_{sm}^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\vec{A}_m = 0 \qquad (3.2)$$

where $L_{\varphi m}$ and L_{Am} are linear differential operators, φ_m and $\vec{A}_m$ are the scalar and the vector potential functions respectively, c_{dm}^2 and c_{sm}^2 define the material properties (see Table 4).

If the solutions to these equations satisfy the gauge invariance

$$L_{Am}\left(\nabla\cdot\vec{A}_m\right) = 0 \qquad (3.3)$$

or

$$\nabla\cdot\vec{A}_m = 0 \qquad (3.4)$$

then all field quantities can be obtained from

$$\vec{u}_m = \nabla \varphi_m + \nabla \times \vec{A}_m \tag{3.5}$$

$$\vec{u}_m = \nabla \varphi_m + \nabla \times \vec{A}_m \tag{3.5}$$

$$2\vec{\Omega}_m = \nabla \times \vec{u}_m = \nabla \times \nabla \times \vec{A}_m \tag{3.6}$$

$$\varsigma_{ijm} = \left[-P_m + \lambda_m \left(\nabla \cdot \vec{u}_m\right)\right]\delta_{ij} + \mu_m \left[\vec{e}_i \cdot \left(\vec{e}_j \cdot \nabla\right)\vec{u}_m + \vec{e}_j \cdot \left(\vec{e}_i \cdot \nabla\right)\vec{u}_m\right]$$

and

$$\Delta_m = \nabla \cdot \vec{u}_m = \nabla^2 \varphi_m \tag{3.8}$$

Equations (3.1) to (3.8) can also be applied in studies of the mechanical interactions of layered viscoelastic continua. A classification of the type of problems that can be solved is given below.

The time dependence of the solution can be determined with the aid of linear transform techniques such as Laplace transform or Fourier transform. Whenever we can separate the dependence of the solution on the spatial variables we can determine the space dependence of basic solutions of the differential equations of motion by utilizing again linear transform techniques. Such separation of the dependence on the spatial coordinates is for example possible when we use Cartesian, circular-cylindr ical, elliptic-cylindr ical, parabolic-cylindrical, spherical, and conical coordinates. The true complexities enter usually into the problem through the character of the geometric domain. Domains defined by spheres, cones, and cylinders with circular, elliptic, and parabolic cross sections can be studied by separation of variables. Likewise, we can analyze spherical shells and cylindrical shells with circular, elliptic, and rectangular circumferences in the same manner, at least for certain boundary conditions. We also can give separable solutions for infinite domains with cavities or inclusions in the form of spheres, and cylinders of the types mentioned. Finally, as to the boundary conditions we may prescribe zero displacements or zero stresses on the boundaries, irrespective of whether we are dealing with a single medium or a multilayered medium.

As mentioned before, it is advantageous to delineate a class of general solutions in which the dependence on the spatial variable is separated and to determine the various geometric configurations of the continua and the types of boundary conditions which can be accommodated by the solutions for certain material models. We shall demonstrate this for the case of spherical coordinates.

Depending on whether the medium is a viscous fluid or an elastic solid, we use for the representation of $\vec{A}_m$ that given by (2.35). Generalizing these expressions for $\vec{A}_m$ to allow for a more complex material behavior (see Appendix A) we can write the field quantities in terms of the scalar potentials φ_m, χ_m, and ψ_m which are solutions of

$$L_{\varphi m}\,\varphi_m = 0 \tag{3.9}$$

$$L_{Am}\,\chi_m = 0 \tag{3.10}$$

$$L_{Am}\,\psi_m = 0 \tag{3.11}$$

According to (A7) and (3.5) the components of the displacement vector $\vec{u}$ are then defined by

$$u_{rm} = \frac{\partial\varphi_m}{\partial r} + \frac{\partial^2 r\psi_m}{\partial r^2} - r\nabla^2\psi_m \tag{3.12}$$

$$u_{\theta m} = \frac{1}{r}\frac{\partial\varphi_m}{\partial\theta} + \frac{1}{r}\frac{\partial^2 r\psi_m}{\partial\theta\,\partial r} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r\nabla^2\chi_m\right) \tag{3.13}$$

$$u_{\phi m} = \frac{1}{r\sin\theta}\frac{\partial\varphi_m}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial^2 r\psi_m}{\partial\phi\,\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\nabla^2\chi_m\right) \tag{3.14}$$

With (A7) and (3.6) the components of the rotational vector $2\vec{\Omega}_m$ can be expressed as

$$2\Omega_{rm} = r\nabla^2\left(\nabla^2\chi_m\right) - \frac{\partial^2\left(r\nabla^2\chi_m\right)}{\partial r^2} \tag{3.15}$$

$$2\Omega_{\theta m} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\nabla^2\psi_m\right) - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}\left(r\nabla^2\chi_m\right) \tag{3.16}$$

$$2\Omega_{\phi m} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\nabla^2\psi_m\right) - \frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi\partial r}\left(r\nabla^2\chi_m\right) \tag{3.17}$$

On the basis of (A7), (3.5), and (3.7) we can give the components of the stress tensor ς_{ijm} as

$$\varsigma_{rrm} = \left[P_m + \lambda_m\nabla^2\varphi_m + 2\mu_m\frac{\partial^2 P_m}{\partial r^2}\right] + 2\mu_m\left[\frac{\partial^3 r\psi_m}{\partial r^3} - \frac{\partial}{\partial r}\left(r\nabla^2\psi_m\right)\right] \tag{3.18}$$

$$\begin{aligned}\varsigma_{r\theta m} = {} & \mu_m\left[\frac{2}{r}\frac{\partial^2\varphi_m}{\partial\theta\partial r} - \frac{2}{r^2}\frac{\partial\varphi_m}{\partial\theta}\right] \\ & + \mu_m\left[\frac{2}{r}\frac{\partial^3 r\psi_m}{\partial\theta\partial r^2} - \frac{2}{r^2}\frac{\partial^2 r\psi_m}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\nabla^2\psi_m\right)\right] \\ & + \mu_m r\frac{\partial}{\partial r}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(r\nabla^2\chi_m\right)\right]\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
\varsigma_{r\phi m} = {} & \mu_m \left[\frac{2}{r\sin\theta}\frac{\partial^2\varphi_m}{\partial\phi\partial r} - \frac{2}{r^2\sin\theta}\frac{\partial\varphi_m}{\partial\phi} \right] \\
& + \mu_m \left[\frac{2}{r\sin\theta}\frac{\partial^3 r\psi_m}{\partial\theta\partial r^2} - \frac{2}{r^2\sin\theta}\frac{\partial^2 r\psi_m}{\partial\phi\partial r} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r\nabla^2\psi_m\right) \right] \\
& + \mu_m r\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(r\nabla^2\chi_m\right) \right]
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
\varsigma_{\theta\theta m} = {} & \left[P_m + \lambda_m\nabla^2\varphi_m + 2\mu_m\left(\frac{1}{r^2}\frac{\partial^2\varphi_m}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi_m}{\partial r} \right) \right] \\
& + 2\mu_m \left[\frac{1}{r^2}\frac{\partial^3 r\psi_m}{\partial r\partial\theta^2} + \frac{1}{r}\frac{\partial^2 r\psi_m}{\partial r^2} - \nabla^2\psi_m \right] \\
& - 2\mu_m \left[\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(r\nabla^2\chi_m\right) \right) \right]
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
\varsigma_{\theta\phi m} = {} & \mu_m \left[\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^2\theta}\frac{\partial\varphi_m}{\partial\phi} \right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\varphi_m}{\partial\phi\partial\theta} \right] \\
& + \mu_m \left[\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^2\theta}\frac{\partial^2 r\psi_m}{\partial\phi\partial r} \right) + \frac{1}{r^2\sin\theta}\frac{\partial^3 r\psi_m}{\partial\phi\partial\theta\partial r} \right] \\
& + \mu_m \left[\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(r\nabla^2\chi_m\right) \right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left(r\nabla^2\chi_m\right) \right]
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
\varsigma_{\theta\theta m} = {} & \left[P_m + \lambda_m\nabla^2\varphi_m + 2\mu_m\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi_m}{\partial\phi^2} + \frac{1}{r}\frac{\partial\varphi_m}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\varphi_m}{\partial\theta} \right) \right] \\
& + 2\mu_m \left[\frac{1}{r^2\sin^2\theta}\frac{\partial^3 r\psi_m}{\partial\phi^2\partial r} + \frac{1}{r}\frac{\partial^2 r\psi_m}{\partial r^2} + \frac{\cot\theta}{r^2}\frac{\partial^2 r\psi_m}{\partial\theta\partial r} - \nabla^2\psi_m \right] \\
& + 2\mu_m \left[\frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi\partial\theta}\left(r\nabla^2\chi_m\right) - \frac{\cot\theta}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(r\nabla^2\chi_m\right) \right]
\end{aligned} \tag{3.23}
$$

By adding (3.18), (3.21), and (3.23) we obtain the mean of the normal stresses

$$
\begin{aligned}
\frac{\varsigma_{rrm}+\varsigma_{\theta\theta m}+\varsigma_{\phi\phi m}}{3} &= -P_m + \left(\lambda_m + \frac{2}{3}\mu_m\right)\nabla^2\varphi_m \\
&\quad + \frac{2}{3}\mu_m\left[\nabla^2\left(\frac{\partial r\psi_m}{\partial r}\right) - \frac{\partial}{\partial r}\left(r\nabla^2\psi_m\right) - 2\nabla^2\psi_m\right] \\
&= -P_m + \left(\lambda_m + \frac{2}{3}\mu_m\right)\nabla^2\varphi_m
\end{aligned}
\tag{3.24}
$$

where $\left[\nabla^2\left(\frac{\partial r\psi_m}{\partial r}\right) - \frac{\partial}{\partial r}\left(r\nabla^2\psi_m\right) - 2\nabla^2\psi_m\right]$ can be proved to be identically equal to zero. The physical and mathematical interpretation of (3.24) for the cases of a viscous fluid ($m = 1$) and an elastic solid ($m = 2$) are given for example in references 9 to 13.

The terms containing $\nabla^2\varphi_m$, $\nabla^2\psi_m$, or $\nabla^2\chi_m$ in (3.12) to (3.23) can be replaced by $\frac{1}{c_{dm}^2}\frac{\partial^2\varphi_m}{\partial t^2}$, $\frac{1}{c_{sm}^2}\frac{\partial^2\psi_m}{\partial t^2}$, and $\frac{1}{c_{sm}^2}\frac{\partial^2\chi_m}{\partial t^2}$, according to Table 4 and equations (3.9) to (3.11).

All of the field quantities defined in terms of the potential functions φ_m, χ_m, and ψ_m can be applied for nonsymmetric motions as well as for axially symmetric nontorsional and torsional motions. The relative contributions of φ_m, χ_m, and ψ_m to the various types of motion can be assessed by a direct inspection of the components of the displacement vector.

In the case of axially symmetric nontorsional motion we can give the components of the displacement vectors as

$$u_{rm} = u_{rm}(r,\theta,t)$$

$$u_{\theta m} = u_{\theta m}(r,\theta,t)$$

$$u_{\phi m} = 0$$

According to (3.12), (3,13), and (3.14) we have in such a case

$$\varphi_m = \varphi_m(r,\theta,t)$$

$$\psi_m = \psi_m(r,\theta,t)$$

$$\chi_m = 0$$

Thus the corresponding field quantities are defined by

$$u_{rm} = \frac{\partial\varphi_m}{\partial r} + \frac{\partial^2 r\psi_m}{\partial r^2} - r\nabla^2\psi_m \tag{3.25}$$

$$u_{\theta m} = \frac{1}{r}\frac{\partial\varphi_m}{\partial\theta} + \frac{1}{r}\frac{\partial^2 r\psi_m}{\partial\theta\,\partial r} \tag{3.26}$$

$$u_{\phi m} = 0 \tag{3.27}$$

$$2\Omega_{rm} = 0 \tag{3.28}$$

$$2\Omega_{\theta m} = 0 \tag{3.29}$$

$$2\Omega_{\phi m} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\nabla^2\psi_m\right) \tag{3.30}$$

$$\varsigma_{rrm} = \left[P_m + \lambda_m \nabla^2 \varphi_m + 2\mu_m \frac{\partial^2 \varphi_m}{\partial r^2} \right]$$
$$+ 2\mu_m \left[\frac{\partial^3 r\psi_m}{\partial r^3} - \frac{\partial}{\partial r}\left(r \nabla^2 \psi_m \right) \right] \tag{3.31}$$

$$\varsigma_{r\theta m} = \mu_m \left[\frac{2}{r} \frac{\partial^2 \varphi_m}{\partial \theta \partial r} - \frac{2}{r^2} \frac{\partial \varphi_m}{2\theta} \right]$$
$$+ \mu_m \left[\frac{2}{r} \frac{\partial^3 r\psi_m}{\partial \theta \partial r^2} - \frac{2}{r^2} \frac{\partial^2 r\psi_m}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial}{\partial \theta} (r \nabla^2 \psi_m) \right] \tag{3.32}$$

$$\varsigma_{r\phi m} = 0 \tag{3.33}$$

$$\varsigma_{\theta\theta m} = \left[P_m + \lambda_m \nabla^2 \varphi_m + 2\mu_m \left[\frac{1}{r^2} \frac{\partial^2 \varphi_m}{\partial \theta^2} + \frac{1}{r} \frac{\partial \varphi_m}{\partial r} \right] \right]$$
$$+ 2\mu_m \left[\frac{1}{r^2} \frac{\partial^3 r\psi_m}{\partial r \partial \theta^2} + \frac{1}{r} \frac{\partial^2 r\psi_m}{\partial r^2} - \nabla^2 \psi_m \right] \tag{3.34}$$

$$\varsigma_{\theta\phi m} = 0 \tag{3.35}$$

$$\varsigma_{\phi\phi m} = \left[P_m + \lambda_m \nabla^2 \varphi_m + 2\mu_m \left[\frac{1}{r} \frac{\partial \varphi_m}{\partial r} + \frac{\cot\theta}{r^2} \frac{\partial \varphi_m}{\partial \theta} \right] \right]$$
$$+ 2\mu_m \left[\frac{1}{r} \frac{\partial^2 r\psi_m}{\partial r^2} + \frac{\cot\theta}{r^2} \frac{\partial^2 r\psi_m}{\partial \theta \partial r} - \nabla^2 \psi_m \right] \tag{3.36}$$

In the case of axially symmetric torsional motion, the components of the displacement vector are

$$u_{rm} = 0$$

$$u_{\theta m} = 0$$

$$u_{\phi m} = u_{\phi m}(r, \theta, t)$$

Such a displacement pattern is enforced if we require

$$\varphi_m = 0$$

$$\psi_m = 0$$

$$\chi_m = \chi_m(r, \theta, t)$$

Accordingly the field quantities can be expressed in terms of $\chi_m(r, \theta, t)$ alone.

$$u_{rm} = 0 \tag{3.37}$$

$$u_{\theta m} = 0 \tag{3.38}$$

$$u_{\phi m} = \frac{1}{r}\frac{\partial}{\partial\theta}(r\nabla^2\chi_m) \tag{3.39}$$

$$2\Omega_{rm} = r\nabla^2\nabla^2\chi_m - \frac{\partial^2(r\nabla^2\chi_m)}{\partial r^2} \tag{3.40}$$

$$2\Omega_{\theta m} = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(r\nabla^2\chi_m) \tag{3.41}$$

$$2\Omega_{\phi m} = 0 \tag{3.42}$$

$$\varsigma_{rrm} = -P_m \tag{3.43}$$

$$\varsigma_{r\theta m} = 0 \tag{3.44}$$

$$\varsigma_{r\phi m} = \mu_m r \frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial \theta}(r\nabla^2 \chi_m)\right] \tag{3.45}$$

$$\varsigma_{\theta\theta m} = P_m \tag{3.46}$$

$$\varsigma_{\theta\phi m} = \mu_m \left[\frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(r\nabla^2 \chi_m)\right]\right] \tag{3.47}$$

$$\varsigma_{\phi\phi m} = -P_m \tag{3.48}$$

For radially symmetric motion, we have as the components of the displacement vector:

$$u_{rm} = u_{rm}(r,t)$$
$$u_{\theta m} = 0$$
$$u_{\phi m} = 0$$

The associated scalar potentials are in this case

$$\varphi_m = \varphi_m(r,t)$$
$$\psi_m = 0$$
$$\chi_m = 0$$

and the corresponding field quantities can be expressed in terms of $\varphi_m(r,t)$:

$$u_{rm} = \frac{\partial \varphi_m}{\partial r} \tag{3.49}$$

$$u_{\theta m} = 0 \tag{3.50}$$

$$u_{\phi m} = 0 \tag{3.51}$$

$$2\Omega_{rm} = 0 \tag{3.52}$$

$$2\Omega_{\theta m} = 0 \tag{3.53}$$

$$2\Omega_{\phi m} = 0 \tag{3.54}$$

$$\varsigma_{rrm} = -P_m + \lambda_m \nabla^2 \varphi_m + 2\mu_m \frac{\partial^2 \varphi_m}{\partial r^2} \tag{3.55}$$

$$\varsigma_{r\theta m} = 0 \tag{3.56}$$

$$\varsigma_{r\phi m} = 0 \tag{3.57}$$

$$\varsigma_{\theta\theta m} = -P_m + \lambda_m \nabla^2 \varphi_m + 2\mu_m \frac{1}{r}\frac{\partial \varphi_m}{\partial r} \tag{3.58}$$

$$\varsigma_{\theta\phi m} = 0 \tag{3.59}$$

$$\varsigma_{\phi\phi m} = -P_m + \lambda_m \nabla^2 \varphi_m + 2\mu_m \frac{1}{r}\frac{\partial \varphi_m}{\partial r} \tag{3.60}$$

From the above three special cases of motions, we infer that φ_m characterizes dilatational motion, ψ_m represents nontorsional shearing motion, while χ_m represents torsional shearing motion of the medium.

CHAPTER 4

APPLICATIONS OF THE GENERAL SOLUTIONS TO SPECIFIC PROBLEMS

In Chapter 3 it was shown that all field quantities of a continuum can be obtained in terms of three independent scalar potential functions φ_m, χ_m and ψ_m which are solutions to the governing potential equations (3.9), (3.10) and (3.11). In this chapter it will be shown how these results can be applied to specific cases.

For purposes of illustration we simplify the procedures and seek harmonic time solutions of the governing equations (3.9), (3.10) and (3.11) for each material model. These equations are then further reduced into three corresponding scalar Helmholtz's equations which involve two physical quantities k_{dm} and k_{sm} which themselves vary with the material models. A list of k_{dm} and k_{sm} for each model is given in Table 6.

The k_{dm} and k_{sm} in Table 6 can be interpreted physically as dilatational wave numbers and shear wave numbers which in general are frequency dependent. The simplest forms of these wave numbers are the cases of elastic solids and viscous fluids. For materials other than these two, they are quite complicated. Once a boundary value problem is solved for a particular kind of material, one can replace the appropriate k_{dm} and k_{sm} from Table 6 and obtain the solution of the same boundary value problem for another material.

Now let us explore in more detail the physical meaning of the wave numbers k_{dm} and k_{sm} by considering the two fundamental materials, namely the elastic solid and the viscous fluid. The interpretation of these wave numbers for other materials can be generalized from these results.

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TABLE 6

HARMONIC SOLUTION CHARACTERIZED BY k^2_{dm} AND k^2_{sm}

For solutions defined by

$$\varphi_m = \phi_m e^{i\omega t}$$
$$\vec{A}_m = \left[\nabla \times (\vec{e}_z \psi_m) + \nabla \times \nabla \times (\vec{e}_z \chi_m)\right] e^{i\omega t}$$

for general cylindrical coordinates

$$\vec{A}_m = \left[\nabla \times (\vec{e}_r r \psi_m) + \nabla \times \nabla \times (\vec{e}_r r \chi_m)\right] e^{i\omega t}$$

for spherical coordinates

$$u_m = \nabla \varphi_m + \nabla \times \vec{A}_m$$

provided that ϕ_m, ψ_m and χ_m satisfy

$$(\nabla^2 + k^2_{dm})\phi_m = 0$$
$$(\nabla^2 + k^2_{sm})\psi_m = 0$$
$$(\nabla^2 + k^2_{sm})\chi_m = 0$$

where $k^2_{dm} = \dfrac{\omega^2}{c^2_{dm}}$ $\qquad k^2_{sm} = \dfrac{\omega^2}{c^2_{sm}}$

we obtain the following expressions k^2_{dm} and k^2_{sm} for the various material models considered

Viscous fluid $m = 1$	$k^2_{d1} = \dfrac{\omega^2}{c^2_{d1}} = \dfrac{\omega^2}{c^2_0 + \dfrac{1}{\rho_o}(\eta' + 2\eta)i\omega}$ $k^2_{s1} = \dfrac{\omega^2}{c^2_{s1}} = \dfrac{\omega^2}{\dfrac{\eta}{\rho_o} i\omega}$

Elastic solid $m = 2$	$k_{d2}^2 = \frac{\omega^2}{c_{d2}^2} = \frac{\omega^2}{\left(\frac{\lambda + 2\mu}{\rho_o}\right)}$ $k_{s2}^2 = \frac{\omega^2}{c_{s2}^2} = \frac{\omega^2}{\left(\frac{\mu}{\rho_o}\right)}$
Maxwell fluid $m = 3$	$k_{d3}^2 = \frac{\omega^2}{c_{d3}^2} = \frac{\omega^2}{c_0^2 + \frac{1}{\frac{\rho_o}{\lambda} + \frac{\rho_o}{\eta' i\omega}} + \frac{2}{\frac{\rho_o}{\mu} + \frac{\rho_o}{\eta i\omega}}}$ $k_{s3}^2 = \frac{\omega^2}{c_{s3}^2} = \omega^2 \left(\frac{\rho_o}{\mu} + \frac{\rho_o}{\eta i\omega}\right)$
Kelvin solid $m = 4$	$k_{d4}^2 = \frac{\omega^2}{c_{d4}^2} = \frac{\omega^2}{c_0^2 + \frac{1}{\rho_o}(\lambda + \eta' i\omega) + \frac{2}{\rho_o}(\mu + \eta i\omega)}$ $k_{s4}^2 = \frac{\omega^2}{c_{s4}^2} = \frac{\omega^2}{\frac{1}{\rho_o}(\mu + \eta i\omega)}$

3 parameter $m = 5$	$k_{d5}^2 = \frac{\omega^2}{c_{d5}^2} = \frac{\omega^2}{c_0^2 + \frac{1}{\frac{\rho_o}{\lambda_1} + \frac{\rho_o}{\lambda_2 + \eta'_2 i\omega}} + \frac{1}{\frac{\rho_o}{\mu_1} + \frac{\rho_o}{\mu_2 + \eta_2 i\omega}}}$ $k_{s5}^2 = \frac{\omega^2}{c_{s5}^2} = \omega^2 \left(\frac{\rho_o}{\mu_1} + \frac{\rho_o}{\mu_2 + \eta_2 i\omega} \right)$
3 parameter $m = 6$	$k_{d6}^2 = \frac{\omega^2}{c_{d6}^2} = \frac{\omega^2}{c_0^2 + \frac{1}{\frac{\rho_o}{\eta'_1 i\omega} + \frac{\rho_o}{\lambda_2 + \eta'_2 i\omega}} + \frac{2}{\frac{\rho_o}{\eta_1 i\omega} + \frac{\rho_o}{\mu_2 + \eta_2 i\omega}}}$ $k_{s6}^2 = \frac{\omega^2}{c_{s6}^2} = \omega^2 \left(\frac{\rho_o}{\eta_1 i\omega} + \frac{\rho_o}{\mu_2 + \eta_2 i\omega} \right)$

4 parameter $m = 7$	$k_{d7}^2 = \frac{\omega^2}{c_{d7}^2} = \frac{\omega^2}{c_0^2 + \frac{1}{\frac{\rho_o}{\lambda_1} + \frac{\rho_o}{\eta'_1 i\omega} + \frac{\rho_o}{\lambda_2 + \eta'_1 i\omega}} + \frac{2}{\frac{\rho_o}{\mu_1} + \frac{\rho_o}{\eta_1 i\omega} + \frac{1}{\mu_2 + \eta_2 i\omega}}}$ $k_{s7}^2 = \frac{\omega^2}{c_{s7}^2} = \omega^2 \left[\frac{\rho_o}{\mu_1} + \frac{\rho_o}{\eta_1 i\omega} + \frac{\rho_o}{\mu_2 + \eta_2 i\omega} \right]$
4 parameter $m = 8$	$k_{d8}^2 = \frac{\omega^2}{c_{d8}^2} = \frac{\omega^2}{c_0^2 + \frac{1}{\frac{\rho_o}{\lambda_1 \eta'_1 i\omega} + \frac{\rho_o}{\lambda_2 + \eta'_2 i\omega}} + \frac{2}{\frac{\rho_o}{\mu_1 + \eta_1 i\omega} + \frac{\rho_o}{\mu_2 + \eta_2 i\omega}}}$ $k_{s8}^2 = \frac{\omega^2}{c_{s8}^2} = \omega^2 \left[\frac{\rho_o}{\mu_1 + \eta i\omega} + \frac{\rho_o}{\mu_2 + \eta_2 i\omega} \right]$

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As can be seen later from the examples described, the wave numbers k_{dm} and k_{sm} will appear in the frequency equations. They can be cast into a dimensionless form by multiplying them with a representative geometrical length b of the materials bk_{dm}, bk_{sm}. As such they depend on the physical and geometric characteristics (frequency, viscosity, density, constitutive law, and the geometric parameters). An expansion of the solution with respect to small or large values of the parameters bk_{dm} and bk_{sm} provides two extreme cases of the general solution. For the time being we shall restrict ourselves to those cases where b is finite. The cases in which the physical quantities are kept constant and the geometrical quantities are allowed to vary will be dealt with later.

In the case of a viscous fluid, we have according to Table 6

$$k_{d1} = \frac{\omega}{c_0} \frac{\frac{\sqrt{2}}{2}}{\sqrt{1+\left(\frac{\nu'+2\nu}{c_0}\right)^2\left(\frac{\omega}{c_0}\right)^2}} \left\{ \left[\sqrt{1+\left(\frac{\nu'+2\nu}{c_0}\right)^2\left(\frac{\omega}{c_0}\right)^2}+1\right]^{1/2} - 1\left[\sqrt{1+\left(\frac{\nu'+2\nu'}{c_0}\right)^2\left(\frac{\omega}{c_0}\right)^2}-1\right]^{1/2} \right\}$$

$$k_{s1} = \sqrt{\frac{\omega}{\nu}}\frac{\sqrt{2}}{2}(1-i)$$

The expressions for k_{d1} can be approximated by series expansion for

$$\left(\frac{\nu'+2\nu}{c_0}\right)\left(\frac{\omega}{c_0}\right) = \frac{\left(\frac{\nu'+2\nu}{b^2}\right)}{\left(\frac{c_0}{b}\right)} \frac{\omega}{\left(\frac{c_0}{b}\right)} >> 1$$

and

$$\frac{\frac{\upsilon' + 2\upsilon}{b^2}}{\left(\frac{c_0}{b}\right)} \frac{\omega}{\left(\frac{c_0}{b}\right)} << 1$$

In the first case k_{d1} is approximately equal to

$$k_{d1} \cong \frac{\sqrt{2}}{2}\sqrt{\frac{\omega}{(\upsilon' + 2\upsilon)}}(1 - i)$$

In the second expansion k_{d1} is approximately equal to

$$k_{d1} \cong \left(\frac{\omega}{c_0}\right)$$

The physical interpretations of these two expansions are listed below.

k_{d1} Approximately Equal to	Possible Conditions	Physical Statement
$\frac{\sqrt{2}}{2}\sqrt{\frac{\omega}{(\upsilon'+2\upsilon)}}(1-i)$ small $\to 0$ (long wavelength)	$\omega \sim \frac{c_0}{b} << \frac{\upsilon'+2\upsilon}{b^2}$	Viscous effect dominant over compressible effect. Frequency response in the order of $\left(\frac{c_0}{b}\right)$
$\frac{\sqrt{2}}{2}\sqrt{\frac{\omega}{(\upsilon'+2\upsilon)}}(1-i)$ finite (medium wavelength)	$\omega \sim \frac{\upsilon'+2\upsilon}{b^2} >> \frac{c_0}{b}$	Viscous effect dominant over compressible effect. Frequency response in the order of $\left(\frac{\upsilon'+2\upsilon}{b^2}\right)$
$\frac{\sqrt{2}}{2}\sqrt{\frac{\omega}{(\upsilon'+2\upsilon)}}(1-i)$ large (short wavelength)	$\omega >> \frac{\upsilon'+2\upsilon}{b^2} \sim \frac{c_0}{b}$	Viscous effect same order with compressible. High frequency response.

$\left(\frac{\omega}{c_0}\right)$ small $\to 0$ (long wavelength)	$\omega << \frac{c_0}{b} \sim \frac{\upsilon' + 2\upsilon}{b^2}$	Viscous effect and compressible effect same order of magnitude. Low frequency response.
$\left(\frac{\omega}{c_0}\right)$ finite (medium wavelength)	$\omega \sim \frac{c_0}{b} >> \frac{\upsilon' + 2\upsilon}{b^2}$	Compressible effect dominant over the viscous effect. Frequency response in the order of $\left(\frac{c_0}{b}\right)$
$\left(\frac{\omega}{c_0}\right)$ small (long wavelength)	$\omega \sim \frac{\upsilon' + 2\upsilon}{b^2} << \frac{c_0}{b}$	Compressible effect dominant over the viscous effect. Frequency response in the order of $\left(\frac{\upsilon' + 2\upsilon}{b^2}\right)$

The exact expression of k_{d1} must be utilized when

$$\frac{\upsilon' + 2\upsilon}{c_0}\frac{\omega}{c_0} = \frac{\left(\frac{\upsilon' + 2\upsilon}{b^2}\right)\omega}{\left(\frac{c_0}{b}\right)\left(\frac{c_0}{b}\right)} \sim L$$

that is when either $\left(\frac{\upsilon' + 2\upsilon}{b^2}\right) > \omega > \left(\frac{c_0}{b}\right)$ or $\left(\frac{\upsilon' + 2\upsilon}{b^2}\right) < \omega < \left(\frac{c_0}{b}\right)$

Noticing that $\omega \Big/ \left(\frac{\upsilon' + 2\upsilon}{b^2}\right)$ is equivalent to the Reynolds number of the viscous fluid, but it is not the usual Reynolds number since it pertains to harmonically varying motions: It should rather be called the wave Reynolds number.

The values of k_{s1} in two extreme cases can be obtained immediately when $b\sqrt{\frac{\omega}{\upsilon}} >> 1$ or $b\sqrt{\frac{\omega}{\upsilon}} << 1$ i.e. when the frequency is either high or low.

For an elastic solid, Table 6 gives

$$k_{d2} = \frac{\omega}{\sqrt{\dfrac{\lambda + 2\mu}{\rho_0}}} = \frac{\omega}{c_{d2}} \qquad\qquad k_{s2} = \frac{\omega}{\sqrt{\dfrac{\mu}{\rho_0}}} = \frac{\omega}{c_{s2}}$$

Here the two extreme cases correspond also to high and low frequencies or equivalently to short and long wave solutions $\frac{\omega b}{c_{d2}} >> 1$ and $\frac{\omega b}{c_d} << 1$ and $\frac{\omega b}{c_{s2}} >> 1$ and $\frac{\omega b}{c_{s2}} << 1$. Physical interpretations of k_{dm} and k_{sm} for other material models can also be obtained although the situation becomes quite involved, especially with regard to the dilatational wave number k_{dm} for m = 3, 4, 5, 6, 7, 8,....

Now let us proceed to solve some problems of interest beginning with spherically symmetric motion, axially symmetric torsional motion and axially symmetric nontorsional motion. In each case we shall deal with the interior, exterior, and interior-exterio r (shell type) problems.

4.1. Spherically Symmetric Motion

This motion is defined by equations (3.49), (3.50) and (3.51) which indicate that $u_{rm} = u_{rm}(r,t)$ with $u_{\theta m} = u_{\phi m} = 0$ everywhere at all times. An equivalent definition would be $\varphi_m = \varphi_m(r,t)$ with $\psi_m + \chi_m = 0$ everywhere at all times. Thus one can see that the governing potential equations (3.9), (3.10) and (3.11) are satisfied if

$$\varphi_m = \left[A_m \vartheta_o\left(k_{dm} r\right) + B_m Y_o\left(k_{dm} r\right)\right] e^{i\omega t} \tag{4.1}$$

With (4.1) the displacement vector is according to (3.49) given by

$$u_{rm} = \left[A_m k_{dm} \vartheta'_o\left(k_{dm} r\right) + B_m k_{dm} Y'_o\left(k_{dm} r\right)\right] e^{i\omega t} \tag{4.2}$$

where A_m; B_m, are arbitrary constants and k_{dm} is defined according to Table 6.

In the case of viscous fluid [m = 1] we have

$$u_{r1} = \left[A_1 k_{d1} \vartheta'_o(k_{d1} r) + B_1 k_{d1} Y'_o(k_{d1} r)\right] e^{i\omega t} \tag{4.3}$$

The excess pressure $P_e = (P_1 - P_o)$ can be obtained from (2.25), (2.26), (2.28) and Table 6:

$$\begin{aligned} P_e &= (P_1 - P_o) = c_0^2 (\rho_1 - \rho_o) \\ &= \rho_o \left[(\upsilon' + 2\upsilon)\frac{\partial}{\partial t}\nabla^2 \varphi_1 - \frac{\partial^2 \varphi_1}{\partial t^2}\right] \\ &= -\rho_o c_0^2 \nabla^2 \varphi_1 \\ &= \rho_o c_0^2 k_{d1}^2 \left[A_1 \vartheta_o(k_{d1} r) + B_1 Y_o(k_{d1} r)\right] e^{i\omega t} \end{aligned} \tag{4.4}$$

With (4.1) and (4.4) the remaining nontrivial field quantities can according to (3.55), (3.58) and (3.60) be written as

$$\begin{aligned} \varsigma_{rr1} + P_o = {} & A_1 k_{d1}^2 \left[-(\rho_o c_0^2 + \eta' i\omega)\vartheta_o(k_{d1} r) + 2i\omega\eta \vartheta''_o(k_{d1} r)\right] e^{i\omega t} \\ & + B_1 k_{d1}^2 \left[-(\rho_o c_0^2 + i\omega\eta')Y_o(k_{d1} r) + 2i\omega\eta Y_o(k_{d1} r)\right] e^{i\omega t} \end{aligned} \tag{4.5}$$

$$\begin{aligned} \varsigma_{\theta\theta 1} + P_0 = {} & A_1 k_{d1}^2 \left[-(\rho_0 c_0^2 + i\omega\eta')\vartheta_0(k_{d1} r) + 2i\omega\eta \frac{\vartheta'_0(k_{d1} r)}{(k_{d1} r)}\right] e^{i\omega t} \\ & + B_1 k_{d1}^2 \left[-(\rho_0 c_0^2 + i\omega\eta')Y_0(k_{d1} r) + 2i\omega\eta \frac{Y'_0(k_{d1} r)}{(k_{d1} r)}\right] e^{i\omega t} \end{aligned} \tag{4.6}$$

$$\begin{aligned} \varsigma_{\phi\phi 1} + P_0 = {} & A_1 k_{d1}^2 \left[-(\rho_0 c_0^2 + i\omega\eta')\vartheta_0(k_{d1} r) + 2\omega i\eta \frac{\vartheta'_0(k_{d1} r)}{(k_{d1} r)}\right] e^{i\omega t} \\ & + B_1 k_{d1}^2 \left[-(\rho_0 c_0^2 + i\omega\eta')Y_0(k_{d1} r) + 2\omega i\eta \frac{Y'_0(k_{d1} r)}{(k_{d1} r)}\right] e^{i\omega t} \end{aligned} \tag{4.7}$$

Adding (4.5), (4.6) and (4.7), we obtain

$$\frac{\varsigma_{rr1} + \varsigma_{\theta\theta 1} + \varsigma_{\phi\phi 1}}{3} = -\left(P_o + P_e\right) + i\omega\left(\eta' + \frac{2}{3}\eta\right)\Delta_1 \tag{4.8}$$

The physical interpretation of equation (4.8) implies that the dilatational pressure $i\omega\left(\eta' + \frac{2}{3}\eta\right)\Delta_1$ in a viscous compressible fluid is equal to the algebraic sum of the static pressure P_o and the excess pressure P_e plus the averaged normal viscous stresses at any point of the fluid at any time. It should be clarified that if the Stokes relation holds, i.e. $\eta' + \frac{2}{3}\eta = 0$ it does not mean that the fluid is incompressible or the flow irrotational; the fluid is still viscous and compressible and the dilatation Δ_1 may not vanish identically at all.

In the case of the elastic solid, $m = 2$, the radial displacement according to (4.2) is

$$u_{r2} = \left[A_2 k_{d2} \vartheta'_o\left(k_{d2} r\right) + B_2 k_{d2} Y'_o\left(k_{d2} r\right)\right] e^{i\omega t} \tag{4.9}$$

Other nonzero field quantities are

$$\begin{aligned} \varsigma_{rr2} = {} & A_2 k_{d2}^2 \left[-\lambda\vartheta_o\left(k_{d2} r\right) + 2\mu\vartheta''_o\left(k_{d2} r\right)\right] e^{i\omega t} \\ & + B_2 k_{d2}^2 \left[-\lambda Y_o\left(k_{d2} r\right) + 2\mu Y''_o\left(k_{d2} r\right)\right] e^{i\omega t} \end{aligned} \tag{4.10}$$

$$\begin{aligned} \varsigma_{\theta\theta 2} = {} & A_2 k_{d2}^2 \left[-\lambda\vartheta_o\left(k_{d2} r\right) + 2\mu\frac{\vartheta'_o\left(k_{d2} r\right)}{\left(k_{d2} r\right)}\right] e^{i\omega t} \\ & + B_2 k_{d2}^2 \left[-\lambda Y_o\left(k_{d2} r\right) + 2\mu\frac{Y'_o\left(k_{d2} r\right)}{\left(k_{d2} r\right)}\right] e^{i\omega t} \end{aligned} \tag{4.11}$$

$$\varsigma_{\phi\phi 2} = A_2 k_{d2}^2 \left[\lambda \vartheta_o \left(k_{d2} r\right) + 2\mu \frac{\vartheta'_o \left(k_{d2} r\right)}{\left(k_{d2} r\right)} \right] e^{i\omega t}$$

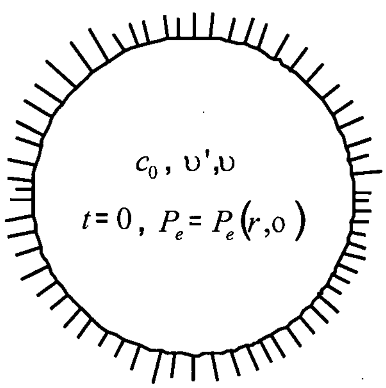
$$+ B_2 k_{d2}^2 \left[\lambda Y_o \left(k_{d2} r\right) + 2\mu \frac{Y'_o \left(k_{d2} r\right)}{\left(k_{d2} r\right)} \right] e^{i\omega t} \tag{4.12}$$

Adding (4.10), (4.11) and (4.13), we obtain

$$\frac{\varsigma_{rr2} + \varsigma_{\theta\theta 2} + \varsigma_{\phi\phi 2}}{3} = \left(\lambda + \frac{2\mu}{3} \right) \nabla_2 \tag{4.13}$$

Applications of the harmonic solutions (4.1) to (4.13) can be illustrated with the following problems.

4.1.1. <u>Spherically symmetric vibration and attenuation of a viscous, compressible fluid enclosed by a rigid spherical boundary and subjected to an initial finite excess pressure applied at the center of the fluid.</u>



The boundary conditions require that the field quantities be finite at $r = 0$ for any $t > 0$,. thus $B_1 = 0$ and at $r = a$ we have $u_{r1}(a,t) = 0$, which gives with (4.3)

$$\vartheta'_o\left(k_{d1}a\right)=\frac{d}{d\left(k_{d1}a\right)}\left[\frac{\sin\left(k_{d1}a\right)}{\left(k_{d1}a\right)}\right]=0$$

i.e.

$$\tan\left(k_{d1}a\right)=\left(k_{d1}a\right) \tag{4.14}$$

This is the characteristic equation of the problem where $\left(k_{d1}\right)$ is the wave number and as such related to the wavelength λ_{d1} by $k_{d1}=\frac{2\pi}{\lambda_{d1}}$. The root $\left(k_{d1}a\right)$ of equation (4.14) is well known [see for example, Lamb[46)]]:

$$\left(k_{d1}a\right)=1.4303\pi\,,\quad 2.450\pi\,,\quad 3.4709\pi\,,\quad 4.4774\pi\,,\quad 5.4818\pi \tag{4.15}$$

$$\frac{\lambda_{d1}}{2a}=0.6992\qquad 0.4067\qquad 0.2881\qquad 0.2233\qquad 0.1824 \tag{4.16}$$

Since k_{d1} is related to ω as shown in Table 6 and is so defined in order to obtain the solution from the field quantities given above, one can solve for ω in terms of $\left(k_{d1}a\right)=\alpha_j$. Thus

$$\frac{\omega_j}{\left(\frac{c_0\alpha_j}{a}\right)}=\left[1-\left(\frac{\left(\upsilon'+2\upsilon\right)\alpha_j}{2ac_0}\right)^2\right]^{1/2}+i\frac{\left(\upsilon'+2\upsilon\right)\alpha_j}{2ac_0} \tag{4.17}$$

where $j=1,\,2,\,3,\,4,\ldots$.
For each α_j given by (4.15) there is a ω_j such that the boundary condition $u_{r1}\left(a,k\right)=0$ is satisfied.

With the initial condition

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$$P_e(r,t)\Big/_{t=0} = P_e(r,o) \tag{4.18}$$

and (4.4), we obtain

$$P_e(r,o) = \sum_{j=1}^{\infty} A_{1j} \frac{\rho_o c_0^2 \alpha_j^2}{a^2} \vartheta_o \left(\alpha_j \frac{r}{a}\right) \tag{4.19}$$

Since the α_j 's are the roots of $\vartheta'_o(\alpha_j)=0$ we can determine the A_{1j} 's from

$$\begin{aligned} A_{1j} &= \frac{a^2}{\rho_o c_0^2 \alpha_j^2} \frac{\int_0^a r P_e(r,o) \vartheta_o(k_{d1} r) dr}{\int_0^a r \vartheta_o^2 (k_{d1} r) dr} \\ &= \frac{a^2}{\rho_o c_0^2 \alpha_j^2} \frac{\int_0^a r P_e(r,o) \vartheta_o(k_{d1} r) dr}{\frac{a^2}{2} \vartheta_o^2 (k_{d1} a)} \end{aligned} \tag{4.20}$$

In particular, when the initial excess pressure is distributed and represented by a function

$$\begin{aligned} Pe(r,o) &= \Delta P = const. \qquad & 0 \angle r \angle \in \\ &= 0 & \in \angle r \angle a \end{aligned} \tag{4.21}$$

then

$$A_{1j} = \frac{a^2}{\rho_o c_0^2 \alpha_j^2} \Delta P \frac{\sin^2\left(\frac{\alpha_j}{2}\frac{\in}{a}\right)}{\sin^2\left(\frac{\alpha_j}{2}\right)\cos^2\left(\frac{\alpha_j}{2}\right)} \tag{4.22}$$

Thus all field quantities of the viscous compressible fluid inside the rigid inclusion are uniquely determined:

$$u_{r1}(r,t) = \sum_{j=1}^{\infty} \frac{a}{\rho_o c_0^2 \alpha_j} \Delta P \frac{\sin^2\left(\frac{\alpha_j}{2}\frac{\in}{a}\right)}{\sin^2\left(\frac{\alpha_j}{2}\right) - \cos^2\left(\frac{\alpha_j}{2}\right)} \vartheta'_0\left(\alpha_j \frac{r}{a}\right) e^{i\omega_j t} \tag{4.23}$$

$$P_e(r,t) = \sum_{j=1}^{\infty} \Delta P \frac{\sin^2\left(\frac{\alpha_j}{2}\frac{\in}{a}\right)}{\sin^2\left(\frac{\alpha_j}{2}\right) - \cos^2\left(\frac{\alpha_j}{2}\right)} \vartheta_0\left(\alpha_j \frac{r}{a}\right) e^{i\omega_j t} \tag{4.24}$$

$$\varsigma_{rr1} + P_o = \sum_{j=1}^{\infty} \frac{\Delta P}{\rho_o c_0^2} \frac{\sin^2\left(\frac{\alpha_j}{2}\frac{\in}{a}\right)}{\sin^2\left(\frac{\alpha_j}{2}\right) - \cos^2\left(\frac{\alpha_j}{2}\right)}$$

$$x \left[\left(\rho_o c_0^2 + i w_j \eta'\right) \vartheta_o\left(\alpha_j \frac{r}{a}\right) + 2 i w_j \eta \frac{\vartheta''_o\left(\alpha_j \frac{r}{a}\right)}{\alpha_j \frac{r}{a}} \right] e^{i w_j t} \tag{4.25}$$

$$\varsigma_{\theta\theta 1} + P_o = \sum_{j=1}^{\infty} \frac{\Delta P}{\rho_o c_0^2} \frac{\sin^2\left(\frac{\alpha_j}{2}\frac{\in}{a}\right)}{\sin^2\left(\frac{\alpha_j}{2}\right) - \cos^2\left(\frac{\alpha_j}{2}\right)}$$

$$x \left[\left(\rho_o c_0^2 + i w_j \eta'\right) \vartheta_o\left(\alpha_j \frac{r}{a}\right) + 2 i w_j \eta \frac{\vartheta'_o\left(\alpha_j \frac{r}{a}\right)}{\alpha_j \frac{r}{a}} \right] e^{i w_j t} \tag{4.26}$$

$$\varsigma_{\phi\phi l} + P_o = \sum_{j=1}^{\infty} \frac{\Delta P}{\rho_o c_0^2} \frac{\sin^2\left(\frac{\alpha_j}{2}\frac{\in}{a}\right)}{\sin^2\left(\frac{\alpha_j}{2}\right)\cos^2\left(\frac{\alpha_j}{2}\right)}$$

$$x\left\{\left(\rho_o c_0^2 + iw_j\eta\right)\vartheta_o\left(\alpha_j\frac{r}{a}\right) + 2iw_j\eta\frac{\vartheta'_o\left(\alpha_j\frac{r}{a}\right)}{\left(\alpha_j\frac{r}{a}\right)}\right\}e^{iw_j t} \qquad (4.27)$$

where ω_j is given by (4.17) and α_j represents the roots of (4.14).

The solutions are valid in the region $\in \leq r \leq a$ for $0 \leq t \leq \infty$, where $\in$ is chosen such that ς_{rr1}, $\varsigma_{\phi\phi 1}$ and $\varsigma_{\theta\theta 1}$ will be finite at $r = \in$ and remain finite as we let $\in$ approach zero.

A detailed physical interpretation of these results is of interest. We observe from (4.17) that the motion of the viscous fluid in the inclusion can be classified as underdamped if

$$1 - \left(\frac{\upsilon' + 2\upsilon}{ac_0}\frac{\alpha_j}{2}\right)^2 > 0 \qquad (4.28)$$

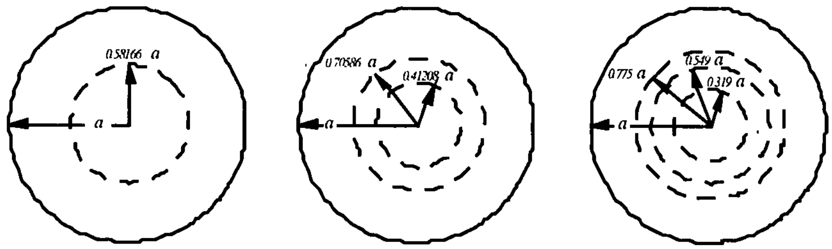
as critical-damp ed if

$$1 - \left(\frac{\upsilon' + 2\upsilon}{ac_0}\frac{\alpha_j}{2}\right)^2 = 0 \qquad (4.29)$$

and as overdamped if

$$1 - \left(\frac{\upsilon' + 2\upsilon}{ac_0}\frac{\alpha_j}{2}\right)^2 < 0 \qquad (4.30)$$

The ratio of the α_j values represents the positions of the spherical nodes. As shown by Lamb[46)] for the inviscid fluid, in the second mode there are two spherical nodes which are located at $\frac{1.4303}{2.4590}a = 0.5816592\ a$ and a. Thus one can further predict that the third mode should have three nodes, located at $\frac{1.4303}{3.4709}a = 0.4120833\ a$, $\frac{2.450}{3.4709}a = 0.7058679\ a$ and a and that the fourth mode should have four nodes, located at $\frac{1.4303}{4.4774}a = 0.3194487\ a$, $\frac{2.4590}{4.4774}a = 0.5492026\ a$, $\frac{3.4709}{4.4774}a = 0.775204\ a$ and a, etc.



$\alpha_2 = 2.4590\pi$ $\alpha_3 = 3.4709\pi$ $\alpha_4 = 4.4774\pi$

From (4.28), (4.29) and (4.30) one can see that as α_j increases, i.e. as we increase the order of the mode, the motion of the viscous fluid begins with an underdamped wave motion, becomes a critically-damped one and finally reaches an overdamped character. The cause of the damping is, of course, the υ' and υ. If $(\upsilon' + 2\upsilon) = 0$ the motion of the fluid is undamped, thus the wave motion of an inviscid fluid in a perfectly rigid inclusion never decays.

Equation (4.17) can be rewritten in the form

$$\omega_j = \pm\frac{2\pi}{T} + i\frac{2\pi}{\eta} \tag{4.31}$$

Then the time factor $e^{iw_j t}$ becomes

$$e^{-\frac{2\pi}{\eta}t \pm i\frac{2\pi}{T}t} \tag{4.32}$$

where

$$2\pi f = \frac{2\pi}{T} = \left(\frac{c_0}{a}\right)\alpha_j \left[1 - \left(\frac{\upsilon' + 2\upsilon}{ac_0}\right)^2 \left(\frac{\alpha_j}{2}\right)^2\right]^{1/2} \tag{4.33}$$

$$\frac{2\pi}{\eta} = \frac{\upsilon' + 2\upsilon}{a^2}\frac{\alpha_j^2}{2} \tag{4.34}$$

f is the frequency of the underdamped wave motion for a given mode α_j and T is the corresponding period. $\frac{2\pi}{\eta}$ is the decayed factor of all field quantities at any given position for a given mode α_j for the case of underdamped motion. From (4.32) one can see that the magnitudes of all field quantities at a particular position are attenuated by a factor of $e^{-1} = \frac{1}{2.7321}$ at a time interval of

$$t_d = \frac{\eta}{2\pi} = \frac{a^2}{\upsilon' + 2\upsilon}\frac{2}{\alpha_j^2} = \frac{2}{(\upsilon' + 2\upsilon)k_{d1}^2} = \frac{\lambda_{d1}^2}{2\pi^2(\upsilon' + 2\upsilon)} \tag{4.35}$$

t_d is called "modulus of decay" or "time constant" and (4.35) holds for underdamped and critical-damped motion.

In the case of overdamped motion, i.e. when

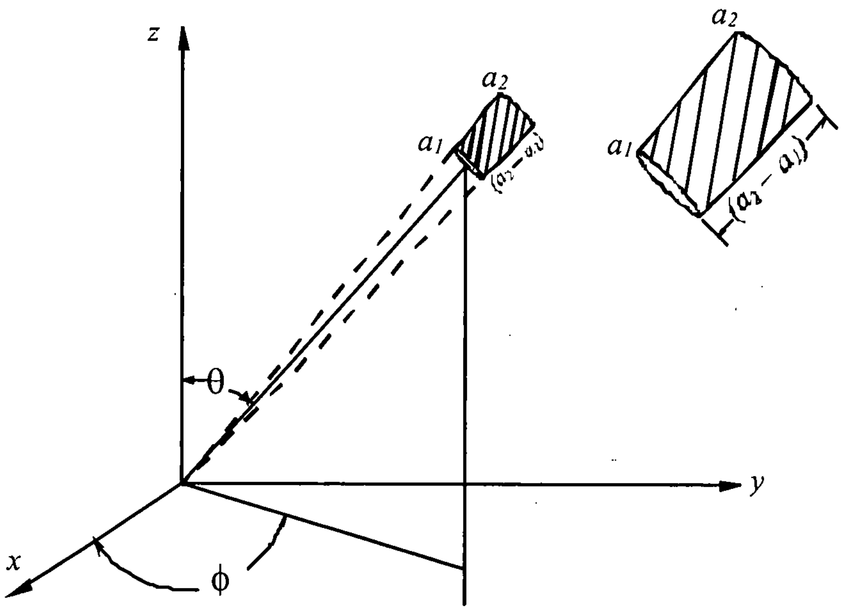
$$1-\left(\frac{\upsilon'+2\upsilon}{ac_0}\frac{\alpha_j}{2}\right)^2<0$$

harmonic wave motions cease and the "modulus of decay" becomes

$$t_d=\frac{\eta}{2\pi}=\frac{a}{c_0\alpha}\left\{\frac{\upsilon'+2\upsilon}{ac_0}\frac{\alpha_j}{2}\pm\left[\left(\frac{\upsilon'+2\upsilon}{ac_0}\frac{\alpha_j}{2}\right)^2-1\right]^{1/2}\right\} \tag{4.36}$$

4.1.2. Viscous effects in fluid cones

The theory of rigid conical tubes filled with an inviscid fluid was originally treated by Rayleigh[47)]. One can see that it is a part of the solution of spherically symmetric motion. Since $u_\phi = u_\theta = 0$ and $u_r = u_r(r,t)$, and the fluid motion is in the radial direction only, one can form a tube bound by two spherical radii a_1 and a_2 as shown below.



Rayleigh treated the problem with two particular boundary conditions, namely: (1) both ends of the tube are open; (2) both ends of the tube are closed. The same problem is extended here to account for viscosity and compressibility of the fluid. Also we allow for one more set of boundary conditions by considering one end closed and the other end opened.

a) Both ends opened:

In this case we require that $Pe(a,t)=0$ and $Pe(a_2,t)=0$. From (4.4) we have

$$A_l \vartheta_o(k_{dl}a_l) + B_l Y_o(k_{dl}a_l) = 0 \tag{4.37}$$

$$A_l \vartheta_o(k_{dl}a_2) + B_l Y_o(k_{dl}a_2) = 0 \tag{4.38}$$

Since $A_1 \neq 0$ and $B_1 \neq 0$ we obtain as frequency equation

$$\vartheta_o(k_{dl}a_l)Y_o(k_{dl}a_2) - \vartheta_o(k_{dl}a_2)Y_o(k_{dl}a_l) = 0 \tag{4.39}$$

which can be reduced to

$$\sin k_{dl}(a_2 - a_l) = 0$$

or

$$k_{dl}(a_2 - a_l) = \frac{2\pi}{\lambda_{dl}}(a_2 - a_l) = j\pi \qquad j = 1.2.3.4\ldots \tag{4.40}$$

Thus for each $k_{d1} = \dfrac{j\pi}{a_2 - a_1}$, there is a corresponding ω_j from (4.17) satisfying the boundary conditions for opened ends:

$$\omega_j = \frac{c_o}{a_2 - a_1} j\pi \left\{ \left[1 - \left(\frac{\upsilon' + 2\upsilon}{(a_2 - a_1)} \frac{j\pi}{2c_o} \right)^2 \right]^{1/2} + i \frac{\upsilon' + 2\upsilon}{(a_2 - a_1)} \frac{j\pi}{2c_o} \right\} \quad (4.41)$$

where $j = 1, 2, 3, 4, \ldots$.

The nontrivial solution of (4.37) and (4.38) is

$$\frac{A_1}{Y_o\left(\frac{j\pi a_2}{a_2 - a_1}\right)} = \frac{B_1}{-\vartheta_o\left(\frac{j\pi a_2}{a_2 - a_1}\right)} = c_j \quad (4.42)$$

This means that all field quantities can be obtained in terms of c_j:

$$\varphi_1 = \sum_{j=1}^{\infty} \frac{c_j (a_2 - a_1)^2}{(j\pi)^2 a_2 r} \sin\left[\left(\frac{a_2 - r}{a_2 - a_1}\right) j\pi\right] e^{i\omega_j t} \quad (4.43)$$

$$P_o = \sum_{j=1}^{\infty} c_j \frac{\rho_o c_o^2}{a_2 r} \sin\left[\left(\frac{a_2 - r}{A_2 - a_1}\right) j\pi\right] e^{i\omega_j t} \quad (4.44)$$

$$u_{r1} = \sum_{j=1}^{\infty} \left\{ -\frac{c_j (A_2 - a_1)}{j\pi a_2 r} \sin\left[\left(\frac{a_2 - r}{a_2 - a_1}\right) j\pi\right] \left[\cot\frac{j\pi (a_2 - r)}{(a_2 - a_1)} + \frac{a_2 - a_1}{j\pi r}\right] \right\} e^{i\omega_j t} \quad (4.45)$$

$$\varsigma_{rr1} + P_o = \sum_{j=1}^{\infty} C_j \left\{ -\left[\rho_o c_o^2 + i\omega_j(\eta' + 2\eta)\right] \frac{\sin\left(\frac{j\pi (a_2 - r)}{(a_2 - a_1)}\right)}{a_2 r} \right.$$

$$\left. + \frac{2i\omega_j \eta}{a_2 r} \left[\frac{2\sin\left(\frac{j\pi (a_2 - r)}{a_2 - a_1}\right)}{\left(\frac{j\pi}{a_2 - a_1} r\right)^2} - \frac{2\cos\frac{j\pi (a_2 - r)}{a_2 - a_1}}{\left(\frac{j\pi}{a_2 - a_1} r\right)} \right] \right\} e^{i\omega_j t} \quad (4.46)$$

$$\varsigma_{\theta\theta 1} + P_o = \sum_{j=1}^{\infty} c_j \left\{ -\left[\rho_o c_o^2 + \eta' i\omega_j\right] \frac{\sin j\pi \frac{(a_2 - r)}{a_2 - a_1}}{a_2 r} \right.$$

$$\left. + \frac{2i\omega_j \eta}{a_2 r} \left[\frac{\sin \frac{j\pi(a_2 - r)}{a_2 - a_1}}{\left(\frac{j\pi}{a_2 - a_1} r\right)^2} - \frac{\cos \frac{j\pi(a_2 - r)}{a_2 - a_1}}{\left(\frac{j\pi}{a_2 - a_1} r\right)} \right] \right\} e^{i\omega_j t} \qquad (4.47)$$

$$\varsigma_{\phi\phi 1} + P_o = \sum_{j=1}^{\infty} c_j \left\{ -\left[\rho_o c_o^2 + \eta' i\omega_j\right] \frac{\sin \frac{j\pi(a_2 - r)}{a_2 - a_1}}{a_2 r} \right.$$

$$\left. - \frac{2i\omega_j \eta}{a_2 r} \left[\frac{\sin \frac{j\pi(a_2 - r)}{a_2 - a_1}}{\left(\frac{j\pi}{a_2 - a_1} r\right)^2} - \frac{\cos \frac{j\pi(a_2 - r)}{a_2 - a_1}}{\left(\frac{j\pi}{a_2 - a_1} r\right)} \right] \right\} e^{i\omega_j t} \qquad (4.48)$$

Taking the sum of (4.46), (4.47) and (4.48), we obtain

$$\varsigma_{rr1} + \varsigma_{\theta\theta 1} + \varsigma_{\phi\phi 1} + 3P_o = \sum_{j=1}^{\infty} c_j \left\{ -\left(3\rho_o c_o^2 + i\omega_j (3\eta' + 2\eta)\right) \frac{\sin \frac{j\pi(a_2 - r)}{a_2 - a_1}}{a_2 r} \right\} e^{i\omega_j t}$$

$$= -3P_e + i\omega_j (3\eta' + 2\eta) \nabla^2 \varphi_1$$

or

$$\frac{\varsigma_{rr1} + \varsigma_{\theta\theta 1} + \varsigma_{\phi\phi 1}}{3} = -P_1 + i\omega_j\left(\eta' + \frac{2}{3}\eta\right)\Delta_1 \qquad (4.49)$$

This is in agreement with (4.8).

b) Both ends closed:

In this case we have $u_{r1}(a,t) = 0$ and $u_{r2}(a_2,t) = 0$, and with (4.3)

$$A_l k_{dl} \vartheta_0'(k_{dl} a_l) + B_l k_{dl} Y_0'(k_{dl} a_l) = 0 \qquad (4.50)$$

$$A_l k_{dl} \vartheta_0'(k_{dl} a_2) + B_l k_{dl} Y_0'(k_{dl} a_2) = 0 \qquad (4.5)$$

Again we must have $A_1 \neq 0,\ B_1 \neq 0$ for a nontrivial solution, or

$$\vartheta_0'(k_{dl} a_l) + Y_0'(k_{dl} a_2) - Y_0'(k_{dl} a_l)\vartheta_0'(k_{dl} a_2) = 0 \qquad (4.52)$$

After some manipulation, the last equation can be simplified and written as

$$\frac{\tan k_{d1}(a_2 - a_1)}{k_{d1}(a_2 - a_1)} = \frac{1}{1 + \dfrac{a_2 a_1}{(a_2 - a_1)^2}[k_{d1}(a_2 - a_1)]^2} \qquad (4.53)$$

If we take $a_1 = 0$ in this form of the characteristic equation it reduces exactly to (4.14). For given values of a_1 and a_2, $k_{d1}(a_2 - a_1)$ can be determined from (4.53). If $k_{d1}(a_2 - a_1) = \alpha_j$, then for each α_2, the corresponding frequency ω_j is defined by (4.17):

$$\frac{\omega_j}{\left(\dfrac{c_o \alpha_j}{a_2 - a_1}\right)} = \left[1 - \left(\frac{\upsilon' + 2\upsilon}{2} \frac{\alpha_j}{c_o (a_2 - a_1)}\right)^2\right]^{1/2} + i \frac{\upsilon' + 2\upsilon}{2} \frac{\alpha_j}{c_o (a_2 - a_1)} \qquad (4.54)$$

Since (4.54) represents a complex number, it is convenient to represent it graphically together with the expression

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$$\frac{\&}{p_e / \rho_o} = \frac{-\rho_e}{p_e} = \frac{\omega_j}{\left(\frac{c_o}{a_2 - a_1}\alpha_j\right)} \frac{\alpha_j}{c_o(a_2 - a_1)} e^{-i\frac{\pi}{2}} \tag{4.55}$$

which is obtained from (4.4) and (2.1). We note that $\rho_o \& / p_e$ lags behind $\omega_j / \frac{c_o}{a_2 - a_1}\alpha_j$ with a phase angle $\pi/2$ and that the amplitude shrinks with a factor $\alpha_j / c_o(a_2 - a_1)$ with respect to the expression (4.54). When $a_1 = 0$, (4.54) reduces to (4.17).

The physical interpretation of these results is as follows. After obtaining $\alpha_j = k_{d1}(a_2 - a_1)$ from (4.53) for given values of a_1 and a_2, the mode shape of the vibrating fluid between two concentric rigid spherical boundaries can be determined immediately by taking the ratio among the α_j values as demonstrated in example 4.1.1. Corresponding to each α_j the magnitude of ω_j can be determined from $\omega_j / \frac{c_o \alpha_j}{a_2 - a_1} = 1$. If $(\upsilon' + 2\upsilon)$ is known, it follows that θ_j can be determined and thus the magnitude and direction of $\rho\ \rho_e / p_e$. The projection of ω_j along the real axis gives the value of $\frac{2\pi}{T_j}$ and that along the imaginary axis gives the value of $\frac{2\pi}{\tau_j}$. $\frac{2\pi}{T_j}$ is the frequency of vibration for the particular mode α_j and $\frac{2\pi}{\tau_j}$ gives the "decay time" for such a mode. In particular, when the Stokes relation $\upsilon' + 2\upsilon = 0$ holds, then $\theta_j = 0$, the fluid becomes purely elastic, and there will be no decay of the motion at all.

According to the preceding physical interpretation, one can perform an experiment to determine $(\upsilon' + 2\upsilon)$, provided that the mode shape corresponding to an α_j value can be induced. Since α_j is known theoretically from (4.53) for given a_2 and a_1 and with T_j determined experimentally, we can evaluate $(\upsilon'+2\upsilon)$ from

$$\left[1-\left(\frac{\upsilon'+2\upsilon}{2}\,\frac{\alpha_j}{c_o(a_2-a_2)}\right)^2\right]^{1/2}=\frac{2\pi}{T_j}\,\frac{1}{\left(\dfrac{c_o\alpha_j}{a_2-a_1}\right)}$$

The quantities θ_j, η_j, and ρ_e/p_e can thus be determined from the expressions given in the graph above.

The values of $\alpha_j = k_{d1}(a_2 - a_1)$ can be discussed in some detail for two extreme cases: $a_1 \neq 0$; and, $a_1 >> (a_2 - a_1)$, $a_2 >> (a_2 - a_1)$ and $a_2 - a_1 = h =$ finite. For $a_1 = 0$ the values of α_j are the same as in example 4.1.1. In the second case the fluid cone degenerated into a viscous fluid vibrating between two almost parallel, plane rigid boundaries. Denoting $a_2 + a_1 = b = 2a$, equation (4.53) can be rewritten as

$$\frac{\tan k_{d1}h}{k_{d1}h}=\frac{1}{1+\dfrac{(k_{d1}h)^2}{4}\left[\left(\dfrac{b}{h}\right)^2-1\right]} \qquad (4.56)$$

If $(b/h)\rightarrow\infty$ and $k_{d1}h$ is finite, then $\tan k_{d1}h \rightarrow 0$. Therefore, $k_{d1}h = j\pi$. If $b/h >> 1$, the first approximate solution of (4.56) can be obtained by setting $k_{d1}h = j\pi + \epsilon$ where $\epsilon << 1$ and $\tan\epsilon \cong \epsilon$. Then (4.53) is reduced to $(j\pi+\epsilon)\,\epsilon \cong 4\left(\frac{h}{b}\right)^2$ and we have

$$\in \cong \frac{4}{j\pi}\left(\frac{h}{b}\right)^2 - \frac{16}{(j\pi)^3}\left(\frac{h}{b}\right)^4 + \ldots\ldots$$

Thus for $b/h >> 1$, $k_{d1}h$ can be given as

$$\alpha_j = k_{d1}h = j\pi + \in = j\pi + \frac{4}{j\pi}\left(\frac{h}{b}\right)^2 - \frac{16}{(j\pi)^3}\left(\frac{h}{b}\right)^4 + \ldots \tag{4.57}$$

and the corresponding wavelength λ_{d1} is approximately

$$\begin{aligned}\lambda_{d1} &= \frac{2\pi}{k_{d1}} \cong \frac{2\pi h}{j\pi + \in} \cong \frac{2\pi h}{j\pi}\left(1 - \frac{\in}{j\pi}\right) \\ &= \frac{2h}{j}\left[1 - \frac{4}{(j\pi)^2}\left(\frac{h}{b}\right)^2 + \frac{16}{(j\pi)^4}\left(\frac{h}{b}\right)^4 - \ldots\right]\end{aligned} \tag{4.58}$$

Since we consider the problem of standing waves (vibrations) which are attenuated with time, the results of (4.57) and (4.58) are identical with those obtained by Rayleigh[47)] for the case of an inviscid fluid. When (4.57) $k_{d1}h = \alpha_j$ is substituted into (4.54), the frequency is clearly a complex number except when $\upsilon' + 2\upsilon = 0$ or when $\upsilon' = \upsilon = 0$ (Rayleigh's solution)[47)].

c) One end closed and the other opened:

Here the boundary conditions are $u_r(a_1, t) = 0$ and $P_e(a_2, t) = 0$, or with (4.3) and (4.4)

$$A_l k_{dl} \vartheta'_0 (k_{dl} a_l) + B_l k_{dl} Y'_0 (k_{dl} a_l) = 0 \tag{4.59}$$

$$A_l \vartheta_0 (k_{dl} a_2) + B_l Y_0 (k_{dl} a_2) = 0 \tag{4.60}$$

In this case we also have $A_1 \neq 0$, $B_1 \neq 0$, and thus obtain as frequency equation the relation

$$\vartheta'_0 (k_{d1} a_1) Y_0 (k_{d1} a_2) - Y'_0 (k_{d1} a_1) \vartheta_0 (k_{d1} a_2) = 0 \tag{4.61}$$

which can be simplified to

$$\frac{\tan k_{d1}(a_2 - a_1)}{k_{d1}(a_2 - a_1)} = \frac{a_1}{(a_1 - a_1)} = \frac{a_2}{a_2 - a_1}\left[- \frac{a_1}{a_2}\right] \tag{4.62}$$

For given values of a_1 and a_2 with $a_2 > a_1$, we can determine $k_{d1}(a_2 - a_1)$ from (4.62). If $k_{d1}(a_2 - a_1) = \alpha_j$ then-for each α_j value, the corresponding frequency ω_j can be determined from (4.54). When the end opened is at $r = a_1$ and the closed end at $r = a_2$, the characteristic equation (4.62) is

$$\frac{\tan k_{d1}(a_2 - a_1)}{k_{d1}(a_2 - a_1)} = \frac{a_2}{a_2 - a_1} \tag{4.63}$$

The solutions of (4.62) and (4.63) can be evaluated by plotting $\tan k_{d1}(a_2 - a_1)$, $\frac{a_2}{a_2 - a_1} k_{d1}(a_2 - a_1)$ and $-\frac{a_1}{a_2 - a_1} k_{d1}(a_2 - a_1)$ versus $k_{d1}(a_2 - a_1)$ as shown below.

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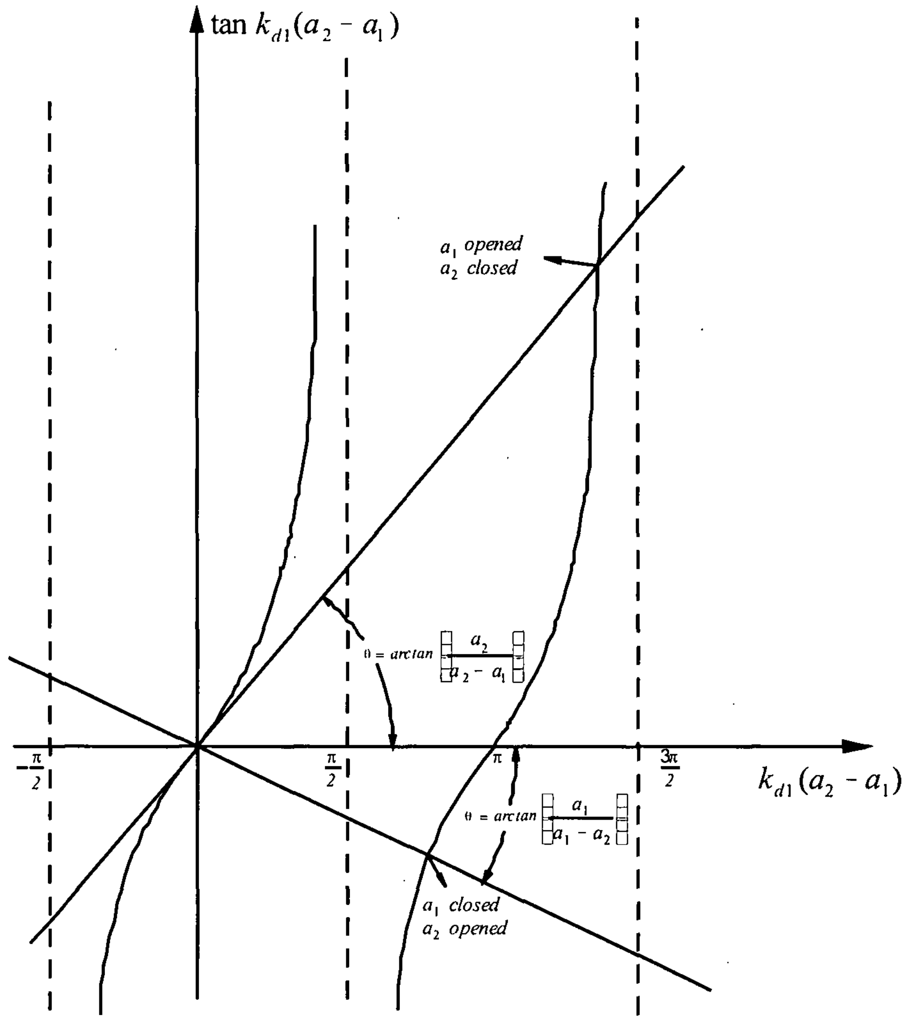

tan $k_{d1}(a_2 - a_1)$
a_1 opened
a_2 closed
$\theta = \arctan \left[\frac{a_2}{a_2 - a_1} \right]$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
$k_{d1}(a_2 - a_1)$
$\theta = \arctan \left[\frac{a_1}{a_1 - a_2} \right]$
a_1 closed
a_2 opened

One can observe that the intersections of $\tan k_{d1}(a_2 - a_1)$ with $k_{d1}(a_2 - a_1)\left(\frac{a_1}{a_2 - a_1}\right)$ and $k_{d1}(a_2 - a_1)\left(\frac{a_2}{a_2 - a_1}\right)$ provide two distinct solutions corresponding to the two alternatives. For each $\alpha_j = k_{d1}(a_2 - a_1)$ the decay factor is less when the cone is open at $r = a_1$ and closed at $r = a_2$ (loudspeaker analogy). From the figure, one can see that the maximum of the decay factor ratio for the two alternatives is e^{π}. The decay factor ratio depends on the values of a_2 and a_1 and ranges from e^{π} and $e^{0.4303\,\pi}$. However with increasing this ratio reaches e^{π} independent of a_2 and a_1.

In concluding this discussion of the viscous effects in fluid cones it should be noted that the present solutions are valid only insofar as they allow for slip in the radial direction along the wall of the tube. Strictly speaking, once the viscous effect is considered the solutions are exact only for viscous fluid contained within two concentric spherical boundaries. A formal solution for the conical pipe containing viscous fluid is rather involved and the present example serves to provide some insight into the dynamic behavior of such bodies. The field quantities are not written out for this case since they can be easily obtained as for examples a) and b).

4.1.3. <u>Spherically symmetric free vibration of a spherical elastic solid, a spherical viscous fluid drop and a viscoelastic sphere.</u>

The boundary conditions considered are: $\varsigma_{rrm}(r = a,\ t) = 0$, $\varsigma_{r\phi m}(r = a,\ t) = 0$, and $\varsigma_{r\phi m}(r = a,\ t) = 0$. Since $\varsigma_{r\theta m} = \varsigma_{r\phi m} = 0$ is automatically satisfied for spherically symmetric motion, $\varsigma_{rrm}(r = a, t) = 0$ is the only condition to be enforced.

For an elastic solid $(m = 2)$ we have according to (4.10) and with $B_2 = 0$ (to achieve finiteness of the field quantities at the origin $r = 0$ the characteristic equation

$$-\lambda\vartheta_o(k_{d2}a) + 2\mu\vartheta_o''(k_{d2}a) = 0 \tag{4.64}$$

which can be simplified to

$$1 - \left[\frac{1}{(k_{d2}a} - \frac{\lambda + 2\mu}{4\mu}(k_{d2}a)\right]\tan(k_{d2}a) = 0 \tag{4.65}$$

This result was also obtained in another way by Love[34]. He expressed $\frac{4\mu}{\lambda + 2\mu}$ in terms of Poisson's ratio ς as $\frac{2(1 - 2\varsigma)}{1 - \varsigma}$. With $-1 < \varsigma < \frac{1}{2}$ we have $3 > \frac{4\mu}{\lambda + 2\mu} > 0$. Love gave the six lowest roots of (4.65) for a Poisson's ratio $\varsigma = 0.25$: $k_{d2}a = 0.8160\pi$, 1.9285π, 2.9359π, 3.9658π, 4.9728π and 5.9774π.

In the case of a viscous fluid ($m = 1$) we set $B_1 = 0$ in the harmonic solution (4.5) to ascertain finiteness of the field quantities at the origin r = 0. The boundary condition $\varsigma_{rr1}(r = a,t) = 0$ leads to the characteristic equation for a spherical viscous fluid drop:

$$1 - \left[\frac{1}{(k_{d1}a)} - \frac{\rho c_o^2 + i\omega(\eta' + 2\eta)}{i\omega\eta}(k_{d1}a)\right]\tan(k_{d1}a) = 0 \tag{4.66}$$

Note that (4.66) can also be obtained by inspection from (4.65) and Table 6.

Similarly we have in the case of a viscoelastic material

$$1 - \left[\frac{1}{(k_{dm}a)} - \frac{(k_{dm}a)}{4\left(\frac{k_{dm}}{k_{sm}}\right)^2}\right]\tan(k_{dm}a) = 0 \tag{4.67}$$

where k_{dm} and k_{sm} are listed in Table 6 for a given material model m. Equation (4.67) reduces to that obtained by Bland[42)] when we substitute k_{dm}, and k_{sm}, from Tables 3 and 4.

4.1.4. Spherically symmetric free vibration of an elastic shell, a viscous fluid shell (bubble) and a viscoelastic shell.

We consider here shells which are geometrically defined by two concentric spheres with radii $r = a_1$ and $r = a_2\,(a_2 > a_1)$. For traction-free boundaries we have

$$\varsigma_{rrm}(r = a_1, t) = \varsigma_{r\theta m}(r = a_1, t) = \varsigma_{r\phi m}(r = a_1, t) = 0$$

$$\varsigma_{rrm}(r = a_2, t) = \varsigma_{r\theta m}(r = a_2, t) = \varsigma_{r\phi m}(r = a_2, t) = 0$$

As indicated before, $\varsigma_{r\theta m} = \varsigma_{r\phi m} = 0$ is automatically satisfied for spherically symmetric motion and we therefore have to require only $\varsigma_{rrm}(r = a_1, t) = 0$ and $\varsigma_{rrm}(r = a_2, t) = 0$.

In the case of an elastic shell $(m = 2)$ the above boundary conditions imply according to (4.10)

$$A_2 k_{d2}^2\left[-\lambda \vartheta_0(k_{d2}a_1) + 2\mu \vartheta_0''(k_{d2}a_1)\right] + B_2 k_{d2}^2\left[-\lambda Y_0(k_{d2}a_1) + 2\mu Y_0''(k_{d2}a_1)\right] = 0 \quad (4.68)$$

and

$$A_2 k_{d2}^2\left[-\lambda \vartheta_0(k_{d2}a_2) + 2\mu \vartheta_0''(k_{d2}a_2)\right] + B_2 k_{d2}^2\left[-\lambda Y_0(k_{d2}a_2) + 2\mu Y_0''(k_{d2}a_2)\right] = 0 \quad (4.69)$$

Seeking the nontrivial solutions we must have $A_2 \neq 0$, $B_2 \neq 0$, or

$$\frac{-\lambda Y_0(k_{d2}a_2) + 2\mu Y_0''(k_{d2}a_2)}{-\lambda \vartheta_0(k_{d2}a_2) + 2\mu \vartheta_0''(k_{d2}a_2)} = \frac{-\lambda Y_0(k_{d2}a_1) + 2\mu Y_0''(k_{d2}a_1)}{-\lambda \vartheta_0(k_{d2}a_1) + 2\mu \vartheta_0''(k_{d2}a_1)} \quad (4.70)$$

(4.70) is the characteristic equation for spherically symmetric free vibration of a spherical shell with an inner radius a_1 and an outer radius a_2. This equation can be simplified and written as

$$(k_{d2}a_2) + arc\tan\left(\frac{1}{k_{d2}a_2} - \frac{\lambda + 2\mu}{4\mu}(k_{d2}a_2)\right)$$

$$= (k_{d2}a_1) + arc\tan\left(\frac{1}{k_{d2}a_1} - \frac{\lambda + 2\mu}{4\mu}(k_{d2}a_1)\right) \qquad (4.71)$$

A numerical example of the solution of (4.71) is given below.

$a_1 = 1.81$ $\qquad$ $a_2 = 2.54$ $\qquad$ Poisson's ratio $\qquad$ $\varsigma = 0.34$

The roots $k_{d2}a_2$ are

$\frac{\omega a_1}{c_{d2}} = k_{d2}a_2$ = 7.8791995, 15.62335, 23.397896, 31.179907,

38.964892, 46.751358, 54.538672, 62.326515,

70.114712, 77.903153, ...

For a geometrical parametric study of the characteristic equation (4.71) we consider the following two cases.

(a) $a_2 - a_1 = h, \left(\frac{h}{2a}\right) << 1$.

This means the shell is very thin and $a_2 \approx a_1 = a$.

(b) $a_2 - a_1 = h$ where h is finite and $a_2 \to \infty$, $a_1 \to \infty$, and $a_2 + a_1 = 2a$. We again have $\left(\frac{h}{2a}\right) << 1$ but now the shell is approximately a plate of finite thickness h.

In the case (a) we arrive at a first approximation for the roots of (4.71) by expanding the expressions containing a_2 into a Taylor series with respect to a_1:

$$\left(\frac{h}{a}\right)\left\{\left[(k_{d2}a)^2 - (3c - c^2)\right] + \frac{c\,(k_{d2}a)^4 + 2c^2(k_{d2}a)^2 - c^2(3c - c^2)}{(k_{d2}a)^4 + (c^2 - 2c)(k_{d2}a)^2 + c^2}\left(\frac{h}{a}\right) + 0\left(\frac{h}{a}\right)^2 + \dots\dots\right\} = 0 \qquad (4.72)$$

where $c = \dfrac{4\mu}{\lambda + 2\mu} = \dfrac{2(1 - 2\varsigma)}{1 - \varsigma}$

Equation (4.72) is the characteristic equation for the spherically symmetric free vibration of a very thin shell of radius a and thickness h. The first order solution of this equation gives

$$\frac{\omega a}{c_{d2}} = k_{d2}a = \sqrt{(3c - c^2)} = \sqrt{\frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)}} \qquad (4.73)$$

The period of vibration of the thin shell is

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \pi a\sqrt{\frac{3}{\mu}\left(\frac{\lambda + 2\mu}{3\lambda + 2\mu}\right)} \qquad (4.74)$$

which is identical to the result obtained by Love[34)].

For the case (b) we rearrange (4.71)

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$$\tan k_{d2}\left(a_2 - a_1\right) = \frac{\dfrac{1}{k_{d2}a_1} - \dfrac{1}{k_{d2}a_2} + \dfrac{1}{c}\left(k_{d2}a_2 - k_{d2}a_1\right)}{1 + \left(\dfrac{1}{k_{d2}a_1} - \dfrac{1}{c}k_{d2}a_1\right)\left(\dfrac{1}{k_{d2}a_2} - \dfrac{1}{c}k_{d2}a_2\right)} \tag{4.75}$$

With $a_2 - a_1 = h$, $a_2 + a_1 = b = 2a$, this can be further simplified:

$$\tan\left(k_{d2}h\right) = \frac{\left(k_{d2}h\right)^2 + 1 + \dfrac{1}{4c}\left[\left(k_{d2}b\right)^2 - \left(k_{d2}h\right)^2\right]}{\dfrac{\left(k_{d2}b\right)^2 - \left(k_{d2}h\right)^2}{4} + 1 - \dfrac{1}{2c}\left[\left(k_{d2}b\right)^2 + \left(k_{d2}h\right)^2\right] + \dfrac{1}{4c^2}\left[\left(k_{d2}b\right)^2 + \left(k_{d2}h\right)^2\right]^2} \tag{4.76}$$

As $b/h \to \infty$ with $\left(k_{d2}h\right)$ finite, the right-hand side of (4.76) approaches zero, thus $\tan\left(k_{d2}h\right) \to 0$. Therefore $k_{d2}\pi = j\pi$ where $j = 1,2,3,4,\ldots$. If $b/h >> 1$ the first approximation of (4.76) can be obtained by setting $\left(k_{d2}h\right) = j\pi + \in$ where $\in << 1$ and $\tan \in\, = \,\in$. Equation (4.76) is then reduced to

$$\in \cong \frac{c}{\left(j\pi + \in\right)^2}\left(\frac{h}{b}\right)^2 \qquad \text{or} \qquad \in \cong \frac{c}{\left(j\pi\right)^2}\left(\frac{h}{b}\right)^2 - \frac{2c^2}{\left(j\pi\right)^5}\left(\frac{h}{b}\right)^4 + \ldots$$

For $b/h >> 1$ or $h/b << 1$ we can approximate $k_{d2}h$ as

$$\left(k_{d2}h\right) = j\pi + \frac{c}{\left(j\pi\right)^2}\left(\frac{h}{b}\right)^2 - \frac{2c^2}{\left(j\pi\right)^5}\left(\frac{h}{b}\right)^4 + \ldots. \tag{4.77}$$

The corresponding wavelength λ_{d2} and the corresponding period of vibration T are given by

$$\lambda_{d2} = \frac{2\pi}{k_{d2}} = \frac{2\pi h}{j\pi + \in} \cong \frac{2\pi h}{j\pi}\left[1 - \frac{\in}{j\pi}\right] = \frac{2h}{j}\left[1 - \frac{c}{(j\pi)^3}\left(\frac{h}{b}\right)^2 + \frac{2c^2}{(j\pi)^6}\left(\frac{h}{b}\right)^4 + \ldots\right] \quad (4.78)$$

and

$$T = \frac{2\pi}{\omega} \cong \frac{2\pi h}{c_{d2}}\frac{1}{(j\pi + \in)} = \frac{2h}{jc_{d2}}\left[1 - \frac{c}{(j\pi)^3}\left(\frac{h}{b}\right)^2 + \frac{2c^2}{(j\pi)^6}\left(\frac{h}{b}\right)^4 + \ldots\right] \quad (4.79)$$

A comparison of the two geometrical parametric studies of the characteristic equation (4.71) is listed in the chart below.

$1 \gg \left(h/2a\right)$	
$h \to 0,\ a$ finite	h finite, $a \to \infty$
$k_{d2} = \frac{1}{a}\sqrt{3c - c^2}$ $c = \frac{4\mu}{\lambda + 2\mu} = \frac{2(1-2\varsigma)}{1-\varsigma}$	$k_{d2} = \frac{1}{h}\left[j\pi + \frac{c}{(j\pi)^2}\left(\frac{h}{2a}\right)^2 - \frac{2c^2}{(j\pi)^5}\left(\frac{h}{2a}\right)^4 + \ldots\right]$ $c = \frac{4\mu}{\lambda + 2\mu} = \frac{2(1-2\varsigma)}{1-\varsigma}$ $j = 1, 2, 3, 4, \ldots$ integers
$T = \frac{2\pi a}{c_{d2}}\frac{1}{\sqrt{3c - c^2}}$	$T = \frac{2h}{jc_{d2}}\left[1 - \frac{c}{(j\pi)^3}\left(\frac{h}{2a}\right)^2 + \frac{2c^2}{(j\pi)^5}\left(\frac{h}{2a}\right)^4 - \ldots\right]$ $j = 1, 2, 3, 4, \ldots$ integers
Long period, low frequency	Short period, high frequency
Thin spherical shell (Physically this represents the breathing of the spherical shell in the radial direction.)	Plate of finite thickness (Physically this represents the thickness stretch of the shell in the radial direction. In particular, as the radius approaches infinity, it becomes the thickness stretch vibration of a plate.)

This geometric study of the characteristic equation (4.71) can also be formalized in the following way. Since $a_2 - a_1 = h$, $a_2 + a_1 = b = 2a$ and $\in = \left(\frac{h}{2a}\right) = \left(\frac{h}{b}\right)$ we obtain

$$a_2 = a(1+\in) = \frac{h}{2}\left(\frac{1}{\in} + 1\right)$$

$$a_1 = a(1+\in) = \frac{h}{2}\left(\frac{1}{\in} - 1\right)$$

Substituting $a_2(a_1,\in)$, $a_1(a,\in)$ and $a_2(h,\in)$, $a_1(h,\in)$ into (4.71), we can then write the characteristic equation (4.71) as a function of $(a,\in)$ or as a function of $(h,\in)$. Using Taylor's series expansion for both expressions of (4.71) in terms of ascending powers of $\in$ and noting that the coefficients of 1, $\in$, $\in^2$, $\in^3$, ...must be zero we obtain the same solutions shown in the chart.

In view of the fact that the characteristic equation (4.71) is the correct exact solution for the free vibration of a spherical shell of inner radius a_1 and outer radius a_2, the two degenerated solutions just obtained can be regarded as "singular perturbation" solutions of the characteristic equation. It is called "singular" or "irregular" because there is only one limit $\in \to 0$ but two distinct solutions are produced. Other parametric physical quantities are not affected, as they appear in the degenerated solutions. Systematic techniques and detailed reviews of the method of perturbation can be found in books and articles by Lagerstrom et al. [18)], Van Dyke[19)], Cole[20)] and Chang[21)].

In the case of a viscous fluid shell $(m = 1)$ the characteristic equation can be obtained from (4.5) or simply by invoking the correspondence principle and using Table 6 and (4.71). As characteristic equation we obtain:

$$\left(k_{d1}a_2\right) + arc\tan\left[\frac{1}{\left(k_{d1}a_2\right)} - \frac{c_0^2 + \left(\upsilon' + 2\upsilon\right)i\omega}{4\upsilon i\omega}\left(k_{d1}a_2\right)\right]$$

$$= \left(k_{d1}a_1\right) + arc\tan\left[\frac{1}{k_{d1}a_1} - \frac{c_0^2 + \left(\upsilon' + 2\upsilon\right)i\omega}{4\upsilon i\omega}\left(k_{d1}a_1\right)\right] \quad (4.80)$$

The difference between (4.71) and (4.80) is the coefficient $c_m = \frac{4k_{dm}^2}{k_{sm}^2}$ which is a constant for the elastic solid $c_2 = \frac{4\mu}{\lambda + 2\mu}$ and frequency dependent for the viscous fluid $c_1 = \frac{4\upsilon' i\omega}{c_0^2 + \left(\upsilon' + 2\upsilon\right)\omega}$. Thus (4.80) is more complicated to solve for the unknown ω (frequency). In (4.71) ω is a real number, whereas in (4.80) ω is complex. Although $c_1 = \frac{4\mu' i\omega}{c_0^2 + \left(\upsilon' + 2\upsilon\right)\omega}$ is frequency dependent, the geometrical parametric study of (4.80) will not be not be affected by this and we can proceed as in the case of (4.71).

If we replace k_{d2} in (4.73) by $k_{d1} = \frac{\omega}{\sqrt{c_0^2 + i\left(\upsilon' + 2\upsilon\right)\omega}}$ and c_2 by $\frac{4\upsilon' i\omega}{c_0^2 + i\left(\upsilon' + 2\upsilon\right)\omega}$, the characteristic equation of a thin spherical viscous fluid shell of radius a vibrating in a spherically symmetric breathing mode is

$$k_{d1}a = \frac{\omega a}{\sqrt{c_0^2 + i\left(\upsilon' + 2\upsilon\right)\omega}} = \left[\left(\frac{4\upsilon i\omega}{c_0^2 + \left(\upsilon' + 2\upsilon\right)i\omega}\right)3 - \frac{4\upsilon i\omega}{c_0^2 + \left(\upsilon' + 2\upsilon\right)i\omega}\right]^{1/2}$$

From this relation we deduce

$$\omega = i\frac{a^2c_0^2 + 4\mu\left(3\upsilon' + 2\upsilon\right)}{2\left(\upsilon' + 2\upsilon\right)a^2} \pm \frac{a^2c_0^2 + 4\mu\left(3\upsilon' + 2\upsilon\right)}{2\left(\upsilon' + 2\upsilon\right)a^2}i\left[1 - \frac{48\upsilon a^2c_0^2\left(\upsilon' + 2\upsilon\right)}{\left(a^2c_0^2 + 4\upsilon\left(3\upsilon' + 2\upsilon\right)\right)^2}\right]^{1/2}$$

$$=\frac{2\pi}{T}+i\frac{2\pi}{\eta} \tag{4.81}$$

We know that the motion is overdamped because

$$1-\frac{48\upsilon a^2c_0^2\left(\upsilon'+2\upsilon\right)}{\left[a^2c_0^2+4\upsilon\left(3\upsilon'+2\upsilon\right)\right]^2}>0$$

Similarly it follows from (4.77) that the characteristic equation of a thin spherical viscous fluid plate of thickness h undergoing thickness stretch vibrations is

$$k_{d1}h=\frac{\omega h}{\sqrt{c_0^2+i\left(\upsilon'+2\upsilon\right)\omega}}\cong j\pi+\frac{1}{\left(j\pi\right)}\left(\frac{h}{2a}\right)^2\frac{4\upsilon i\omega}{c_0^2+\left(\upsilon'+2\upsilon\right)\omega}+\ldots..$$

The corresponding relation for ω can be given as

$$\omega^3-\frac{i}{\left(\upsilon'+2\upsilon\right)}\left[c_0^2+\frac{1}{h^2}\left[j\pi\left(\upsilon'+2\upsilon\right)+\frac{4\upsilon}{\left(j\pi\right)^2}\left(\frac{h^2}{2a}\right)^2\right]\right]\omega^2$$

$$-\frac{2j\pi c_0^2}{\left(\upsilon'+\upsilon\right)h^2}\left[j\pi\left(\upsilon'+2\upsilon\right)+\frac{4\upsilon}{\left(j\pi\right)^2}\left(\frac{h}{2a}\right)^2\right]\omega+\frac{i}{\left(\upsilon'2\upsilon\right)}\frac{\left(j\pi\right)^2c_0^2}{h^2}=0 \tag{4.82}$$

The stability of the solution can be examined without solving equation (4.82) for ω . It is well known that for a solution of the form $e^{\lambda t}$ where is characterized by $\lambda^3+\alpha\lambda^2+B\lambda+\partial=0$ is stable (i.e. real part of λ 's are negative) if $\alpha>0$, $\beta>0$, $\partial>0$ and $\alpha\beta-\partial>0$. We now let $\lambda=i\omega$ and note that our characteris tic equation (4.82) assumes the form

$$\omega^3 - i\alpha\omega^2 - \beta\omega + i\partial = 0$$

in which α, β and ∂ are all real and positive. Hence the thickness stretch vibrations of a spherical viscous fluid plate are also stable.

In the case of a viscoelastic shell $(m = 3,4,5,6,...)$ the characteristic equation of spherical symmetric free vibration of the shell of inner radius a_1 and outer radius a_2 can again be obtained by means of the correspondence principle, using Table 6 and equation (4.71):

$$(k_{dm}a_2) + \arctan\left(\frac{1}{(k_{dm}a_2)} - \frac{1}{c_m}(k_{dm}a_2)\right)$$

$$= (k_{dm}a_1) + \arctan\left(\frac{1}{(k_{dm}a_1)} - \frac{1}{c_m}(k_{dm}a_1)\right) \tag{4.83}$$

where

$$c_m = \frac{4k_{dm}^2}{k_{sm}^2}$$

Performing the same geometrical parametric study as before, we obtain as degenerate solutions of (4.83) the breathing mode solution

$$k_{dm}a = \sqrt{(3c_m - c_m^2)} \tag{4.84}$$

and the thickness stretching mode solution

$$k_{dm}h = \left(j\pi + \frac{c_m}{(j\pi)^2}\left(\frac{h}{2a}\right)^2 - \frac{2c_m^2}{(j\pi)^5}\left(\frac{h}{2a}\right)^4 + \ldots\right) \tag{4.84}$$

$j = 1,2,3,4,K$ integers

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4.1.5. <u>Spherically symmetric free vibration of a spherical elastic shell containing a viscous fluid.</u>

We now consider a spherical elastic shell with an inner radius $r = a_1$ and outer radius $r = a_2$. The field quantities must be finite at $r = 0$ for any $t > 0$ and therefore we must have $B_1 = 0$. The displacements of the viscous fluid and of the elastic shell are continuous at the interface, while the outer surface of the shell is considered stress-free. Hence, with (4.3), (4.5), (4.9) and (4.10), the boundary conditions are

$$u_{r1}(a_1, t) - u_{r2}(a_1, t) = 0 \tag{4.86}$$

$$P_0 + \varsigma_{rr1}(a_1, t) - \varsigma_{rr2}(a_1, t) = 0 \tag{4.87}$$

$$\varsigma_{rr2}(a_2, t) = 0 \tag{4.88}$$

and the characteristic equation of the problem stated is

$$\begin{vmatrix} k_{d1}\vartheta_0'(k_{d1}a_1) & -k_{d2}\vartheta_0'(k_{d2}a_1) & -k_{d2}Y_0'(k_{d2}a_1) \\ k_{d1}^2\begin{bmatrix} -(\rho_0 c_0^2 + \eta i\omega)\vartheta_0(k_{d1}a_1) \\ + 2i\omega\eta\vartheta_0''(k_{d1}a_1) \end{bmatrix} & -k_{d2}^2\begin{bmatrix} -\lambda\vartheta_0(k_{d2}a_1) \\ + 2\mu\vartheta_0''(k_{d2}a_1) \end{bmatrix} & -k_{d2}^2\begin{bmatrix} -\lambda Y_0(k_{d2}a_1) \\ + 2\mu Y_0''(k_{d2}a_1) \end{bmatrix} \\ 0 & -k_{d2}^2\begin{bmatrix} -\lambda\vartheta_0(k_{d2}a_2) \\ + 2\mu\vartheta_0''(k_{d2}a_2) \end{bmatrix} & -k_{d2}^2\begin{bmatrix} -\lambda Y_0(k_{d2}a_2) \\ + 2\mu Y_0''(k_{d2}a_2) \end{bmatrix} \end{vmatrix} = 0 \tag{4.89}$$

To examine the effects of the geometric parameters on the roots of (4.89) we examine the three cases:

(a) $a_1 = 0$. In this case (4.89) reduces to the characteristic equation for the free vibrations of an elastic sphere which we discussed in section 4.1.3.

(b) a_1 finite with $a_2 \to \infty$. Geometrically this constitutes a viscous fluid sphere surrounded by an infinite elastic medium.

(c) $a_2 - a_1 = h$ and $a_2 - a_1 = b = 2a$, with $a_2 - a(1+\in) = \frac{h}{2}\left(\frac{1}{\in} + 1\right)$, $a_1 = a(1-\in) = \frac{h}{2}\left(\frac{1}{\in} - 1\right)$, and $\in = \frac{h}{2a} = \frac{h}{b} \to 0$. As before, this will provide us with two approximate solutions of the characteristic equation, one corresponding to a very thin elastic shell of radius a containing a viscous fluid, the other representing the solution to the thickness stretch vibrations of an elastic sheet on a viscous fluid.

4.2. Axially Symmetric Torsional Motion

This motion is defined by (3.37), (3.38) and (3.39), according to which $u_{rm} = u_{\theta m} = 0$, $u_{\theta m} = u_{\phi m}(r,\theta,t) = \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\nabla^2\chi_m\right)$, or $\chi_m = \chi_m(r,\theta,t)$ and $\varphi_m = \psi_m = 0$ everywhere at all times. Thus the governing potential equations (3.9), (3.10) and (3.11) are satisfied if

$$\chi_m = \left[A_m\vartheta_n(k_{sm}r) + B_mY_n(k_{sm}r)\right]e^{i\omega t} \tag{4.90}$$

From (3.39) it follows that the displacement $u_{\phi m}$ is given by

$$u_{\phi m} = -k_{sm}^2\left[A_m\vartheta_n(k_{sm}r) + B_mY_n(k_{sm}r)\right]\frac{dP_n(\cos\theta)}{d\theta}e^{i\omega t} \tag{4.91}$$

With (3.40) and (3.41) the nontrivial components of the rotational vector become

$$2\Omega_{rm} = k_{sm}^2\left(rk_{sm}^2 + 2\frac{d}{dr} + r\frac{d^2}{dr^2}\right)\left[A_m\vartheta_n(k_{sm}r) + B_mY_n(k_{sm}r)\right]P_n(\cos\theta)e^{i\omega t}$$

$$= \frac{k_{sm}^2}{r} n(n+1)\left[A_m \vartheta_n(k_{sm}r) + B_m Y_n(k_{sm}r)\right] P_n(\cos\theta) e^{i\omega t} \quad (4.92)$$

$$2\Omega_{\theta m} = \frac{k_{sm}^2}{r} \frac{d}{dr}\left[A_m r \vartheta_n(k_{sm}r) + B_m r Y_n(k_{sm}r)\right] \frac{dP_n(\cos\theta)}{d\theta} e^{i\omega t} \quad (4.93)$$

and according to (3.43), (3.45), (3.46), (3.47) and (3.48) the nontrivial components of the stress tensor ς_{ijm} are

$$\varsigma_{rrm} = \varsigma_{\theta\theta m} = \varsigma_{\phi\phi m} = -P_m \quad (4.94)$$

$$\varsigma_{r\phi m} = -\mu_m k_{sm}^2 r \frac{d}{dr}\left[A_m \frac{\vartheta_n(k_{sm}r)}{r} + B_m \frac{Y_n(k_{sm}r)}{r}\right] \frac{dP_n(\cos\theta)}{d\theta} e^{i\omega t} \quad (4.95)$$

$$\varsigma_{\theta\phi m} = -\mu_m \frac{k_{sm}^2}{r}\left[A_m \vartheta_n(k_{sm}r) + B_m Y_n(k_{sm}r)\right] \sin\theta \frac{d}{d\theta}\left[\frac{1}{\sin\theta} \frac{dP_n(\cos\theta)}{d\theta}\right] e^{i\omega t} \quad (4.96)$$

k_{dm}, k_{sm} can be obtained from Table 6 for any of the materials listed.

Applications of the harmonic solutions (4.90) to (4.96) to torsional motion will be illustrated in four specific problems.

4.2.1. Torsional vibration and attenuation of a viscous fluid, an elastic solid, and a viscoelastic material in a rigid spherical inclusion.

The boundary conditions require that the field quantities be finite at $r = 0$ for any time $t > 0$ and thus that $B_m = 0$. At $r = a$ we have $u_{\phi m}(a,t) = 0$ which yields as characteristic equation

$$\vartheta_n(k_{sm}a) = 0 \quad (4.97)$$

The roots $k_{sm}a = \alpha_{jn}$ of (4.97) are well known and are for example given in the mathematical handbook edited by Abramowitz and Stegun[48)]. The first four roots for $n = 0,1,2,3$ are given below:

$k^a_{sm} = \alpha_n$	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$j = 1$	3.141593	4.493409	5.763459	6.987932
$j = 2$	6.283185	7.725252	9.095011	10.417119
$j = 3$	9.424778	10.904122	12.322941	13.698023
$j = 4$	12.566370	14.066194	15.514603	16.923621

Using Table 6 and the field quantities given previously, one can resolve ω in terms of $k_{sm}a = \alpha_{jn}$ for each material m. For a viscous fluid $(m = 1)$ we find for example from Table 6

$$(k_{s1}a)^2 = \alpha^2_{jn} = \frac{c\omega^2 a^2}{\frac{\eta}{\rho_o} i\omega}$$

that is

$$\omega_{jn} = i\frac{\eta}{\rho_o}\frac{\alpha^2_{jn}}{a^2} \tag{4.98}$$

When ω_{jn} is substituted back into the solutions with the time factor $e^{i\omega jnt}$ it is clear that the corresponding field quantities are exponentially decaying with time like $e\frac{\eta}{\rho_o}\frac{\alpha_{jn}}{a^2}t$. The positions of nodes in the radial direction for each fixed meridianal mode defined by n can be obtained by taking the ratio of successive α_{jn} as illustrated from example 1.1. The position of the meridianal nodes can be obtained from $\frac{dp_n(\cos\theta)}{d\theta} = 0$ which gives

$$
\begin{array}{cccccc}
 & n=0 & n=1 & n=2 & n=3 & n=4 \\
\theta = \theta_0 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{c} 0 \\ \\ \pi \end{array} & \begin{array}{c} 0 \\ \\ \pi \end{array} & \begin{array}{c} 0 \\ \pi/2 \\ \pi \end{array} & \begin{array}{c} 0 \\ \cos^{-1}\left(\pm\frac{1}{5}\right) \\ \pi \end{array} & \begin{array}{c} 0 \\ \cos^{-1}\left(\pm\frac{3}{7}\right) \\ \pi \end{array}
\end{array}
$$

From (4.98) we conclude that the torsional motion of a viscous fluid within a rigid spherical inclusion is always overdamped.

In the case of an elastic solid $(m = 2)$ we infer from Table 6

$$(k\omega a) = \alpha_{jn} = \frac{\omega a}{\sqrt{\frac{\mu}{\rho_0}}} = \frac{\omega a}{c_s} \tag{4.99}$$

For a given radial (j) mode and meridianal (n) mode the torsional vibration of an elastic sphere in a rigid inclusion is undamped and has a frequency $f = \frac{\omega}{2\pi} = \frac{c_s\alpha_{jn}}{2\pi a}$.

In the case of a viscoelastic material defined by the Maxwell model $(m = 3)$ we obtain from Table 6

$$(k_{s3}a)^2 = \alpha_{jn}^2 = \omega^2\left(\frac{\rho_0}{\mu} + \frac{\rho_0}{i\omega}\right)a^2$$

or

$$\omega^2 - i\frac{\mu}{\eta}\omega - \frac{\mu}{\rho_0}\frac{\alpha_{jn}^2}{a^2} = 0 \tag{4.100}$$

The roots of (4.100) are

$$\frac{\omega}{\left(\frac{\mu}{2\eta}\right)} = i + \left(\frac{4\eta^2}{\rho_0 \mu}\frac{\alpha_{jn}^2}{a^2} - 1\right)^{1/2} \tag{4.101}$$

and

$$\frac{\omega}{\left(\frac{\mu}{2\eta}\right)} = i - \left(\frac{4\eta^2}{\rho_0 \mu}\frac{\alpha_{jn}^2}{a^2} - 1\right)^{1/2} \tag{4.102}$$

The motion of the material in the cavity is

underdamped if $\frac{4\eta^2}{\rho_0 \mu}\frac{\alpha_{jn}^2}{a^2} - 1 > 0$,

critical damped if $\frac{4\eta^2}{\rho_0 \mu}\frac{\alpha_{jn}^2}{a^2} - 1 = 0$,

and, overdamped if $\frac{4\eta^2}{\rho_0 \mu}\frac{\alpha_{jn}^2}{a^2} - 1 > 0$.

The frequency of underdamped vibration is

$$f = \frac{1}{4\pi}\frac{\mu}{\eta}\left(\frac{4\eta^2}{\rho_0 \mu}\frac{\alpha_{jn}^2}{a^2} - 1\right)^{1/2} \tag{4.103}$$

and the corresponding attenuation factor is

$$td = \frac{2\eta}{\mu} \tag{4.104}$$

For overdamped motion the attenuation factor (time constant) is given by

$$td = \frac{\frac{2\eta}{\mu}}{1 \pm \sqrt{1 - \frac{4\eta^2}{\rho_0 \mu} \frac{\alpha_{jn}^2}{a^2}}} \tag{4.105}$$

In the case of a viscoealstic material of the Kelvin type $(m = 4)$ we deduce from Table 6

$$(k_{s4} a)^2 = \alpha_{jn}^2 = \frac{\omega^2 a^2}{\frac{1}{\rho_0}(\mu + \eta i\omega)}$$

or

$$\omega^2 - i\frac{\eta}{\rho_0}\frac{\alpha_{jn}^2}{a^2}\omega - \frac{\mu}{\rho_0}\frac{\alpha_{jn}^2}{a^2} = 0 \tag{4.106}$$

The roots of (4.106) are

$$\frac{\omega}{\left(\frac{\eta}{2\rho_0}\frac{\alpha_{jn}^2}{a^2}\right)} = i + \left(\frac{4\rho_0 \mu a^2}{\eta^2 \alpha_{jn}^2} - 1\right)^{1/2} \tag{4.107}$$

$$\frac{\omega}{\left(\frac{\eta}{2\rho_0}\frac{\alpha_{jn}^2}{a^2}\right)} = i - \left(\frac{4\rho_0\mu a^2}{\eta^2\alpha_{jn}^2} - 1\right)^{1/2} \tag{4.108}$$

and the motion of the material in the spherical cavity is

underdamped if $\frac{4\rho_0\mu a^2}{\eta^2\alpha_{jn}^2} - 1 > 0$,

critical damped if $\frac{4\rho_0\mu a^2}{\eta^2\alpha_{jn}^2} - 1 = 0$,

and, overdamped if $\frac{4\rho_0\mu a^2}{\eta^2\alpha_{jn}^2} - 1 < 0$.

The frequency of underdamped vibration is

$$f = \frac{1}{2\pi}\frac{\eta}{2\rho_0}\frac{\alpha_{jn}^2}{a^2}\left(\frac{4\rho_0\mu a^2}{\eta^2\alpha_{jn}^2} - 1\right)^{1/2} \tag{4.109}$$

and the corresponding attenuation factor (time constant) is

$$td = \frac{2\rho_0 a^2}{\eta\alpha_{jn}^2} \tag{4.110}$$

For overdamped motion the latter assumes the form

$$td = \frac{2\rho_0 a^2}{\eta\alpha_{jn}^2}\frac{1}{1 \pm \sqrt{1 - \frac{4\rho_0\mu a^2}{\eta^2\alpha_{jn}^2}}} \tag{4.111}$$

From these four cases of materials, we see that the pure viscous fluid in torsional motion is damped. Other materials of viscoelastic type in torsional motion are either underdamped, critical-damped or overdamped according to the physical quantities of the material models and the radius of the cavity as well as the vibrational mode characterized by α_{jn}.

A comparison of the results for the Maxwell material with those for the Kelvin material reveals some interesting aspects. The attenuation factor for a Maxwell material does not depend on the wave number α_{jn} while that for the Kelvin material depends on α_{jn}^2 in addition to the physical parameters of the model. Using the subscripts $m = 3$ for the Maxwell material and $m = 4$ for the Kelvin material in all physical quantities we find from (4.100) and (4.106) that the dynamic behavior is the same for both materials if

$$\frac{\mu_3}{\rho_3} = \frac{\mu_4}{\rho_4} \tag{4.112}$$

and

$$\frac{\mu_3}{\eta_3} = \frac{\eta_4}{\rho_4}\left(\frac{\alpha_{jn}^2}{a^2}\right) \tag{4.113}$$

This means that the displacements, rotations, and strains will be the same for these two material models, but the corresponding stresses will be different. This may be referred to as the "Kinematic Equivalence of Viscoelastic Models."

4.2.2. Free torsional vibration of a spherical elastic solid, a spherical viscous fluid drop and a viscoelastic sphere.

As boundary conditions we have here $P_m + \varsigma_{rrm}(r = a, t) = 0$, $\varsigma_{r\theta m}(r = a, t) = 0$ and $\varsigma_{r\phi m}(r - a, t) = 0$. Since $\varsigma_{rrm} = \varsigma_{r\theta m} = 0$ is automatically satisfied for axially symmetrical torsional motion $\varsigma_{r\phi m}(r = a, t) = 0$ is the only nontrivial boundary condition. For finiteness of the field quantities at the origin $r = 0$ we again require $B_m = 0$. The characteristic equation assumed the form

$$\frac{d}{da}\left[\frac{\vartheta_n(k_{sm}a)}{a}\right] = 0$$

By means of the recurrence relations of spherical Bessel functions it can be simplified to

$$(n-1)\vartheta_n(\alpha_{jn}) - \alpha_{jn}\vartheta_{n+1}(\alpha_{jn}) = 0 \tag{4.114}$$

where $\alpha_{jn} = k_{sm}a$. Using Rayleigh's formula

$$\vartheta_n(\alpha_{jn}) = (\alpha_{jn})^n\left(-\frac{1}{\alpha_{jn}}\frac{d}{d\alpha_{jn}}\right)^n\left(\frac{\sin\alpha_{jn}}{\alpha_{jn}}\right) = (\alpha_{jn})^n\psi_n(\alpha_{jn})$$

(4.114) can be rewritten as

$$(\alpha_{jn})^n\left[(n-1)\psi_n + \alpha_{jn}\psi'_n\right] = 0 \tag{4.115}$$

Equation (4.115) is identical with the result given by Love[34)].

In particular, when $n = 1$ we find from (4.114)

$$\vartheta_{n+1}(\alpha_{j1}) = \left(\frac{3}{\alpha_{j1}^3} - \frac{1}{\alpha_{j1}}\right)\sin\alpha_{j1} - \frac{3}{\alpha_{j1}^2}\cos\alpha_{j1} = 0$$

or

$$\tan \alpha_{j1} = \frac{3\alpha_{j1}}{3 - \alpha_{j1}^2} \tag{4.116}$$

This result was also obtained by Love[34]. The roots of (4.116) are

$$\alpha_{11} = 1.8346 \qquad \alpha_{31} = 3.9225$$
$$\alpha_{21} = 2.8950 \qquad \alpha_{41} = 4.9385, \ldots$$

All results described in this example are essentially identical to those of example 4.2.1 with the exception that the α_{jn} values were there the roots of $\vartheta_n(\alpha_{jn}) = 0$, while they are now the roots of (4.114).

4.2.3. <u>Free torsional vibration of an elastic shell, a viscous fluid shell (bubble) and a viscoelastic shell of spherical shape.</u>

Again, we let a_1 denote the inner radius and a_2 the outer radius. The boundary conditions are

$$P_m + \varsigma_{rrm}(r = a_1, t) = \varsigma_{r\theta m}(r = a_1, t) = \varsigma_{r\phi m}(r = a_1, t) = 0$$

$$P_m + \varsigma_{rrm}(r = a_2, t) = \varsigma_{r\theta m}(r = a_2, t) = \varsigma_{r\phi m}(r = a_2, t) = 0$$

As indicated before, $P_m + \varsigma_{rrm} = \varsigma_{r\theta m} = 0$ is automatically satisfied for axially symmetric torsional motion, leaving $\varsigma_{r\phi m}(r = a_1, t) = 0$ and $\varsigma_{r\phi m}(r = a_2, t) = 0$ as the required conditions. With (4.95) and these boundary conditions, the characteristic equation of the problem is

$$\begin{vmatrix} \left[(n-1)\vartheta_n(\lambda_{sm}a_1) - (\lambda_{sm}a_1)\vartheta_{n+1}(\lambda_{sm}a_1)\right] & \left[(n-1)Y_n(\lambda_{sm}a_1) - (\lambda_{sm}a_1)Y_{n+1}(\lambda_{sm}a_1)\right] \\ \left[(n-1)\vartheta_n(\lambda_{sm}a_2) - (\lambda_{sm}a_2)\vartheta_{n+1}(\lambda_{sm}a_2)\right] & \left[(n-1)Y_n(\lambda_{sm}a_2) - (\lambda_{sm}a_2)Y_{n+1}(\lambda_{sm}a_2)\right] \end{vmatrix} = 0 \tag{4.117}$$

For the lowest nontrivial circumferential torsional mode (= 1), (4.117) can be further simplified:

$$k_{sm}a_1 - \arctan\frac{3k_{sm}a_1}{3-(k_{sm}a_1)^2} = k_{sm}a_2 - \frac{3k_{sm}a_2}{3-(k_{sm}a_2)^2} \tag{4.118}$$

A geometrical parametric study of (4.118) similar to that of (4.71) also points out some interesting features of the motions. We consider two limiting cases:

(a) $a_2 - a_1 = h$ and $a_2 \to a_1 = a$ with $\left(\frac{h}{2a}\right) << 1$.

This means that the shell is very thin and has a finite radius a.

(b) $a_2 - a_1 = h$ with $a_2 \to \infty$, $a_1 \to \infty$, and $a_2 + a_1 = 2a = b \to \infty$.

In this case the shell approaches a plate of finite thickness h.

Expanding a_2 with respect to a_1 into a Taylor series we obtain in case (a)

$$\frac{(k_{sm}a)^4}{9+3(k_{sm}a)^2+(k_{sm}a)^4} + \frac{\left[36+6(k_{sm}a)^2\right](k_{sm}a)^3}{\left[9+3(k_{sm}a)^2+(k_{sm}a)^4\right]^2}\left(\frac{h}{a}\right)$$

$$+\,O\left(\frac{h}{a}\right)^2 + \,........= 0 \tag{4.119}$$

The first order solution is trivial $(\omega = 0)$ but the second order solution yields

$$(k_{sm}a)^2 = -\,6 \tag{4.120}$$

The motions corresponding to the roots of (4.120) are however not compatible with physical reality, which means that only the trivial solution may be admissible.

Rearranging (4.118), we obtain

$$\tan k_{sm}h = \frac{9k_{sm}\left(a_2 - a_1\right) + 3k_{sm}^2\, a_1\, a_2\left(a_2 - a_1\right)}{9 - 3k_{sm}^2\left(a_2^2 - a_1^2\right) + 9\,k_{sm}^2\, a_2\, a_1 + k_{sm}^4\, a_1^2\, a_2^2} \qquad (4.121)$$

With $2\frac{a}{h} = \frac{b}{h} \to \infty$ and $\left(k_{sm}h\right)$ finite, the right-hand side of (4.121) approaches zero. Therefore $\left(k_{sm}h\right) = j\pi$ where $j = 1, 2, 3, 4, \ldots.$ If $\frac{b}{h} >> 1$ the first approximation of (4.121) can be obtained by setting $\left(k_{sm}h\right) = j\pi + \in$, where $\in << 1$ and $\tan \in = \in$. Then (4.121) reduces to

$$\in = \frac{12}{j\pi + \in}\left(\frac{h}{b}\right)^2\left[1 + \left(\frac{h}{b}\right)^2 + \ldots\ldots\right]$$

or

$$\in = \frac{12}{j\pi}\left(\frac{h}{b}\right)^2\left[1 - \frac{12}{\left(j\pi\right)^2}\left(\frac{h}{b}\right)^2 + \ldots\ldots\right]$$

Thus $\left(k_{sm}h\right)$ can be obtained for $\frac{b}{h} >> 1$ from

$$k_{sm}h = j\pi + \frac{12}{j\pi}\left(\frac{h}{b}\right)^2 - \frac{144}{(j\pi)^3}\left(\frac{h}{b}\right)^4 + \ldots\ldots \qquad (4.122)$$

Equations (4.98) to (4.111) are still valid for this particular solution for if $n = 1$, if a is substituted for h and α_{jn}, for α_{j1} as expressed in the right-hand side of (4.122).

4.2.4. Free torsional vibration of a viscoelastic spherical shell containing a viscous fluid or other viscoelastic materials.

The field quantities of the core materials are finite at $r = 0$ at any time $t = 0$, thus $B_1 = 0$.. The displacements of the shell and that of the core are continuous at the inter-spherical surface, and the outer surface of the viscoelastic shell is considered stress-free. From (4.91) and (4.95) with $B_1 = 0$ the boundary conditions are obtained as

$$u_{\phi 1}(a_1, t) - u_{\phi m}(a_1, t) = 0 \tag{4.123}$$

$$\varsigma_{r\phi 1}(a_1, t) - \varsigma_{r\phi m}(a_1, t) = 0 \tag{4.124}$$

$$\varsigma_{r\phi m}(a_1, t) = 0 \tag{4.125}$$

They yield the characteristic equation

$$\begin{vmatrix} -k_{s1}^2 \vartheta_n(k_{s1}a_1) & k_{sm}^2 \vartheta_n(k_{sm}a_1) & k_{sm}^2 Y_n(k_{sm}a_1) \\ -\dfrac{\mu_1 k_{s1}}{a_1}\left[(n-1)\vartheta_n(k_{s1}a_1) - (k_{s1}a_1)\vartheta_{n+1}(k_{s1}a_1)\right] & \dfrac{\mu_m k_{sm}^2}{a_1}\left[(n-1)\vartheta_n(k_{sm}a_1) - (k_{sm}a_1)\vartheta_{n+1}(k_{sm}a_1)\right] & \dfrac{\mu_m k_{sm}^2}{a_1}\left[(n-1)Y_n(k_{sm}a_1) - (k_{sm}a_1)Y_{n+1}(k_{sm}a_1)\right] \\ & \dfrac{\mu_m k_{sm}^2}{a_2}\left[(n-1)\vartheta_n(k_{sm}a_2) - (k_{sm}a_2)\vartheta_{n+1}(k_{sm}a_2)\right] & \dfrac{\mu_m k_{sm}^2}{a_2}\left[(n-1)Y_n(k_{sm}a_2) - (k_{sm}a_2)Y_{n+1}(k_{sm}a_2)\right] \end{vmatrix} = 0 \tag{4.126}$$

4.3. Axially Symmetric Nontorsional Motion

This motion is defined by equations (3.25), (3.26) and (3.27) which require that $u_{rm} = u_{rm}(r, \theta, t)$, $u_{\theta m} = u_{\theta m}(r, \theta, t)$, and $u_{\phi m} = 0$ everywhere at all times, or $\varphi_m = \varphi_m(r, \theta, t)$, $\psi_m = \psi_m(r, \theta, t)$, and $\chi_m = 0$ everywhere at all times. Thus one can see that the governing potential equations (3.9), (3.10) and (3.11) are satisfied provided that

$$\varphi_m = \left[A_{1m}\vartheta_n\left(k_{dm}r\right) + B_{1m}Y_n\left(k_{dm}r\right)\right]P_n\left(\cos\theta\right)e^{i\omega t} \tag{4.127}$$

and

$$\psi_m = \left[A_{2m}\vartheta_n\left(k_{sm}r\right) + B_{2m}Y_n\left(k_{sm}r\right)\right]P_n\left(\cos\theta\right)e^{i\omega t} \tag{4.128}$$

From (3.25) and (3.26) using (4.127) and (4.128) it follows that the components of the displacement vector can be written in the form

$$\begin{Bmatrix} \dfrac{U_{rm}}{P_n\left(\cos\theta\right)} \\ \\ \dfrac{U_{\theta m}}{\dfrac{d}{d_\theta}P_n\left(\cos\theta\right)} \end{Bmatrix} = \left[\left[U_m\vartheta_n\left(r\right)\right]\left[U_mY_n\left(r\right)\right]\right] \begin{Bmatrix} A_{1m} \\ A_{2m} \\ B_{1m} \\ B_{2m} \end{Bmatrix} e^{iwt} \tag{4.129}$$

where

$$\left[U_m\vartheta_n\left(r\right)\right] = \begin{bmatrix} k_{dm}\vartheta'_n\left(k_{dm}r\right) & \dfrac{n\left(n+1\right)}{r}\vartheta_n\left(k_{sm}r\right) \\ \dfrac{1}{r}\vartheta_n\left(k_{dm}r\right) & k_{sm}\vartheta'_n\left(k_{sm}r\right) + \dfrac{1}{r}\vartheta_n\left(k_{sm}r\right) \end{bmatrix} \tag{4.129a}$$

$$\left[U_mY_n\left(r\right)\right] = \begin{bmatrix} k_{dm}Y'_n\left(k_{dm}r\right) & \dfrac{n\left(n+1\right)}{r}Y_n\left(k_{sm}r\right) \\ \dfrac{1}{r}Y_n\left(k_{dm}r\right) & k_{sm}Y'_n\left(k_{sm}r\right) + \dfrac{1}{r}Y_n\left(k_{sm}r\right) \end{bmatrix} \tag{4.129b}$$

According to (3.41), (3.42), (4.127) and (4.128) the components of the pertinent stresses may be given as:

$$\begin{Bmatrix} \dfrac{\varsigma_{rrm}+P_m}{2\mu_m P_n(\cos\theta)} \\[2ex] \dfrac{\varsigma_{r\theta m}}{2\mu_m \dfrac{d}{d_\theta}P_n(\cos\theta)} \end{Bmatrix} = \frac{1}{r^2}\begin{bmatrix} [\varsigma_m \vartheta_n(r)] & [\varsigma_m Y_n(r)] \end{bmatrix}\begin{Bmatrix} A_{1m} \\ A_{2m} \\ B_{1m} \\ B_{2m} \end{Bmatrix} e^{iwt} \qquad (4.130)$$

where

$$[\varsigma_m \vartheta_n(r)] = \begin{bmatrix} -\dfrac{\lambda_m}{2\mu_m}k_{dm}^2 r^2 \vartheta_n(k_{dm}r) + k_{dm}^2 r^2 \vartheta_n''(k_{dm}r) & n(n=1)[r k_{sm}\vartheta'_n(k_{sm}r) - \vartheta_n(k_{sm}r)] \\[2ex] r k_{dm}\vartheta'_n(k_{dm}r) - \vartheta_n(k_{dm}r) & \left[n^2+n-1-\dfrac{1}{2}(k_{sm}r)^2\right]\vartheta_n(k_{sm}r) - r k_{sm}\vartheta'_n(k_{sm}r) \end{bmatrix} \qquad (4.131)$$

$$[*_m Y_n(r)] = \begin{bmatrix} -\dfrac{\lambda_m}{2\mu_m}k_{dm}^2 r^2 Y_n(k_{dm}r) + k_{dm}^2 r^2 Y_n''(k_{dm}r) & n(n=1)[r k_{sm}Y_n'(k_{sm}r) - Y_n(k_{sm}r)] \\[2ex] r k_{dm}Y_n'(k_{dm}r) - Y_n(k_{dm}r) & \left[n^2+n-1-\dfrac{1}{2}(k_{sm}r)^2\right]Y_n(k_{sm}r) - r k_{sm}Y_n'(k_{sm}r) \end{bmatrix} \qquad (4.132)$$

The other field quantities can be obtained by substituting (4.127) and (4.128) into (3.40), (3.44) and (3.46).

In the following we shall discuss applications of (4.127) through (4.132) to specific problems.

4.3.1. Axially symmetric nontorsional free vibration of a viscous fluid in a rigid spherical inclusion.

The boundary condition of the problem requires that $u_{r1}(r=a,\theta,t)=0$ and $u_{\theta 1}(r=a,\theta,t)=0$. These conditions are met if we equate the determinant of (4.129a) equal to zero for $r=a$. We thus have

$$\frac{k_{d1}a\vartheta_n^1(k_{d1}a)}{\vartheta_n(k_{d1}a)}=\frac{n(n+1)\vartheta_n(k_{d1}a)}{k_{s1}a\vartheta_n^1(k_{s1}a)+\vartheta_n(k_{s1}a)} \tag{4.133}$$

This is the characteristic equation of a viscous fluid in axially symmetric vibration in a rigid spherical inclusion. The corresponding expressions for $(k_{d1}a)$ and $(k_{s1}a)$ are according to Table 6

$$k_{d1}a=\frac{\omega a}{\sqrt{c_0^2+(\upsilon'+2\upsilon)i\omega}}$$

$$k_{s1}a=\sqrt{-i\frac{\omega}{\upsilon}}a=i^{3/2}\sqrt{\frac{\omega}{\upsilon}}\,a$$

There are two degenerated solutions of (4.133) for viscous fluid:

(a) Inviscid solution: υ' and $\upsilon\rightarrow 0$

In this case

$$k_{s1a}=i^{3/2}\sqrt{\frac{c_a\xi\,a}{\upsilon}}\rightarrow\infty$$

and

$$k_{d1}\,a=\xi$$

where the ξ's are the roots of

$$\vartheta_n^1(\xi) = 0 \tag{4.134}$$

(b) Slightly viscous solution: υ' and υ are both small

With this restriction we find

$$k_{s1}a \cong i^{3/2}\sqrt{\frac{c_0 \xi a}{\upsilon} a} \qquad \text{with} \qquad |k_{s1}a| >> 1$$

or

$$k_{d1a}' = \xi + \in$$

where $\in << 1$ and the ξ's are, as before, the roots of $\vartheta_n^1(\xi) = 0$.

Now (4.133) can be approximated by

$$\in \left(\frac{n(n+1)}{\xi} - \xi\right) = (-i)^{1/2} \frac{n(n+1)}{\sqrt{\frac{c_0 \xi a}{\upsilon}}}$$

or

$$\in = \frac{\mp 1 \pm i}{\sqrt{2}} \xi \sqrt{\frac{\upsilon}{c_0 \xi a}} \frac{n(n+1)}{n(n+1) - \xi^2}$$

Solving $k_{d1}a = \xi + \in$ for ω we obtain

$$\frac{\omega a}{c_0} = \xi\left(1 \mp \frac{\sqrt{2}}{2}\sqrt{\frac{\upsilon}{c_0 \xi a}} \frac{n(n+1)}{n(n+1) - \xi^2}\right)$$

$$\pm i\frac{\sqrt{2}}{2}\xi\sqrt{\frac{\upsilon}{c_0\,\xi\,a}\,\frac{n(n+1)}{n(n+1)-\xi^2}} \tag{4.135}$$

The upper and lower signs in (4.135) correspond to $\lfloor n(n+1)-\xi^2\rfloor \gtrless 0$. If $[n(n+1)-\xi^2]>0$, the upper signs hold and in this case the attenuation coefficient (time constant) is

$$t_d=\sqrt{2}\,\frac{n(n+1)-\xi^2}{n(n+1)\sqrt{\xi}}\sqrt{\frac{a^3}{\upsilon c_o}} \tag{4.136}$$

and the frequency of the vibration is given by

$$f=\frac{c_0\xi}{2\pi a}\left(1-\frac{\sqrt{2}}{2}\sqrt{\frac{\upsilon}{c_0\xi a}\,\frac{n(n+1)}{n(n+1)-\xi^2}}\right) \tag{4.137}$$

If $[n(n+1)-\xi^2]<0$, the lower signs hold, for which we obtain as attenuation coefficient

$$t_d=\sqrt{2}\,\frac{\xi^2-n(n+1)}{n(n+1)\sqrt{\xi}}\sqrt{\frac{a^3}{\upsilon c_0}} \tag{4.138}$$

and as frequency

$$f=\frac{c_0\xi}{2\pi a}\left(1-\frac{\sqrt{2}}{2}\sqrt{\frac{\upsilon}{c_0\xi a}\,\frac{n(n+1)}{\xi^2-n(n+1)}}\right) \tag{4.139}$$

where ξ are the roots of (4.134).

According to Abramowitz and Stegun[48)] [Table 10.7, page 468], the lowest value of ξ for a given n always yields $[\xi^2-n(n+1)]>0$. A list of the first four roots for $n=1,2,3,4$ is given below.

ξ	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$j = 1$	2.460536	3.632797	4.762196	5.86842
$j = 2$	6.029292	7.367009	8.653134	9.904306
$j = 3$	9.261402	10.663561	12.018262	13.337928
$j = 4$	12.44526	13.88337	15.279081	16.641787

From these observations, we conc lude that the lower signs in (4.135) leads to an attenuated vibration whose attenuation coefficient is given by (4.138) and whose frequency is defined by (4.139). The upper signs in (4.135) and the subsequeny results from (4.136) and (4.137) correspond to an unstable motion.

The time constants given by (4.138) are identical to those obtained by Lamb[46)]. He also gives an alternate form of (4.134) in terms of Rayleigh's formula. Unfortunately, Lamb made a computational error in the-case of $n = 1$. The lowest mode of vibration ξ should be 2.460536 instead of Lamb's value of 2.081.

4.3.2. <u>Axially symmetric nontorsional free vibration of a spherical elastic solid, a spherical viscous fluid drop and a viscoelastic sphere.</u>

The boundary conditions of this problem are $P_m + \varsigma_{rrm}(r = a, \theta, t) = 0$ and $\varsigma_{r\theta m}(r = a, \theta, t) = 0$. B_{1m} and B_{2m} are set equal to zero to ensure finite field quantities at the origin $r = 0$. The characteristic equation of the problem can be obtained by setting the determinant of (4.131) equal to zero:

$$\begin{vmatrix} \left[\frac{\lambda m}{2\mu m}(k_{dm} a)^2 \vartheta_n(k_{dm} a) + (k_{dm} a)^2 \vartheta''_n(k_{dm} a) \right] & n(n+1)\left[(k_{sm} a)\vartheta'_n(k_{sm} a) - \vartheta_n(k_{sm} a)\right] \\ \left[k_{dm} a\, \vartheta'_n(k_{dm} a) - \vartheta_n(k_{dm} a) \right] & \left[\left(n^2 + n - 1 - \frac{1}{2}(k_{sm} a)^2 \right) \vartheta_n(k_{sm} a) - k_{sm} a\, \vartheta'_n(k_{sm} a) \right] \end{vmatrix} = 0 \qquad (4.\ 140)$$

where λ_m and μ_m are the operators defined in Table 2 with $i\omega$ replacing $\frac{\partial}{\partial t}$ and where k_{dm} and k_{sm} are listed in Table 6 for the various material models (m). In the case of an elastic solid (m = 2), λ_2 and μ_2 are the usual Lamé constants, $k_{d2} = \frac{\omega}{c_{d2}}$ and $k_{s2} = \frac{\omega}{c_{s2}}$, with $c_{d2} = \sqrt{\frac{\lambda_2 + \mu_2}{\rho_0}}$ and $c_{s2} = \sqrt{\frac{\mu_2}{\rho_0}}$ representing respectively the dilatational and shear wave speeds. For the elastic medium the solutions of (4.140) are given in Figure 1. The curves shown are plots of the normalized frequency versus the meridianal wave number. According to this graph the frequency appears to be a continuous function of the wave number; however, the plots are only valid for discrete values of ω for each meridianal wave number n, which is the degree of Legendre's function P_n $(cos\theta)$. For each n there are an infinite number of ω's satisfying (4.140), which means that there are an infinite number of frequency curves.

4.3.3. Axially symmetric nontorsional free vibration of an elastic spherical shell, a viscous fluid spherical shell, and a viscoelastic spherical shell.

The spherical shell under consideration is again defined by two concentric spheres whose radii are a_1, and a_2. The boundary conditions of this problem are given by

$$P_m + \varsigma_{rrm}(r = a_1, \theta, t) = 0$$
$$\varsigma_{r\theta m}(r = a_1, \theta, t) = 0$$
$$P_m + \varsigma_{rrm}(r = a_2, \theta, t) = 0$$
$$\varsigma_{r\theta m}(r = a_2, \theta, t) = 0$$

With (4.130), (4.131) and (4.132) we obtain as characteristic equation

$$\begin{vmatrix} \left[\varsigma_m \vartheta_n (a_1)\right] & \left[\varsigma_m Y_n (a_1)\right] \\ \left[\varsigma_m \vartheta_n (a_2)\right] & \left[\varsigma_m Y_n (a_2)\right] \end{vmatrix} = 0 \tag{4.141}$$

where the matrices $\left[\varsigma_m \vartheta_n (a_l)\right]$ and $\left[\varsigma_m Y_n (a_l)\right]$ are defined by (4.131) and (4.132) for a given viscoelastic material m. Equation (4.141) is a 4×4 determinant and, with Δ_{ij} representing an individual element, it takes the form

$$\begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44} \end{vmatrix} = 0 \tag{4.142}$$

In the case of an elastic solid ($m = 2$) the elements become

$$\Delta_{11}(\alpha_1) = \left[n(n+1) - \frac{1-\varsigma}{1-2\varsigma}\alpha_1^2\right]\vartheta_n(\alpha_1) - 2\alpha_1\vartheta_n'(\alpha_1)$$

$$\Delta_{12}(\alpha_1) = n(n+1)\left[\alpha_1\sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\,\vartheta_n^1\left(\alpha_1\sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\right) - \vartheta_n\left(\alpha_1\sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\right)\right]$$

$$\Delta_{13}(\alpha_1) = \left[n(n+1) - \left(\frac{(1-\varsigma)}{1-2\varsigma}\right)\alpha_1^2\right]Y_n(\alpha_1) - 2\alpha_1 Y_n^1(\alpha_1)$$

$$\Delta_{14}(\alpha_1) = n(n+1)\left[\alpha_1\sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\,Y_n^1\left(\alpha_1\sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\right) - Y_n\left(\alpha_1\sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\right)\right]$$

$$\Delta_{21}(\alpha_1) = \alpha_1\vartheta_n^1(\alpha_1) - \vartheta_n(\alpha_1)$$

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$$\Delta_{22}(\alpha_1) = \left[n^2 + n - 1 - \frac{1-\varsigma}{1-2\varsigma}\alpha_1^2 \right] \vartheta_n \left(\alpha_1 \sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}} \right) - \alpha_1 \sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\, \vartheta_n^1 \left(\alpha_1 \sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}} \right)$$

$$\Delta_{24}(\alpha_1) = \left[n^2 + n - 1 - \frac{1-\varsigma}{1-2\varsigma}\alpha_1^2 \right] Y_n \left(\alpha_1 \sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}} \right) - \alpha_1 \sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}}\, Y_n^1 \left(\alpha_1 \sqrt{\frac{2(1-\varsigma)}{1-2\varsigma}} \right)$$

$$\Delta_{23}(\alpha_1) = \alpha_1 = \alpha_1 Y_n^1(\alpha_1) - Y_n(\alpha_1)$$

and

$$\Delta_{31} = \Delta_{11}(\alpha_2)$$
$$\Delta_{32} = \Delta_{12}(\alpha_2)$$
$$\Delta_{33} = \Delta_{13}(\alpha_2)$$
$$\Delta_{34} = \Delta_{14}(\alpha_2)$$
$$\Delta_{41} = \Delta_{21}(\alpha_2)$$
$$\Delta_{42} = \Delta_{22}(\alpha_2)$$
$$\Delta_{43} = \Delta_{23}(\alpha_2)$$
$$\Delta_{44} = \Delta_{24}(\alpha_2)$$

where $\alpha_2 = \frac{a_2}{a_1}\alpha_1$ and $\alpha_1 = \frac{\omega a_1}{c_d}$. ς is the Poisson's ratio and c_d the dilatational wave speed of the elastic solid.

The frequency of a vibrational mode is normalized by first multiplying it by the inner radius of the spherical shell and then dividing it by the dilatational wave speed of the elastic medium. The frequency spectra are now reviewed for two particular cases. In the case of radial vibration of a sphere the values are identical with those defined by Lover's solution[34)]. In the case of a one-layer thin shell theory used by Baker[49)] and by Naghdi and Kalnins[50)] as far as the two lowest modes are concerned. However, the present investigation indicates that there exists an infinite set of modes of vibration for a one-layer thin shell of finite thickness. All the higher modes, however, tend to have a constant high frequency as the meridianal wave number continues to increase. These facts are demonstrated in Figures 2 and 3 for a thin shell with a mean radius to thickness ratio of 63 and a thick shell with a mean radius to wall thickness ratio of about 3. When the thickness of a shell with a given radius is increased, the higher modes are lowered and approach the same order of magnitude of the two lowest modes. By substituting a particular frequency value from the spectrum into the field quantities, one obtains the displacement and stress fields as functions of the spherical coordinates r, θ and as functions of time. Here the field quantities are normalized with the function θ at a given time, leaving r as the only variable. Thus the profile of the variation of the field quantities in the radial direction can be shown by plotting them as functions of the radial coordinate. The mode shapes can also be given as vectorial sums of the displacements. The modal analysis and the frequency spectra thus obtained are shown in Figures 4 to 8.

4.3.4. Axially symmetric nontorsional free vibration of an elastic spherical shell containing viscous fluid.

The elastic shell under consideration is uniformly thick with inner radius $r=a_1$ and outer radius $r=a_2$. The field quantities are finite at $r=0$ for any $t>0$, thus $B_{11}=0$. The displacements and stresses are continuous at the interspherical surface; the outer surface of the shell is considered stress-free.

The boundary conditions of this problem are

$$u_{r1}(r=a_1,\theta,\ t)-u_{r2}(r=a_1,\theta,\ t)=0$$
$$u_{\theta 1}(r=a_1,\theta,\ t)-u_{\theta 2}(r=a_1,\theta,\ t)=0$$
$$P_1+\varsigma_{rr1}(r=a_1,\theta,\ t)-\varsigma_{rr2}(r=a_1,\theta,\ t)=0$$
$$\varsigma_{r\theta 1}(r=a_1,\theta,\ t)-\varsigma_{r\theta 2}(r=a_1,\theta,\ t)=0$$
$$\varsigma_{rr2}(r=a_2,\theta,\ t)=0$$
$$\varsigma_{r\theta 2}(r=a_2,\theta,\ t)=0$$

From (4.127) to (4.132), with $m = 1$ for the interior of the shell and $m = 2$ for the shell material, the characteristic equation of the problem is

$$\begin{vmatrix} [u_1 \vartheta_n(a_1)] & [u_2 \vartheta_n(a_1)] & [u_2 Y_n(a_1)] \\ 2\mu_1[\varsigma_1 \vartheta_n(a_1)] & 2\mu_2[\varsigma_2 \vartheta_n(a_1)] & 2\mu_2[\varsigma_2 Y_n(a_1)] \\ 0 & 2\mu_2[\varsigma_2 \vartheta_n(a_2)] & 2\mu_2[\varsigma_2 Y_n(a_2)] \end{vmatrix} = 0 \tag{4.143}$$

The characteristic equation (4.143) is considerably more complicated than in previous examples. The elements of the determinant are defined by (4.129a), (4.129b), (4.131) and (4.132). In seeking the high frequency solution of (4.143), we can make use of the asymptotic expansions of the spherical Bessel functions ϑ_n and Y_n for large arguments. By manipulating all the dominant terms in the determinant (4.143), we can prove that the shearing effects and dilatational effects of a spherical shell filled with compressible viscous fluid vibrating at high frequency are almost decoupled. These results are summarized as follows.

(a) Shearing effects

Materials	Solution ω	Physical Statements
Viscous fluid core	$\frac{2\upsilon(n^2+n-1)}{a_1^2} i$	damped solution
	$\frac{\upsilon}{a_1^2}\left(\frac{n}{2}\pi + j\pi\right)^2 i$ where $j = 0, 1, 2, 3\ldots$	damped solution

Spherical elastic shell	$\frac{c_{s2}}{a_2}\sqrt{2\left(n^2+n-1\right)}$	undamped oscillation
	$\frac{c_{s2}}{a_1+a_2}\left(n\pi+\frac{\pi}{2}+2j\pi\right)$ where j = 0, 1, 2, 3…	undamped oscillation

(b) <u>Dilatational effects</u>

Materials	*Solution* ω	Physical Statements
Viscous fluid core	$\left(j\pi+\frac{n}{4}\pi+\frac{\pi}{4}\right)i\left(\frac{\upsilon'+2\upsilon}{a_1^2}\right)\pm\sqrt{4\left(\frac{c_0}{a_1}\right)^2-\left(j\pi+\frac{n}{4}\pi+\frac{\pi}{4}\right)^2\left(\frac{\upsilon'+2\upsilon}{a_1^2}\right)^2}$ where j = 0, 1, 2, 3… when $\frac{c_0/a_1}{\left(\frac{\upsilon'+2\upsilon}{a_1^2}\right)}>\left(j\pi+\frac{n}{4}\pi+\frac{\pi}{4}\right)$	damped oscillation
	when $\frac{c_0/a_1}{\left(\frac{\upsilon'+2\upsilon}{a_1^2}\right)}=\left(j\pi+\frac{n}{4}\pi+\frac{\pi}{4}\right)$	critical-damped motion
	when $\frac{c_0/a_1}{\left(\frac{\upsilon'+2\upsilon}{a_1^2}\right)}<\left(j\pi+\frac{n}{4}\pi+\frac{\pi}{4}\right)$	Overdamped motion
Spherical elastic shell	$\frac{c_{d2}}{(a_1+a_2)}[2j\pi+n\pi+\pi]$ where j = 0, 1, 2, 3…	Undamped oscillation

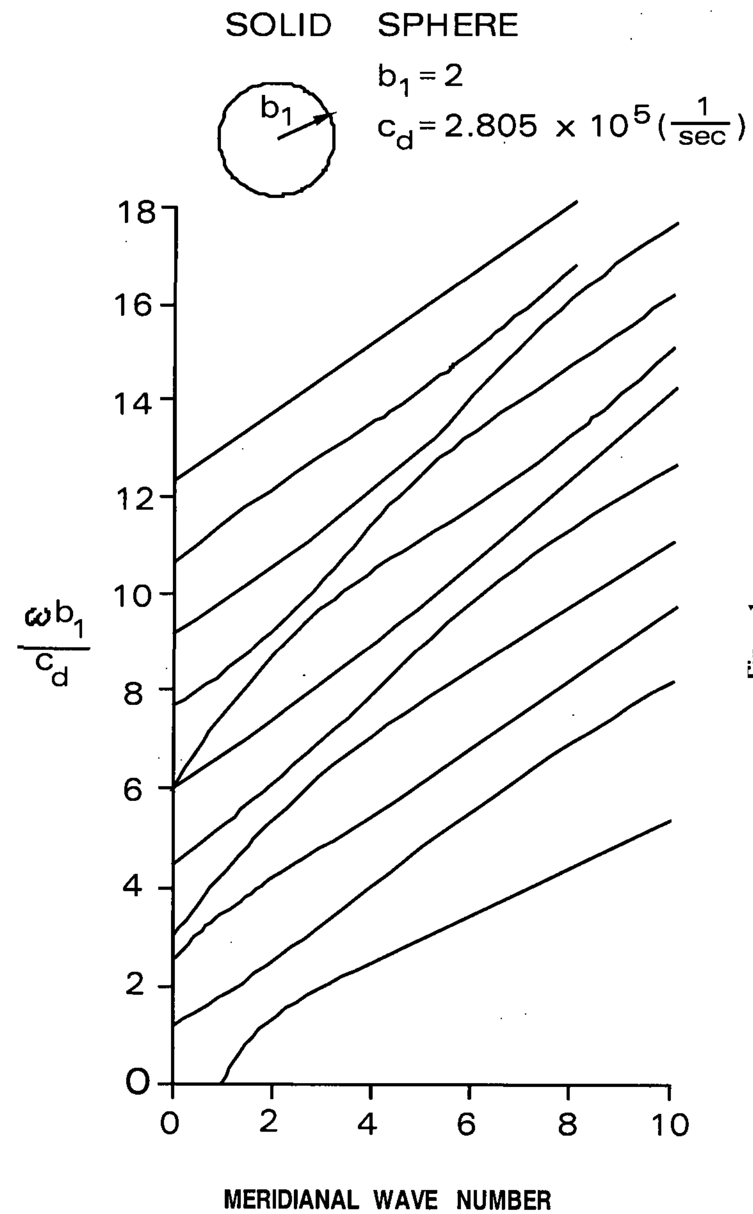

SOLID SPHERE
b1 = 2
b1
cd = 2.805 x 10^5 (1/sec)
18
16
14
12
10
8
6
4
2
0
ωb1/cd
0
2
4
6
8
10
MERIDIANAL WAVE NUMBER

Figure 1

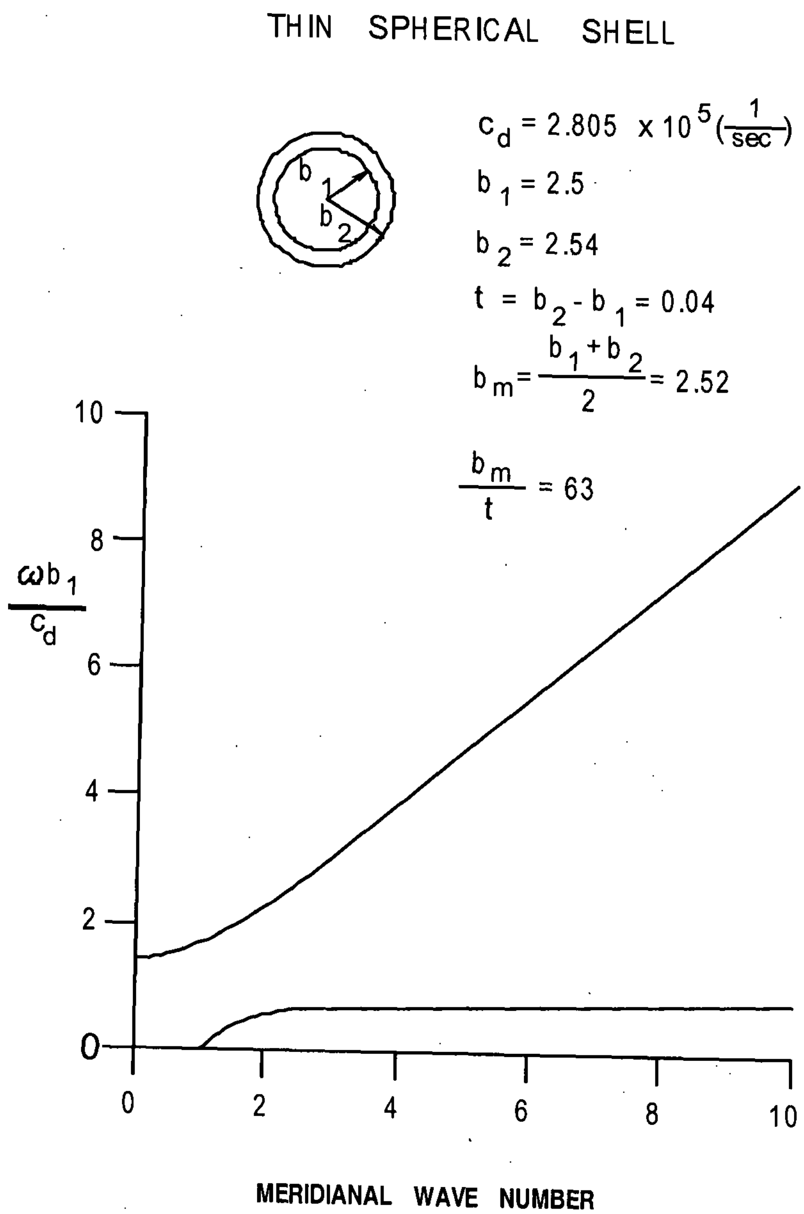

THIN SPHERICAL SHELL
$c_d = 2.805 \times 10^5 (\frac{1}{sec})$
$b_1 = 2.5$
$b_2 = 2.54$
$t = b_2 - b_1 = 0.04$
$b_m = \frac{b_1 + b_2}{2} = 2.52$
$\frac{b_m}{t} = 63$
b_1
b_2
$\frac{\omega b_1}{c_d}$
0
2
4
6
8
10
MERIDIANAL WAVE NUMBER

Figure 2

Po Kee Wong

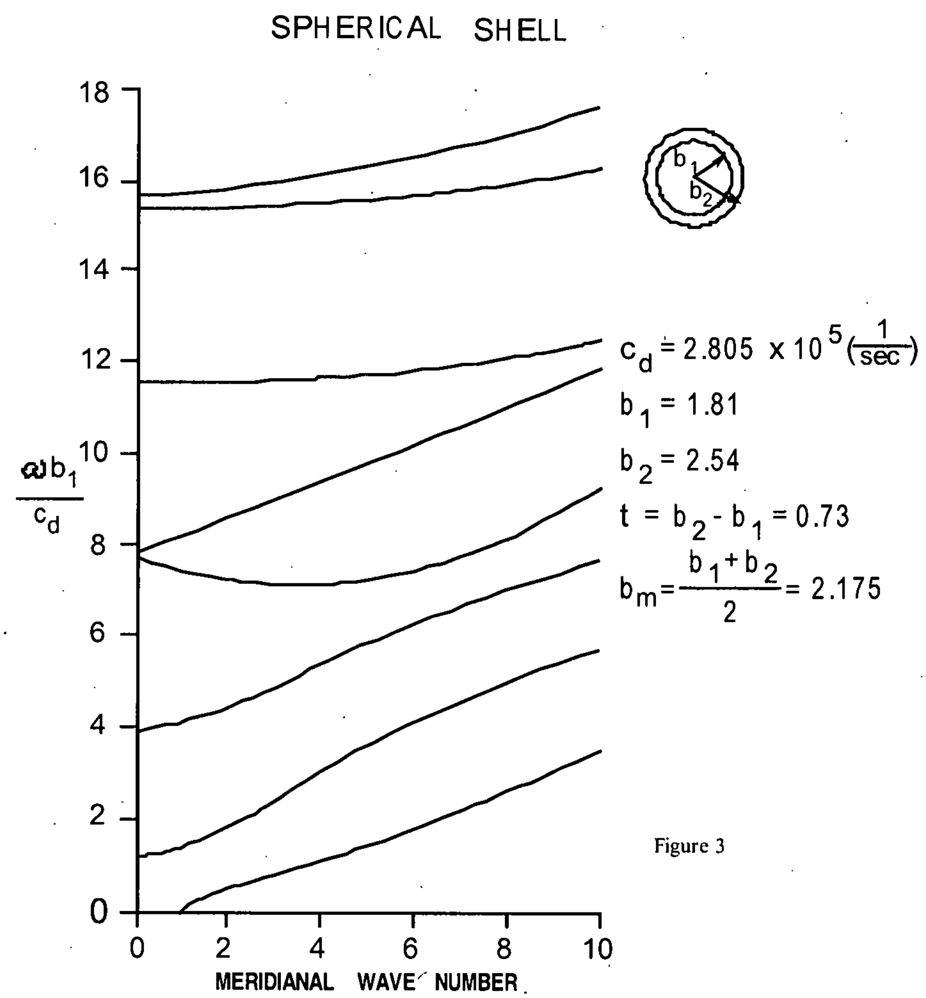


Figure 3

Waves in Viscous Fluids, Elastic Solids, and Viscoelastic Materials

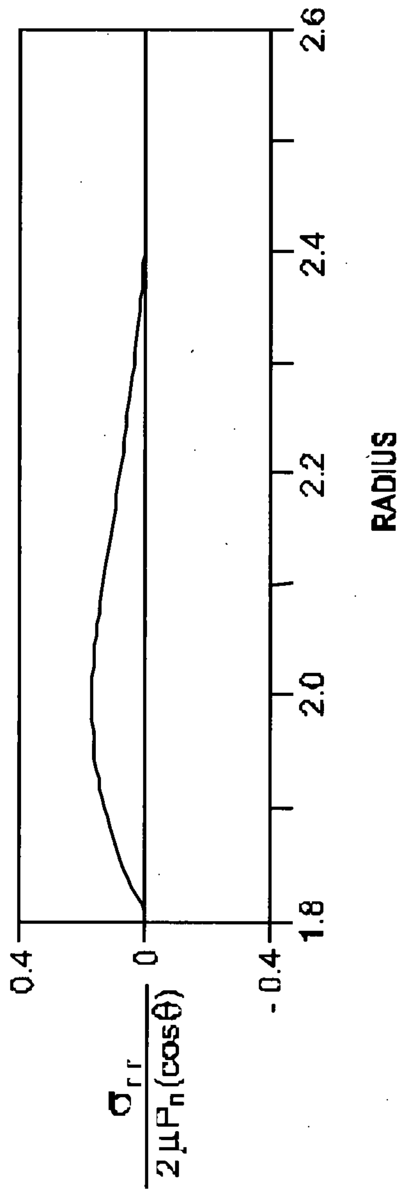


Figure 4

Po Kee Wong

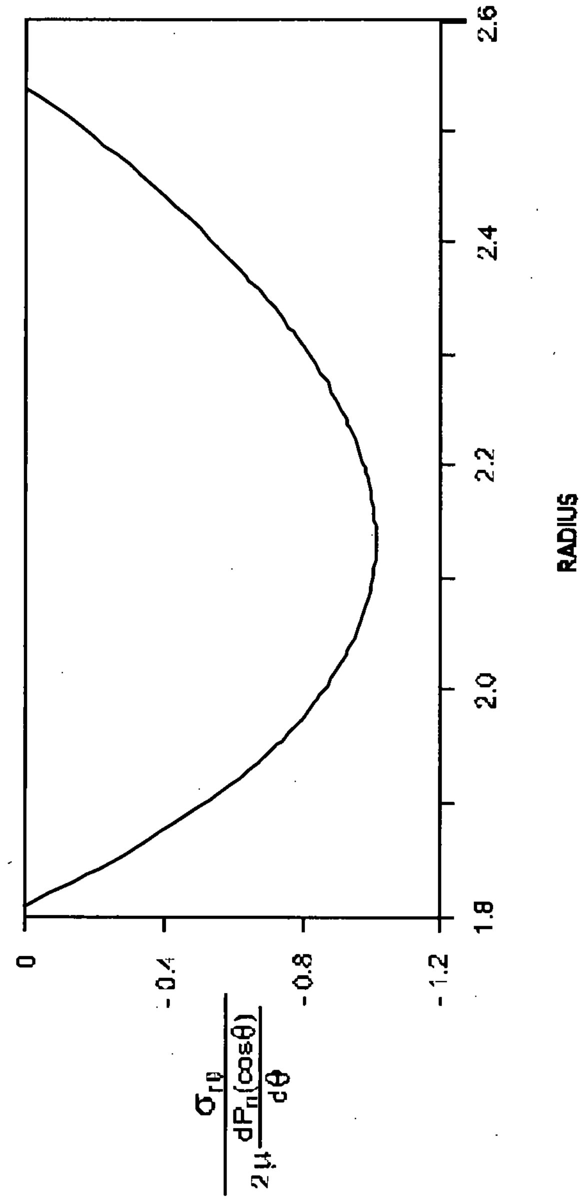


Figure 5

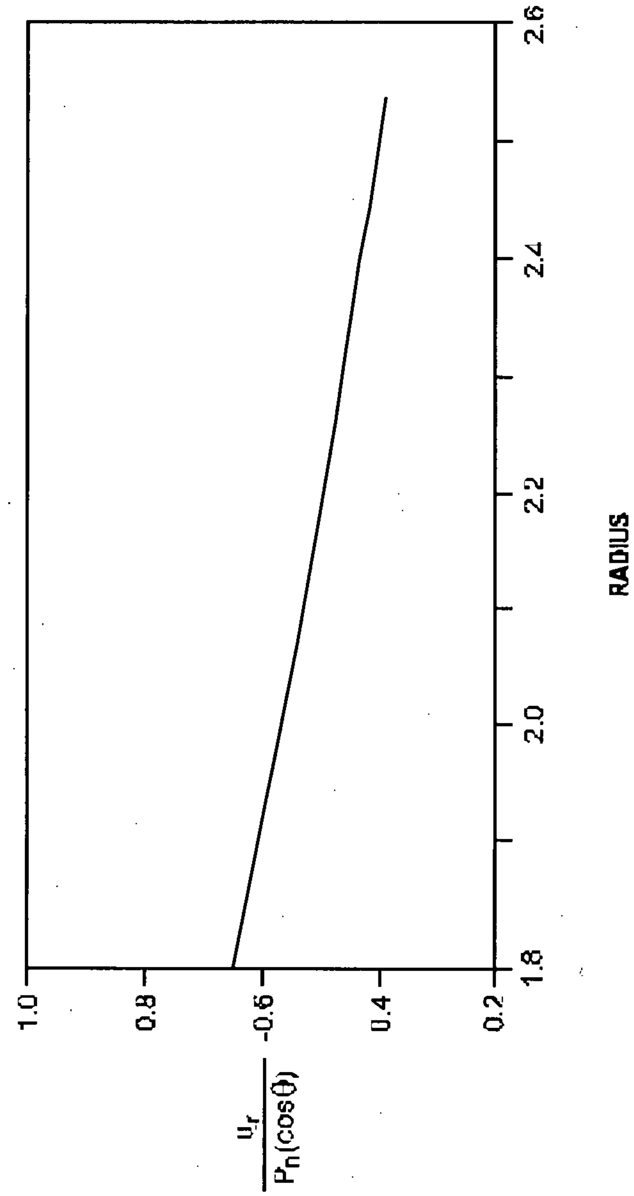


Figure 6

Po Kee Wong

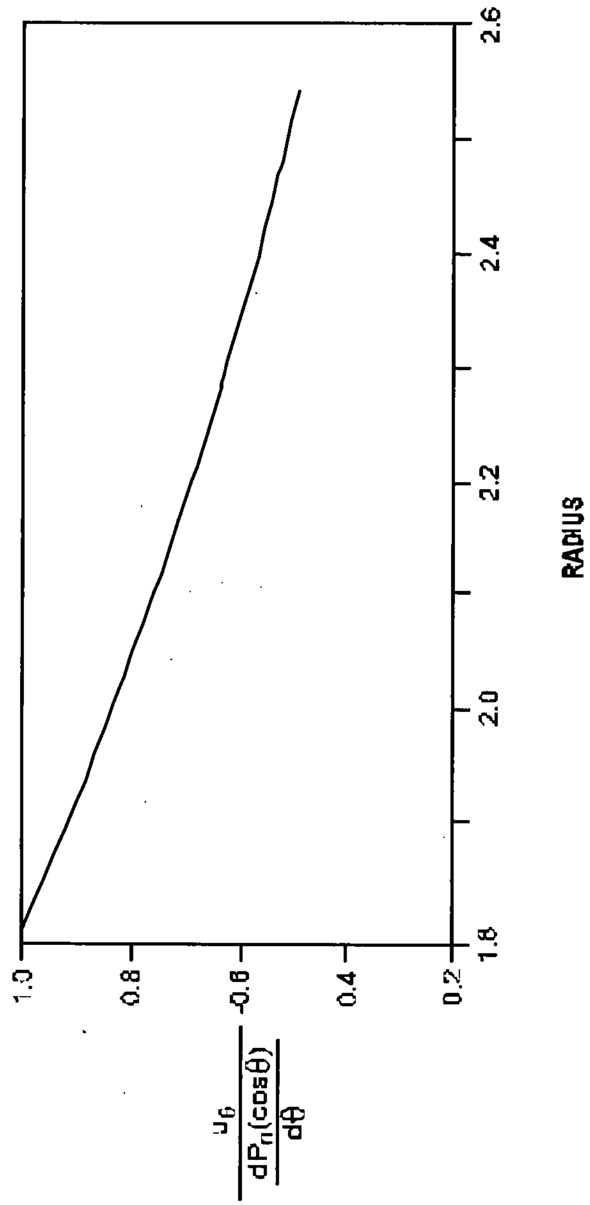
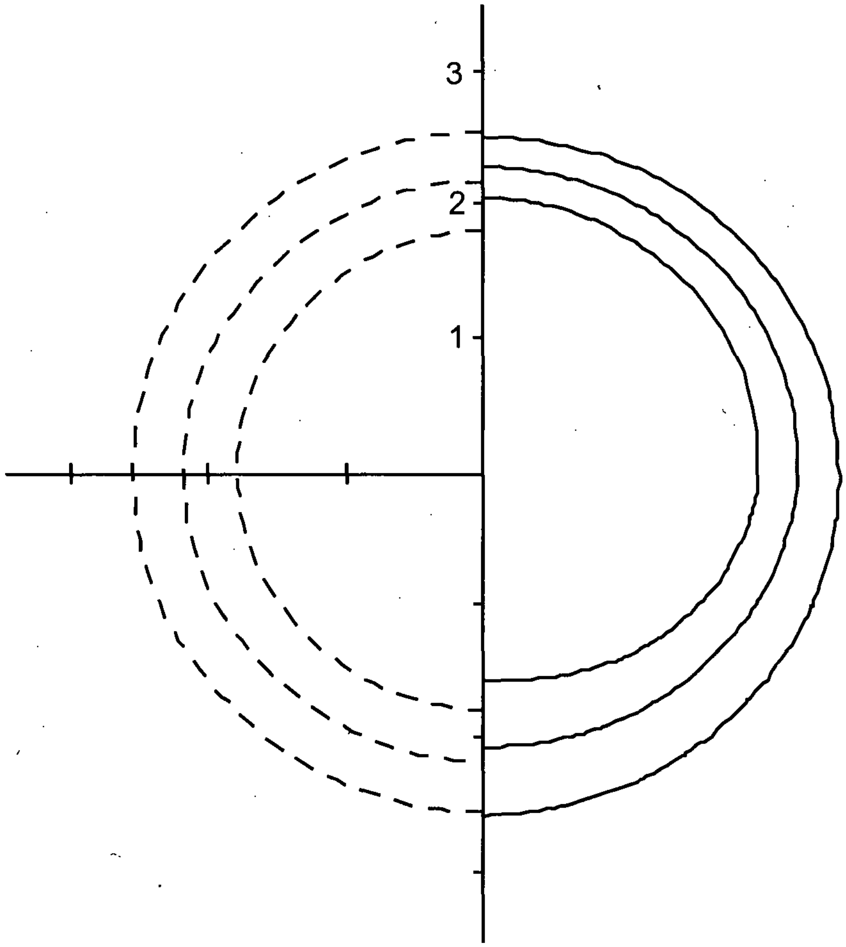


Figure 7



DISPLACEMENT MODE FOR N = 1 AT SECOND FREQUENCY

Figure 8

CHAPTER 5

CONCLUSIONS, DISCUSSION AND SUGGESTIONS

Conclusions

1. The governing displacement equations of motion for a class of three-dimensional homogeneous, isotropic, linear viscoelastic materials are systematically constructed.

2. General solutions of these governing displacement equations are obtained by means of the Lamé-Helmholtz-Stokes potential functions, from which each governing displacement equation of motion is transformed into a scalar and a vector potential equation of motion.

3. The vector potential function is further resolved in terms of two independent scalar potential functions in order to secure a separable solution. This can be done when the vector equation of motion is expressed in terms of spherical, conical, or general cylindrical coordinates. Therefore, the displacement equation of motion for each material model can be transformed into three independent scalar potential equations of motion.

4. The field quantities are obtained in terms of those three independent scalar potential functions, and from these the transient or harmonic solutions for mixed or nonmixed boundary value problems can be obtained.

Discussion and Suggestions

The present theory, as mentioned above, provides a systematic means of constructing and solving the governing displacement equations of motion for a class of three-dimensional, homogeneous, isotropic, linear viscoelastic materials.

The application of the theory to solve some physical problems is demonstrated in thirteen examples which are all based on spherical coordinate systems. However, as noted in the text, the theory can be extended with regard to the following three aspects.

1. Analytical aspect

   a. To provide detailed analytical general solutions and physical examples in other separable coordinate systems, namely, the rectangular, the parabolic, the circular and the elliptic cylindrical coordinate systems, and the conical coordinate system.

   b. To consider thermal and electromagnetic coupling effects in the general solutions for all separable coordinate systems.

   c. To obtain, by means of iteration from the basic linear general solutions, the higher order nonlinear solutions including thermal and electromagnetic coupling effects.

   d. To develop a systematic method of analyzing the physical problems involving nonhomogeneous, anisotropic materials.

   e. To implement electrical and mechanical circuit analogy to simulate complicated material models.

2,3. Numerical and experimental aspects

   a. There are quite a few difficult physical problems which have been analytically solved, but so far no numerical values or experimental data have been obtained for these problems. Examples are: "the circumferential harmonic wave propagation in elastic rods of elliptic cross section"; "the circumferential harmonic wave motions of a viscous fluid in a rigid elliptic inclusion"; and "flexural wave motions in a multilayered circular cylindrical shell." All these problems have been analytically solved, and although they are physically meaningful, with important engineering applications, the numerical results have so far not been obtained.

b. To establish a long-term project for tabulation of well-known functions of complex argument in the course of evaluating the numerical values from the analytical problems. For example, the harmonic solutions of physical problems associated with thermal coupling effects, viscous fluids or viscoelastic materials in a single- or multilayered continuum involve mathematical functions with complex arguments in order to retain the separability of the solutions satisfying the boundary conditions. Also, nonmixed type boundary value problems involve mathematical functions of complex arguments. These can be solved by means of linear transform techniques.

APPENDIX A

SOLUTION OF THE VECTOR DIFFUSION EQUATION AND THE VECTOR WAVE EQUATION IN TERMS OF SCALAR POTENTIALS

The vector diffusion equation and the vector wave equation can be solved in terms of two scalar potentials, which satisfy scalar diffusion equations and scalar wave equations respectively. This will be demonstrated here for the case of spherical coordinates.

If $\psi\,(r,\theta,\phi,t)$ is a scalar function of the spherical coordinates r,θ,ϕ, then it can be readily shown that:

$$\nabla \times (\vec{e}_r\, r\psi) = (\nabla\, r\psi) \times \vec{e}_r \qquad \text{(A1)}$$

and

$$\nabla \times \nabla \times (\vec{e}_r\, r\psi) = \nabla \left(\frac{\partial r\psi}{\partial r} \right) - \vec{e}_r\, r\, \nabla^2 \psi \qquad \text{(A2)}$$

where $\vec{e}_r$ is a unit vector in the r direction.

If $\psi\,(r,\theta,\varphi,t)$ satisfies the scalar diffusion equation and the scalar wave equation

$$\nabla^2 \psi = \left\{ \begin{array}{c} \dfrac{1}{\upsilon}\dfrac{\partial \psi}{\partial t} \\[2ex] \dfrac{1}{c_3^2}\dfrac{\partial^2 \psi}{\partial t^2} \end{array} \right\} \qquad \text{(A3)}$$

then substitution of (A3) into (A2) gives

$$\nabla \times \nabla \times (\vec{e}_r r\psi) = \nabla \left(\frac{\partial r\psi}{\partial r} \right) - \vec{e}_r r \left[\begin{array}{c} \frac{1}{\upsilon} \frac{\partial \psi}{\partial t} \\ \frac{1}{c_3^2} \frac{\partial^2 \psi}{\partial t^2} \end{array} \right] \tag{A4}$$

If ψ satisfies

$$\nabla^2 \psi = \left[\begin{array}{c} \frac{1}{\upsilon} \frac{\partial \psi}{\partial \psi} \\ \frac{1}{c_3^2} \frac{\partial^2 \psi}{\partial t^2} \end{array} \right] \tag{A5}$$

and χ satisfies

$$\nabla^2 \chi = \left[\begin{array}{c} \frac{1}{\upsilon} \frac{\partial t}{\partial t} \\ \frac{1}{c_3^2} \frac{\partial^2 \chi}{\partial t^2} \end{array} \right] \tag{A6}$$

then $\vec{A} = \nabla \times (\vec{e}_r r\psi) + \nabla \times \nabla \times (\vec{e}_r r\chi)$

$$= \nabla \times (\vec{e}_r r\psi) + \nabla \left(\frac{\partial r\chi}{\partial r} \right) - \vec{e}_r r \left[\begin{array}{c} \frac{1}{\upsilon} \frac{\partial \chi}{\partial t} \\ \frac{1}{c_3^2} \frac{\partial^3 \chi}{\partial t^2} \end{array} \right] \tag{A7}$$

and

$$\nabla^2 \vec{A} = \left[\begin{array}{c} \frac{1}{\upsilon} \frac{\partial \vec{A}}{\partial t} \\ \frac{1}{c_3^2} \frac{\partial^2 \vec{A}}{\partial t^2} \end{array} \right] \tag{A8}$$

PROOF: Taking the divergence of (A7) gives

$$\nabla \cdot \vec{A} = 0 \qquad \text{(A9)}$$

Taking the curl of (A7) and making use of (A4), we obtain

$$\nabla \times \vec{A} = \nabla \times \left[\nabla \times \left(\vec{e}_r r \psi \right) + \nabla \left(\frac{\partial r\chi}{\partial r} \right) - \vec{e}_r r \left\{ \begin{matrix} \frac{1}{\upsilon} \frac{\partial \chi}{\partial t} \\ \frac{1}{c_3^2} \frac{\partial^2 \chi}{\partial t^2} \end{matrix} \right\} \right]$$

$$= \nabla \left(\frac{\partial r\psi}{\partial r} \right) - \vec{e}_r r \left\{ \begin{matrix} \frac{1}{\upsilon} \frac{\partial \psi}{\partial t} \\ \frac{1}{c_3^2} \frac{\partial^2 \psi}{\partial t^2} \end{matrix} \right\} - \nabla \times \vec{e}_r r \left\{ \begin{matrix} \frac{1}{\upsilon} \frac{\partial \chi}{\partial t} \\ \frac{1}{c_3^2} \frac{\partial^2 \chi}{\partial t^2} \end{matrix} \right\} \qquad \text{(A10)}$$

The curl of (A10) gives, with (A9) and (A7)

$$\nabla \times \nabla \times \vec{A} \equiv \nabla \cdot \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A} = -\nabla^2 \vec{A}$$

$$= \nabla \times \left[\nabla \left(\frac{\partial r\psi}{\partial r} \right) - \vec{e}_r \left\{ \begin{matrix} \frac{1}{\upsilon} \frac{\partial \psi r}{\partial t} \\ \frac{1}{c_5^2} \frac{\partial^2 \psi r}{\partial t^2} \end{matrix} \right\} - \nabla \times \vec{e}_r \left\{ \begin{matrix} \frac{1}{\upsilon} \frac{\partial r\chi}{\partial t} \\ \frac{1}{c_5^2} \frac{\partial^2 r\chi}{\partial t^2} \end{matrix} \right\} \right]$$

$$= - \left\{ \begin{matrix} \frac{1}{\upsilon} \frac{\partial}{\partial t} \\ \frac{1}{c_5^2} \frac{\partial^2}{\partial t^2} \end{matrix} \right\} \left[\nabla \times \left(\vec{e}_r r \psi \right) + \nabla \times \nabla \times \left(\vec{e}_r r \chi \right) \right] = - \left\{ \begin{matrix} \frac{1}{\upsilon} \frac{\partial \vec{A}}{\partial t} \\ \frac{1}{c_5^2} \frac{\partial^2 \vec{A}}{\partial t^2} \end{matrix} \right\}$$

Hence

$$\nabla \times \nabla \times \vec{A} = -\begin{pmatrix} \dfrac{1}{\upsilon}\dfrac{\partial \vec{A}}{\partial t} \\ \dfrac{1}{c_6^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} \end{pmatrix}$$

and

$$\nabla^2 \vec{A} = \begin{pmatrix} \dfrac{1}{\upsilon}\dfrac{\partial \vec{A}}{\partial t} \\ \dfrac{1}{c_5^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} \end{pmatrix}$$

Therefore, if ψ satisfies $\nabla^2\psi = F\left(\dfrac{\partial}{\partial t}\right)\psi$ and χ satisfies $\nabla^2\chi = F\left(\dfrac{\partial}{\partial t}\right)\chi$ where $F\left(\dfrac{\partial}{\partial t}\right)$ is a function of the operator $\dfrac{\partial}{\partial t}$ only, then $\vec{A} = \nabla x \left(\vec{e}_r r\psi\right) + \nabla \times \nabla \times \left(\vec{e}_r r\chi\right)$ satisfies $\nabla^2\vec{A} = F\left(\dfrac{\partial}{\partial t}\right)\vec{A}$.

APPENDIX B

The well-known Helmholtz's theorem $\vec{u} = \nabla\varphi + \nabla \times \vec{A}$ indicated by equation (2.17) where φ and $\vec{A}$ are potential functions, can be traced back to contributions of Lame and Poisson. These functions φ and $\vec{A}$ are often referred to as Lame's potentials. The traditional proof of the Helmholtz theorem is diversified in the literature of several scientific fields. It can be found in texts of potential theory, electromagnetic theory, vector analysis, partial differential equations, as well as in those of elasticity and fluid mechanics. The traditional proof was obtained in terms of volume integrals for φ and $\vec{A}$ over an infinite domain. These results are in fact particular integral representations of the Poisson's equations $\nabla^2\varphi = \nabla$ and $\nabla^2\vec{A} = -2\vec{\Omega}$, where $\Delta = \nabla \cdot \vec{u}$ and $2\vec{\Omega} = \nabla \times \vec{u}$, $\nabla \cdot \vec{A} = 0$.

An alternate proof of such results is shown in the following sequence: (a) vector algebra; (b) vector identity; (e) commutativeness of divergence and curl with respect to the Laplacian operator; and (d) completion of the proof.

(a) Vector algebra

Given: $\vec{c}, \vec{D}, 2\vec{\Omega}$ as known vectors at any points in the space with their components lying along the coordinate basis of three orthogonal unit vectors $\vec{e}, \vec{e_2}$ and $\vec{e_3}$, and Δ a known scalar at the same point, where $\vec{D} \cdot \vec{C} \neq 0$ and $\vec{D} \perp 2\vec{\Omega}$

Consider: $\vec{u}$ as an unknown vector which satisfies

$$\vec{C} \cdot \vec{u} = \Delta \tag{1}$$

$$\vec{D} \times \vec{u} = 2\vec{\Omega} \tag{2}$$

Then $\vec{u}$ can be determined from both (1) and (2). Such a result can be obtained by taking the cross product of vector $\vec{C}$ and equation (2):

$$\vec{C} \times \left(\vec{D} \times \vec{u}\right) \equiv \vec{D}\left(\vec{C} \cdot \vec{u}\right) - \left(\vec{C} \cdot \vec{D}\right)\vec{u} = \vec{C} \times 2\vec{\Omega} \tag{3}$$

Solve $\vec{u}$ from (3) and replace $\left(\vec{C} \cdot \vec{u}\right)$ from (1), obtaining

$$\vec{u} = \frac{\vec{D}\,\Delta - \vec{C} \times 2\vec{\Omega}}{\left(\vec{C} \cdot \vec{D}\right)} \tag{4}$$

Notice that $\vec{u}$ cannot be determined either from (2) or (3) alone. In the former case, it is because of insufficiency of the simultaneous equations for the components of the vector $\vec{u}$, and in the latter, the determinant is singular.

(b) Vector identity

Since (4) is true for any given vectors $\vec{D}$, $\vec{C}$ and $2\vec{\Omega}$ and any given scalar Δ at any point where these four quantities are finite, then without loss of generality let $C = D = \nabla$ which is defined as a gradient, a vector operator in the usual sense; (1) and (2) become

$$\nabla \cdot \vec{u} = \Delta \tag{5}$$

$$\nabla \times \vec{u} = 2\vec{\Omega} \tag{6}$$

and equivalence of (4) becomes

$$\nabla^2 \vec{u} = \nabla\Delta - \nabla \times 2\vec{\Omega} \tag{7}$$

Substituting Δ from (5) and $2\vec{\Omega}$ from (6) gives the well-known vector identity

$$\nabla^2 \vec{u} = \nabla(\Delta \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) \tag{8}$$

(c) Commutativeness of divergence and curl with respect to the Laplacian ∇^2. Taking the divergence of (8), $\nabla \cdot \left(\nabla^2 \vec{u}\right) = \nabla^2 \left(\nabla \cdot \vec{u}\right) - \nabla \cdot \left[\nabla \times \left(\nabla \times \vec{u}\right)\right]$, and noticing that $\nabla \cdot \left[\nabla \times \left(\nabla \times \vec{u}\right)\right] \equiv 0$ in vector analysis, one obtains

$$\nabla \cdot \left(\nabla^2 \vec{u}\right) = \nabla^2 \left(\nabla \cdot \vec{u}\right) \tag{9}$$

Hence $\nabla \cdot \nabla^2$ and $\nabla^2 \nabla \cdot$ operated on a vector are commutative. Taking the curl of (8), $\nabla \times \left(\nabla^2 \vec{u}\right) = \nabla \times \nabla \left(\nabla \cdot \vec{u}\right) - \nabla \times \left[\nabla \times \left(\nabla \times \vec{u}\right)\right]$, noticing that $\nabla \times \nabla \left(\nabla \cdot \vec{u}\right) \equiv 0$ and using (8) to expand the term

$$-\nabla \times \left[\nabla \times \left(\nabla \times \vec{u}\right)\right] = \nabla^2 \left(\nabla \times \vec{u}\right) - \nabla \left[\nabla \cdot \left(\nabla \times \vec{u}\right)\right] = \nabla^2 \left(\nabla \times \vec{u}\right)$$

one obtains

$$\nabla \times \left(\nabla^2 \vec{u}\right) = \nabla^2 \left(\nabla \times \vec{u}\right) \tag{10}$$

Hence $\nabla \times \nabla^2$ and $\nabla^2 \nabla \times$ are operated on a vector are also commutative.

(d) Completion of the proof.

If

$$\vec{u} = \vec{u}_1 + \vec{u}_2 \tag{11}$$

where $\vec{u}_1$ and $\vec{u}_2$ are functions to be determined satisfying the conditions

$$\nabla \times \vec{u_1} = 0 \tag{12}$$

and

$$\nabla \cdot \overline{u_2} = 0 \tag{13}$$

then substitution of (11) into (7) gives

$$\nabla^2\left(\vec{u_1} + \vec{u_2}\right) = \nabla\Delta - \nabla \times 2\dot{\Omega} \tag{14}$$

Taking the divergence of (14) and noticing that from (9) $\nabla \cdot \nabla^2$ and $\nabla^2\nabla \cdot$ are commutative, from (13) $\nabla \cdot \overline{u_2} = 0$ and from vector analysis $\nabla \cdot \nabla \times 2\dot{\Omega} = 0$, one obtains

$$\nabla^2\left(\nabla \cdot \vec{u_1}\right) = \nabla^2\Delta \tag{15}$$

A sufficient condition for the satisfaction of (15) is that

$$\nabla \cdot \vec{u_1} = \Delta \tag{16}$$

Taking the curl of (14) and noticing from (10) that $\nabla \times \nabla^2$ and $\nabla^2\nabla \times$ are commutative, from (11) that $\nabla \times \vec{u_1} = 0$, from vector analysis that $\nabla \times \nabla\Delta = 0$ and from (8) that $\nabla^2(\nabla \times u_2) = \nabla\left[\nabla \cdot (\nabla \times u_2\right] - \nabla \times \nabla \times (\nabla \times u_2) = -\nabla \times \nabla \times \left(\nabla \times \vec{u_2}\right)$ gives

$$-\nabla \times \nabla \times \left(\nabla \times \overline{u_2}\right) = -\nabla \times \nabla \times 2\dot{\Omega} \tag{17}$$

A sufficient condition for the satisfaction of (17) is that

$$\nabla \times \vec{u_2} = 2\dot{\Omega} \tag{18}$$

Of the two pairs of equations, (12) and (16), and (13) and (18), the first pair contains $\overrightarrow{u_1}$ as an unknown and the second pair contains $\overrightarrow{u_2}$ as an unknown. (12) is satisfied if

$$\overrightarrow{u_1} = \nabla\varphi \tag{19}$$

and it follows from (16)

$$\nabla^2\varphi = \Delta \tag{20}$$

Equation (13) is satisfied if

$$\overrightarrow{u_2} = \nabla \times \vec{A} \tag{21}$$

and it follows from (18) that

$$\nabla \times \left(\nabla \times \vec{A}\right) = 2\vec{\Omega} \tag{22}$$

Simplifying the left-hand side of (22) by means of (8) $\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2\vec{A}$ and choosing

$$\nabla \cdot \vec{A} = 0 \tag{23}$$

equation (22) becomes

$$\nabla^2\vec{A} = -2\vec{\Omega} \tag{24}$$

Substituting (19) and (21) back into (11) completes the proof of

$$\vec{u} = \nabla\varphi + \nabla \times \vec{A}$$

satisfying (7) which represents (5) and (6) provided that $\nabla^2\varphi = \Delta$ and $\nabla^2\vec{A} = -2\vec{\Omega}$ with $\nabla \cdot \vec{A} = 0$.

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From: Po Kee wong [pokwong@verizon.net]
Sent: Saturday, September 30, 2006 2:42 PM
To: '????'
Cc: 'pokwong@verizon.net'; 'pokwong@rcn.com'
Subject: FW: 11 websites where you can obtain my patents; international conference papers and relevant engineering and scientific documents for open review; evaluation and discussion worldwide

Dear Chairman Sheon:

The following websites are consolidated together to facilitate for your filing and tracing of my work that may be of your interest to collect and edit and put them into the file for your book " History of Nan Tao High School ".

While typing the cc of this E-mail to you, President George Bush's E-mail address pops up. This gives me an idea to ask you whether you want also to invite President Bush of USA and President Hu Jin Tao to come to our Centennial Celebration. They can meet and talk informally about the proposed projects of collaborations between two governments to build: (1) A 3rd. identical observatory and; (2) A new satellite ejection station in the "High Plateau Region of China"

Please call me at + 301-585-3453 to discuss about this idea further if you think this is feasible and mutually beneficial to all of us being involved. Please also respond this E-mail with acknowledgement that you have already received all the E-mails of information having sent to you so far.

With my best regards to you all, I am,

Very truly yours,

WONG, Po Kee (1946-1949 Alumni)

Tel: + 301-585-3453

pokwong@verizon.net

pokwong@rcn.com

From: pokwong [mailto:pokwong@rcn.com]
Sent: Friday, September 29, 2006 10:50 PM
To: pokwong@verizon.net; pokwong
Subject:

(1) Search for Po Kee Wong Angles:
http://www.google.com/search?q=+site:mathforum.org+Po+Kee+Wong+Angles&hl=en&lr=&ie=UTF-8&filter=0

You and all members of CSTB are cordially invited to review and evaluate the TRUTH and the IMPACTS from the following websites about the confrontation between Judges and Mathematicians worldwide on the key issue of whether the Judges should be given the Judicial Power by any governmental institutions in the world to rule against the ABSOLUTE TRUTH of MATHEMATICS:

(2) The following information was the first one of the several public documents to have been submitted back to CAFC for the public hearing of the CAFC case number 2006-1324:

http://mathforum.org/kb/thread.jspa?forumID=206&threadID=478694&messageID=1467123

(3) The following information was the second one of the several public documents to have been submitted back to the CAFC for the public hearing of the CAFC case number 2006-1324 IN RE PO KEE WONG:

http://mathforum.org/kb/message.jspa?messageID=1094424&tstart=0

(4) The following was the third of the several public documents to have been submitted back to the CAFC for the public hearing of the CAFC case number 2006-1324 IN RE PO KEE WONG:

http://mathforum.org/kb/thread.jspa?forumID=130&threadID=357907&messageID=1094423

(5) The following was the fourth (4th.) of several public documents that had been submitted back to the CAFC for the Judges to re-consider their current disposition of the CAFC case number 2006-1324 IN RE PO KEE WONG:

http://mathforum.org/kb/message.jspa?messageID=3566959&tstart=0

(6) Comparison of Trajectory Solid with Geometric Solid in Sacttering Theory:

http://adsabs.harvard.edu/abs/2002iaf..confE.559W

(7) U.S. Patent 5084232:
http://patft.uspto.gov/netacgi/nph-Parser?Sect1=PTO1&Sect2=HITOFF&d=PALL&p=1&u=%2Fnetahtml%2FPTO%2Fsrchnum.h

tm&r=1&f=G&l=50&s1=5084232.PN.&OS=PN/5084232&RS=PN/5084232

(8) U.S. Patent 5848377
http://patft.uspto.gov/netacgi/nph-Parser?Sect1=PTO1&Sect2=HITOFF&d=PALL&p=1&u=%2Fnetahtml%2FPTO%2Fsrchnum.htm&r=1&f=G&l=50&s1=5848377.PN.&OS=PN/5848377&RS=PN/5848377

(9) U.S. Patent 6430516
http://patft.uspto.gov/netacgi/nph-Parser?Sect1=PTO1&Sect2=HITOFF&d=PALL&p=1&u=%2Fnetahtml%2FPTO%2Fsrchnum.htm&r=1&f=G&l=50&s1=6430516.PN.&OS=PN/6430516&RS=PN/6430516

(10) Mathforum Search
http://mathforum.org/kb/profile.jspa?userID=47317

(11) ICONE 13 paper 50509
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