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Substitution Measurements on 28-Fuel-Rod Critical Clusters in D₂O and Their Analysis by the Second-Order Perturbation Method

Kiminori Shiba

Power Reactor and Nuclear Fuel Development Corporation
Oarai Engineering Center, Oarai-machi, Ibaraki-ken, Japan

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Material bucklings have been determined as functions of ²³⁵U enrichment in UO₂ (0.7, 1.2, and 1.5 wt% ²³⁵U), PuO₂ enrichment in PuO₂-UO₂ (0.54 and 0.87 wt% PuO₂), fissile content of plutonium (91 and 75% Pu-fissile), lattice pitch ($V_{\text{mod}}/V_{\text{fuel}}$: 7.4 and 9.9), and coolant void fraction. The reference loading of 1.2 wt% ²³⁵U-enriched UO₂ clusters was progressively replaced by the test clusters.

Buckling differences resulting from the substitutions were analyzed by the new second-order (iterative) perturbation method, on the assumption that neutron diffusion is isotropic and that no difference in diffusion coefficients exists between the two lattices. This analysis takes into account the effect of distortion in radial neutron flux distribution in the substituted core without any iterative correction procedure that is usually adopted in the first-order perturbation method. Also, it is not necessary in the case of the present analysis to introduce any usual intermediate region for taking into account the effect of spectrum mismatch between the two lattices.

The material buckling differences between the test and reference lattices, which are in the range of -10.2 to 9.1 m⁻², were determined within 3% of uncertainty.

I. INTRODUCTION

Good neutron economy permits a reactor to make more versatile use of nuclear fuel. Since a heavy-water lattice usually has good neutron economy resulting from small neutron absorption in the heavy-water moderator, it is expected that a heavy-water reactor (HWR) can be operated on either slightly enriched uranium oxide fuel or plutonium-uranium mixed oxide fuel without extensive alterations to lattice specifications.¹ Now, much interest centers on how far material buckling of a heavy-water lattice depends on fuel composition. The aim of the present experiment is to determine this dependence of heavy-water lattices with a 28-fuel-rod cluster, which is used in the FUGEN boiling light-water-cooled, pressure-tube-type HWR (Ref. 2).

Large amounts of fissile material are needed to make a full-scale critical experiment with one-region core on all of the various clusters, which differ from each other in fuel composition. This gives great incentive to apply the progressive substitution method that has been widely utilized³⁻¹⁵ and has the advantage of requiring only small amounts of fissile material.

For present purposes, consequently, the progressive substitution measurements have been

¹W. E. GRAVES, "Analysis of the Substitution Technique for the Determination of D₂O Lattice Bucklings," DP-832, Savannah River Laboratory (1963).

²H. R. LUTZ, T. AUERBACH, W. HEER, and R. W. MEIER, "Exponential and Critical Experiments," 2, 85, International Atomic Energy Agency, Vienna (1964).

³D. C. KING and M. GIBSON, "An Analysis of Some Two-Zone Exponential Experiments on Graphite Moderated Lattices," AEEW-R428, U.K. Atomic Energy Authority, Winfrith (1965).

⁴G. BLAESSER, "Exponential and Critical Experiments," 3, 443, International Atomic Energy Agency, Vienna (1964).

⁵D. S. CRAIG, R. E. GREEN, and A. OKAZAKI, *Trans. Am. Nucl. Soc.*, 9, 126 (1966).

¹H. KATO and Y. MIYAWAKI, *Trans. Am. Nucl. Soc.*, 23, 513 (1976).

²S. SHIMA and S. SAWAI, "The FUGEN Project," *Can. Nucl. Assoc. Ann. Conf.*, p. 204 (1973).

performed on those lattices fueled with natural uranium, 1.5 wt% enriched uranium, or plutonium-uranium mixed oxide, by using the lattices fueled with 1.2 wt% enriched uranium oxide as reference. The PuO₂ enrichments in the mixed oxides are 0.54 and 0.87 wt%. The fissile contents of plutonium are ~91 and ~75%.

Evaluation of material bucklings from substitution measurements usually requires a complicated procedure because the measurements merely give those changes in critical condition that result from partial substitutions of the one-region reference loading with test lattices to be investigated. Experimenters have developed several methods of evaluation, which are classified into two main groups.

The fundamental idea of the first group³⁻⁸ is to normalize some calculational parameters of both reference and test lattices on the basis of experiment by the one-region loading of the reference lattice and then estimate desired quantities of the test lattices from experiments by two-region loading of the reference and test lattices. Evaluations based on this idea are made using few-group diffusion theory combined with homogenization of regions^{3-5,8} or heterogeneous (source-sink) methods.⁶⁻⁸ These methods, however, rely considerably on calculation, and the evaluation of material bucklings is somewhat sensitive to the choice of the calculational parameters used,⁴ leaving some systematic uncertainties in the analyzed results.

Persson et al.^{9,14} proposed the second method, which requires no knowledge of lattice parameters that cannot be determined experimentally. Since this method is based on the first-order perturbation theory, some modifications are adopted to use it in the case of large perturbations.¹⁵ First, the spectrum transient effect due to spectral mismatch between the test and reference lattices is resolved by introducing an intermediate region.

³R. E. GREEN, R. E. KAY, A. OKAZAKI, G. CASINI, H. J. METZDORF, and M. PAILLON, *Trans. Am. Nucl. Soc.*, **9**, 123 (1966).

⁴R. PERSSON, E. BLOMSJÖ, E. ANDERSEN, O. ASPÉLUND, and J. DÖDERLEIN, *Proc. 2nd Int. Conf. Peaceful Uses At. Energy*, Geneva, 1958, **12**, 364, United Nations, New York (1958).

⁵R. PERSSON, C.-E. WIKDAHL, and Z. ZADWORSKI, *Nukleonik*, **4**, 191 (1962).

⁶N. P. BAUMANN, W. F. GRAVES, E. J. HENNELLY, and G. F. O'NEILL, *Nukleonik*, **4**, 200 (1962).

⁷R. PERSSON, *Exponential and Critical Experiments*, **3**, 289, International Atomic Energy Agency, Vienna (1964).

⁸W. B. ROGERS, V. D. VANDELDE, and N. P. BAUMANN, *Trans. Am. Nucl. Soc.*, **8**, 449 (1965).

⁹R. PERSSON, "One-Group Perturbation Theory Applied to Substitution Measurements with Void," AE-248, AB Atomenergi, Sweden (1966).

¹⁰R. PERSSON, *Nukleonik*, **10**, 163 (1967).

Second, the effect of distorted radial flux distribution is treated with the help of iterative correction. However, it is difficult to determine *a priori* the dimensions of this artificially introduced region, nor is the latter iterative correction easy. Furthermore, because of close and essential relation between the spectrum transient and distorted flux, it is doubtful if the resultant two effects can be dealt with separately in the case of a large perturbation resulting from substitutions.

Since the second-order perturbation method dispenses with the assumption that radial flux distribution or neutron spectrum is unperturbed, it is expected that the analysis based on this method requires neither the iterative correction nor the introduction of the intermediate region. Thus, an attempt is made to apply the second-order perturbation method to the analysis of changes in critical conditions resulting from substitutions. For this analysis, we try to make the formation of a simple linear equation that connects buckling differences due to substitutions with generalized statistical weights of substituted regions. The present measurements have been analyzed by means of the simple equation.

II. EXPERIMENTAL PROCEDURE

The DCA critical assembly used in this experiment has no reflector: It is heavy-water moderated and is contained in an aluminum tank (3.01-m i.d., 1.0-cm wall thickness).¹⁶ The moderator level in the reactor tank is measured to an accuracy of 0.01 cm with a level meter in a communicating tube. As shown in Fig. 1, a 28-fuel-rod cluster is separated from the D₂O moderator by an aluminum pressure tube, an air gap, and an aluminum calandria tube. Fuel rods lie with their centers on three concentric circles. Fuel clusters are arranged in a square lattice at a 22.5- or 25.0-cm pitch. The lattice pitch can be altered by changing both upper and lower grid plates.

Pressure tubes are filled with H₂O, air, or proper mixtures of H₂O and D₂O. To simulate coolant void, these mixtures are fixed so that their slowing down powers are equal to those of voided coolant having 30 or 70% void fractions. A small amount of boric acid is added to the mixtures, as shown in Table I, for adjusting absorption in the thermal energy region. Several cross sections of simulated coolants are compared with those of actually voided coolants in Table II. The

¹⁶K. IJIMA, N. AIIHARA, T. HIRAYAMA, T. YONEDA, N. FUKUMURA, and Y. MIYAWAKI, "Handbook of Deuterium Critical Assembly," SN 941 74-22, PNC, University of Tokyo, Japan (1974) (in Japanese).

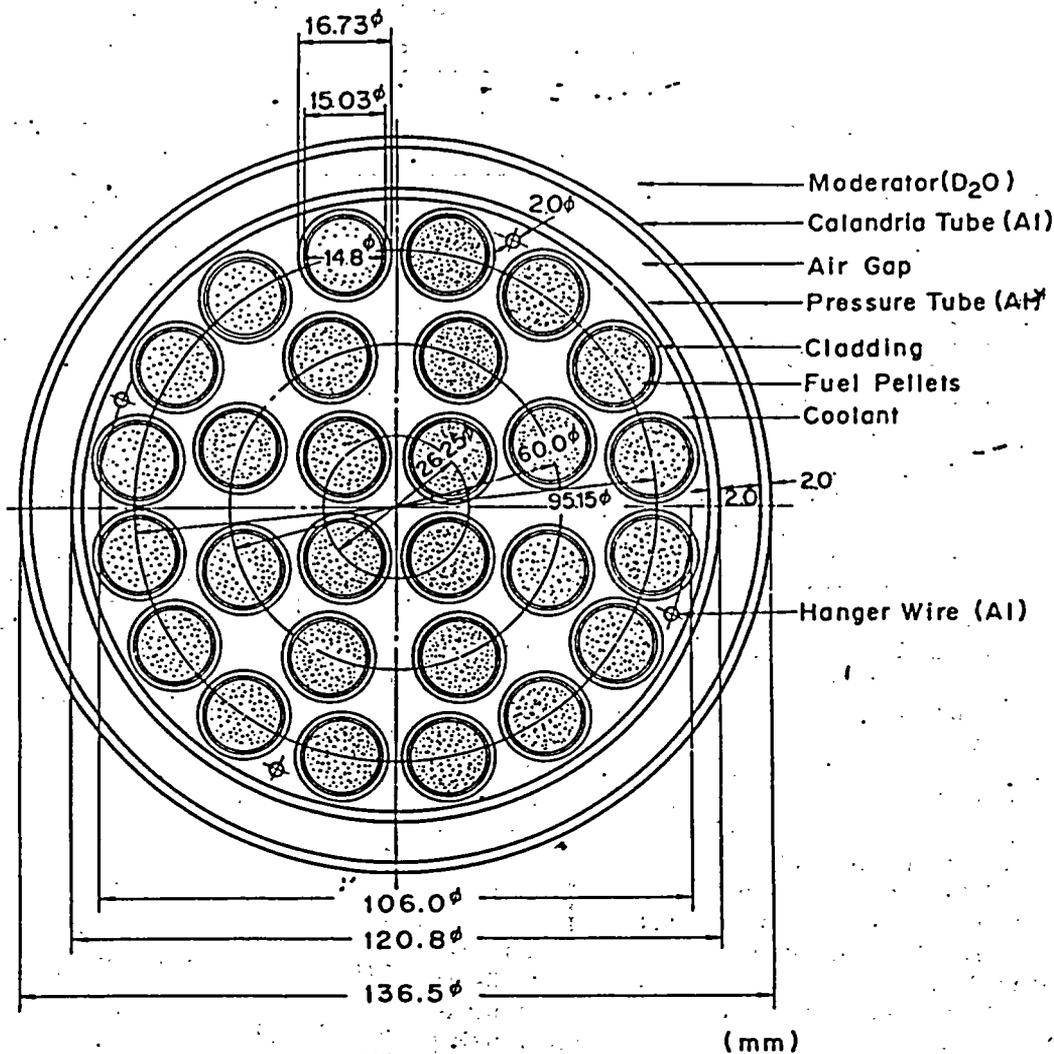


Fig. 1. Cross-sectional view of the fuel assembly.

comparison shows that neutron cross sections, except transport cross sections, are in good agreement between mixtures and actually voided coolants.

Lattices fueled with 1.2 wt% enriched uranium oxide were used for reference. To obtain effective axial extrapolation distances and core radii, flux profile measurements using copper activation were taken in each one-region reference loading. When activating copper wires, liquid coolant levels in pressure tubes were set as close as to that of the D₂O moderator.

The axial flux distributions measured at the center of a fuel cluster and a corner of a unit cell were least-squares fitted to a cosine function to evaluate effective core heights. The least-squares fitting was undertaken by successively dropping data near the core boundaries until fixed values of the effective core height were obtained. The

TABLE I
Composition of Coolants Simulating Void

Simulated Void Fraction (%)	Content (wt%)				Density at 22°C (g/cm ³)
	H ₂ O	D ₂ O	H ₂ BO ₃	Air	
0	100.00	---	---	---	0.9978
30	63.17	36.82	0.00921	---	1.0348
70	18.07	81.91	0.02150	---	1.0840
100	---	---	---	100.00	0.0012

fixed values so determined showed no meaningful dependence on the transversal location in the unit cell. Effective axial extrapolation distances, including the reflector savings due to the upper and lower structures of the reactor, were determined from these fixed values: as 13.6 ± 1.0 and

TABLE II
Comparison of Neutron Cross Sections Between Simulated and Actually Voided-Coolants

Simulated Void Fraction (%)	Ratio ^a of Cross Sections				
	Thermal Region			Epithermal Region	
	Absorption	Scattering	Transport	Slowing Down Power	Transport
30	1.0000	1.0080	1.0203	0.9998	1.1634
70	0.9998	1.0042	1.1103	0.9994	1.8896

^a Ratio = $\frac{\text{cross section of simulated coolant}}{\text{cross section of actually voided coolant}}$

11.3 ± 0.8 cm for 22.5- and 25.0-cm pitch lattices, respectively. They showed no dependence on coolant void fraction beyond the experimental error. The radial flux distribution measured at the centers of the fuel clusters was least-squares fitted to the J_0 function to evaluate the effective core radius. The effective core radius obtained was also independent of coolant void fraction.

Before conversion of critical D₂O heights to axial bucklings, two corrections were made for the critical D₂O heights. The first correction accounts for the presence in the core of perforated aluminum guide tubes for safety rods. (The safety rods themselves were extracted completely at the time of measurement.)

Reactivity worth of these guide tubes was measured in the fully voided lattice of 22.5-cm pitch. An increase in critical D₂O height equivalent to this reactivity was also obtained using experimental reactivity coefficients of D₂O height, and then it was subtracted from the actual D₂O critical height. Possible change in the worth of the guide tubes with coolant void fraction or lattice pitch was ignored in this correction, because it was found that the worth differed only within its measurement error of ~20%. On the other hand, dependence of the worth of the guide tubes on their location was taken into account, using statistical weights at the inserted positions. The resultant magnitude of the correction for the presence of the guide tubes was 1% of the full critical D₂O height.

The second correction accounts for the zero drift of the servo-mechanism utilized in the level meter. This correction shifted the critical D₂O height by 0.5% or less, yielding an additional measurement error of 0.05% in the height.

The experimental error in the resultant material bucklings of the reference lattices was estimated to be ±1.0%, taking ±0.8% for systematic error and ±0.6% at most for statistical error (depending on the number of measurements). The

TABLE III
Material Bucklings of Reference Lattices

Lattice Pitch (cm)	Effective Core Radius (cm)	Simulated Void Fraction (%)	Buckling (m ⁻²)
22.5	$R_c^2 = 2.41$ 154.9 ± 1.9	0	10.62 ± 0.09
		30	10.45 ± 0.10
		70	9.80 ± 0.10
		100	8.73 ± 0.08
25.0	$R_c^2 = 2.57$ 149.4 ± 1.7	0	9.85 ± 0.10
		30	10.21 ± 0.10
		70	10.28 ± 0.10
		100	9.90 ± 0.09

obtained material bucklings of reference lattices are shown in Table III, together with the effective core radii. Bucklings were also normalized to an experimental condition of 22°C and of 99.45 D₂O mol%, using the code METHUSELAH II (Ref. 17). This normalization made the values of the actually measured bucklings change by 1% or less.

In substitution measurements, the central part of the one-region reference loading was progressively replaced by test clusters of natural, 1.5 wt% enriched uranium or plutonium-uranium mixed-oxide fuel, and the resultant critical D₂O heights were measured as a function of the number of substituted fuel clusters. The composition of the fuel clusters investigated is given in Table IV. The various test regions that were used comprised 1, 4, 5, 9, 13, 21, and 25 fuel clusters. Test clusters were arranged around the core axis with fourfold rotation symmetry, as seen in Fig. 2. In the case of the four clusters substitution, the central cluster was again replaced by one reference fuel cluster to keep the fourfold rotation symmetry.

¹⁷A. ALPIAR, "METHUSELAH I, A Universal Assessment Programme for Liquid Moderated Reactor Cells, Using IBM 7090 or STRETCH Computer," AEEW-R135, U.K. Atomic Energy Authority, Winfrith (1963).

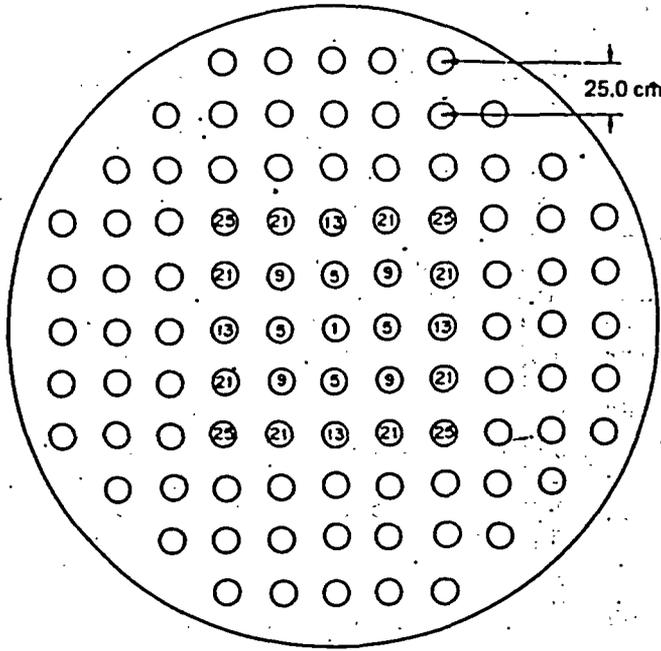


Fig. 2. Loading patterns of test fuel clusters whose positions are represented by numbered circles. Test fuel clusters are placed at circles numbered from 1 to N , in the test region composed of N clusters. (The test fuel clusters are placed at circles marked 5, in the test region composed of four clusters.)

Using the same method in the case of the reference lattices, the correction to the condition of no guide tubes was made for critical D_2O heights with test fuel clusters present. Consider that the magnitude of this correction might be affected by the change in critical D_2O height due to substitution. However, the correction actually obtained was small. It was at most 2% even for the largest substitution of 25 clusters of the lowest enrichment fuel. In this correction, possible but small changes in the worth of the guide tubes themselves, due to the change in their importance in the substituted core, was neglected. The zero drift of the level meter was corrected in a manner similar to the case of one-region reference loading.

Substitution measurements on lattices with liquid coolants sometimes had to be performed under the presence of difference in levels between coolant and moderator for the sake of the structural difficulty. In most cases, coolant level was lower than moderator level. In the lattices of 22.5-cm pitch with light water coolant, exceptionally, the level was higher than moderator level.

Change in critical condition due to the above difference was investigated by means of the first-order one-group perturbation method.¹⁴ The model adopted is a core of one-region reference loading suffering from axial perturbation caused

TABLE IV

Fuels Used in Critical Substitution Measurements on 28-Fuel-Rod Clusters

Abbreviation	Nominal Enrichment (wt%)	Grade of Plutonium	Density of Pellet (g/cm ³)	Material of Cladding	Content of Nuclide in Fuel Pellets (wt%)							
					²³⁵ U	²³⁸ U	²³⁹ Pu	²⁴⁰ Pu	²⁴¹ Pu	²⁴² Pu	Oxygen	
1.2 EU	1.2 (²³⁵ U)	---	10.36 ± 0.05	Al	1.057	86.793	---	---	---	---	---	12.150
1.5 EU	1.5 (²³⁵ U)	---	10.38 ± 0.06	Al	1.317	86.563	---	---	---	---	---	12.120
NU	0.72 (²³⁵ U)	---	10.36 ± 0.04	Al	0.625	87.255	---	---	---	---	---	12.120
0.54 (S) PuO ₂ UO ₂	0.54 (PuO ₂)	Standard	10.17 ± 0.07	Zircaloy 2	0.621	86.782	0.430	4.12 × 10 ⁻²	4.36 × 10 ⁻³	3.03 × 10 ⁻⁴	12.120	
0.87 (S) PuO ₂ UO ₂	0.87 (PuO ₂)	Standard	10.17 ± 0.07	Zircaloy 2	0.619	86.503	0.685	6.58 × 10 ⁻²	6.96 × 10 ⁻³	5.10 × 10 ⁻⁴	12.120	
0.87 (R) PuO ₂ UO ₂	0.87 (PuO ₂)	Reactor	10.25 ± 0.08	Zircaloy 2	0.619	86.493	0.495	1.66 × 10 ⁻¹	7.22 × 10 ⁻²	2.30 × 10 ⁻²	12.120	

by a thin layer on it. The layer contains either moderator or coolant. This leads to

$$\delta B_z^2 = \frac{(D_1 B_{z1}^2 - D_0 B_{z0}^2)W + (D_0 - D_1)B_{z0}^2 U}{D_0 + (D_1 - D_0)U} \quad (1)$$

where

δB_z^2 = change in critical axial buckling

B_{z0}^2 and B_{z1}^2 = respective critical axial bucklings of the one-region reference lattice and of the upper thin layer

D_0 and D_1 = respective diffusion coefficients of the reference lattice and the upper thin layer.

Values of W and U denote statistical weights and are calculated by

$$W = (2/H) \int_0^h \sin^2 B_z Z dZ \quad (2)$$

and

$$U = (2/H) \int_0^h \cos^2 B_z Z dZ \quad (3)$$

where

H = effective height of the perturbed core

B_z^2 = axial buckling corresponding to H

h = thickness of the upper thin layer.

When h is sufficiently smaller than H , the quantities of W and U can be approximated as $W \cong 0$ and $U \cong 2(h/H)$. Therefore, in this case, Eq. (1) is approximated to

$$\Delta H \cong (D_1/D_0 - 1)h \quad (4)$$

where ΔH is the change in the critical D₂O height due to the difference between coolant and moderator levels. This relation was experimentally determined for each one-region reference loading and applied to the critical D₂O heights in the substituted cores for the correction of the difference in levels. Here, the change in the value of D_1/D_0 by substitution was ignored, since the code METHUSELAH II (Ref. 17) predicted that the diffusion coefficient of thermal neutrons differs by 1%, at most, between the test and reference lattices used in the present experiment.

When a large difference is present between the two levels, the right side of Eq. (4) should be expanded to the second-order term of h to estimate the ΔH correctly. By inclusion of this second-order term, the correction for the difference between the levels can be made with an accuracy of 5%, even if it is ignored that the value of ΔH is dependent, to some extent, on the critical D₂O height. Since the value of this correction,

ΔH , is not so large (at most 3% of total D₂O height), additional experimental error due to this correction is <0.15% of the resultant D₂O critical height.

The resultant buckling difference, δB_z^2 , due to each substitution was obtained by converting the corrected D₂O height into axial buckling with the use of the constant effective axial extrapolation distance obtained for each lattice pitch. As mentioned above, dependence of this extrapolation distance on coolant void fraction was found to be negligibly small. Further dependence on fuel composition was also examined by making axial flux distribution measurements in the largely substituted core—37 clusters substitution-by 0.54 wt% enriched PuO₂-UO₂ fuel, 22.5-cm pitch, and 100% void fraction. This resulted in no meaningful difference in the axial extrapolation distances between the substituted region and the reference one.

The critical D₂O heights changed appreciably in the course of the experiment due to the degradation of heavy-water concentration in the coolant mixture because it was in contact with the air. The effect of this degradation to be corrected was estimated at ~0.1% of the critical D₂O height when it elapsed more than one week at maximum between the measurement on the one-region reference loading and on the last substitution.

Since the temperature of the moderator was not controlled, the temperature change within ±1°C was observed during the series of substitution measurements. Uncertainty in buckling differences due to this temperature change was estimated at ~0.01 m⁻² or less.

The resultant experimental error in the present buckling difference, δB_z^2 , was estimated as from 4 (for the largest substitution number) to 8% (for the smallest number), which led to the experimental error of 3% in material buckling difference.

III. PRINCIPLE OF DATA ANALYSIS

If neutron flux in a bare and uniform cylindrical core is at steady state and its axial distribution is postulated as proportional to $\exp(iB_z Z)$, the few-group diffusion equation based on the homogeneous cell model gives

$$B_z^2 \Phi(r) = L \Phi(r) \quad (5)$$

where $\Phi(r)$ is the transversal neutron flux. The gg' element of the matrix operator, L , is defined as

$$L_{gg'} = [\Delta - (\Sigma_a^g + \Sigma_s^g)/D^g] \delta_{gg'} + (\Sigma_s^{g'-g} + \chi^{g'} \nu \Sigma_f^{g'})/D^{g'} \quad (6)$$

g, g' = 1, 2, ..., M

where

Δ = Laplacian operator defined in the transversal plane

D^2 = diffusion coefficient in group g

M = group number,

and where other symbols have their conventional meanings.

If the core is sufficiently large, the assumption of the separability on space and energy is valid except near boundaries. Therefore, solutions to Eq. (5) can be found that have the form

$$\Phi(\mathbf{r}) = \mathbf{a}\psi(\mathbf{r}) \equiv \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^M \end{pmatrix} \psi(\mathbf{r}) \quad (7)$$

Here, \mathbf{a} is a vector of the neutron spectrum, and $\psi(\mathbf{r})$, which is a function of position only, obeys

$$\Delta\psi(\mathbf{r}) + B_z^2\psi(\mathbf{r}) = 0 \quad (8)$$

With the condition that the neutron flux is finite everywhere and single-valued inside the core, the eigenfunctions of Eq. (8) are given by

$$J_m(B, r) \sin(m\phi)$$

or

$$J_m(B, r) \cos(m\phi),$$

where

r = length of \mathbf{r}

ϕ = azimuthal angle of \mathbf{r} , m zero or positive integer

J_m = Bessel function of the first kind of order m .

An application of the boundary condition that the flux goes to zero at the extended boundaries leads to an equation determining B_z :

$$J_m(B, R) = 0, \quad (9)$$

where R is the effective core radius. The eigenfunctions $\psi_k(\mathbf{r})$ are thus

$$\psi_k(\mathbf{r}) = N_{mn} J_m(\lambda_{mn} r/R) \begin{bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{bmatrix}, \quad k = 0, 1, 2, \dots, \quad (10)$$

where λ_{mn} indicates the n 'th zero of J_m . Here, to each k a particular couple (m, n) and a particular selection of either $\sin(m\phi)$ or $\cos(m\phi)$ are associated, according to the rule that the λ_{mn} corresponding to each k increases with k . The normalization factor, N_{mn} , is chosen so that

$$\int_{-\pi}^{\pi} d\phi \int_0^R r dr \psi^2(\mathbf{r}) = 1 \quad (11)$$

The eigenvalue, B_{zk}^2 , associated to $\psi_k(\mathbf{r})$ is

$$B_{zk}^2 = (\lambda_{mn}/R)^2, \quad k = 0, 1, 2, \dots \quad (12)$$

With the help of Eqs. (10) and (12), substitution of $\Phi(\mathbf{r})$ in Eq. (5) by Eq. (7) leads to an equation that determines B_z^2 and \mathbf{a} :

$$(B_z^2 + B_{zk}^2)\mathbf{a} = H\mathbf{a}, \quad (13)$$

where H is the same operator as L , with the exception that it excludes the Laplacian operator, Δ . Since H is the $(M \times M)$ matrix, the number of eigenvalues of H is M . When a particular eigenvalue of H is indicated by B_j^2 , Eq. (13) yields

$$B_{zjk}^2 = B_j^2 - B_{zk}^2, \quad j = 0, 1, \dots, M-1, \quad k = 0, 1, 2, \dots \quad (14)$$

The eigenfunctions, $\Phi_{jk}(\mathbf{r})$, of Eq. (5) associated to B_{zjk}^2 are

$$\Phi_{jk}(\mathbf{r}) = a_j \psi_k(\mathbf{r}), \quad j = 0, 1, \dots, M-1, \quad k = 0, 1, 2, \dots, \quad (15)$$

where a_j is the eigenvector associated to B_j^2 . The j suffix starts from zero for the expression of fundamental a_j as a_0 . When the adjoint spectrum, a_j^* , is determined from the equation adjoint to Eq. (13), the fluxes adjoint to $\Phi_{jk}(\mathbf{r})$ are given by

$$\Phi_{jk}^*(\mathbf{r}) = a_j^* \psi_k(\mathbf{r}), \quad j = 0, 1, \dots, M-1, \quad k = 0, 1, 2, \dots \quad (16)$$

When a_j and a_j^* are normalized so that

$$\sum_{\ell=1}^M a_j^{*\ell} a_j^{\ell} = 1,$$

the orthogonality of $\Phi_{jk}(\mathbf{r})$ and $\Phi_{j'k'}^*(\mathbf{r})$ is given by

$$(\Phi_{j'k'}^*, \Phi_{jk}) = \left(\sum_{\ell=1}^M a_j^{*\ell} a_j^{\ell} \right) \cdot \int \psi_{k'}(\mathbf{r}) \psi_k(\mathbf{r}) d\mathbf{r} = \delta_{j'j} \delta_{k'k} \quad (17)$$

Integration over \mathbf{r} in Eq. (17) is performed on the whole core.

The largest eigenvalue, B_{z00}^2 , among B_{zjk}^2 in Eq. (14) is given by the combination of the values of the largest B_j^2 and the smallest B_{zk}^2 , namely, B_0^2 and B_{r0}^2 , respectively. When B_{z00}^2 is positive, it is possible to control the core height so that it equals π/B_{z00} and then make the core critical. If this is the case, B_{z00}^2 is the axial critical buckling. Since $B_{r0}^2 = (\lambda_{00}/R)^2$, the $\psi_0(\mathbf{r})$ associated to B_{r0}^2 is given by $N_{00} J_0(\lambda_{00} r/R)$ and has the constant azimuthal part.

When a part of the one-region loading of the reference lattice is substituted with a test lattice to be investigated, Eq. (5) is transformed into

$$(\delta B_z^2 + \delta B_z^2) \Phi'(r) = [L + W(r) \cdot \delta L] \Phi'(r) \quad (18)$$

where

δB_z^2 = small change in B_z^2 due to the substitution

$\Phi'(r)$ = perturbed flux

δL = difference in L between the test and reference lattices.

The term $W(r) \cdot \delta L$ in Eq. (18) represents the perturbation due to the substitution. Here, $W(r)$ is defined as

$$W(r) = \begin{cases} 1 & \text{in the substituted region} \\ 0 & \text{in the nonsubstituted region} \end{cases} \quad (19)$$

The perturbation discussed here is restricted to the case where the following can be postulated:

1. The difference in diffusion coefficients between the test and reference lattices is negligible.
2. The perturbation is uniform along the core axis.

An application of the iterative perturbation technique¹⁸ to Eq. (18) can give the perturbed flux, $\Phi'(r)$, associated with the largest eigenvalue and δB_z^2 . Expansion of $\Phi'(r)$ by $\Phi_{jk}(r)$ yields

$$\Phi'(r) = \Phi_{00}(r) + \sum'_{j,k} \epsilon_{jk} \Phi_{jk}(r) \quad (\epsilon_{00} = 1) \quad (20)$$

The prime of symbol,

$$\sum'_{j,k}$$

denotes the elimination of $\Phi_{00}(r)$ from the summation. Inserting Eq. (20) into Eq. (18), operating $\Phi_{jk}^*(r)$ from the left side, and making the inner product gives the equation determining ϵ_{jk} :

$$(\delta B_z^2 + B_{z00}^2 - B_{zjk}^2) \epsilon_{jk} = (\Phi_{jk}^*, P \Phi') \quad (21)$$

The operator P stands for $W(r) \cdot \delta L$. The use of ϵ_{jk} determined from Eq. (21) yields

$$\Phi'(r) = \Phi_{00}(r) + \sum'_{j,k} \frac{(\Phi_{jk}^*, P \Phi')}{\delta B_z^2 + B_{z00}^2 - B_{zjk}^2} \Phi_{jk}(r) \quad (22)$$

When $j = k = 0$, Eq. (21) gives

$$\delta B_z^2 = (\Phi_{00}^*, P \Phi') \quad (23)$$

Iterative use of Eq. (22) can give the perturbed flux to any order of approximation.

Approximation of $\Phi'(r)$ by $\Phi_{00}(r)$ in the right side of Eq. (22) yields

$$\Phi'(r) = \Phi_{00}(r) + \sum_{k=1}^{\infty} \frac{(0k|P|00)}{\delta B_z^2 + B_{z00}^2 - B_{z0k}^2} \Phi_{0k}(r) + \sum_{k=0}^{\infty} \sum_{j=1}^{M-1} \frac{(jk|P|00)}{\delta B_z^2 + B_{z00}^2 - B_{zjk}^2} \Phi_{jk}(r) \quad (24)$$

In Eq. (24), $(jk|P|lm)$ stands for the inner product of $\Phi_{jk}^*(r)$ and $P \Phi_{lm}(r)$. With the use of Eqs. (15) and (16), $(jk|P|lm)$ can be rewritten as

$$(jk|P|lm) = (j|\delta L|l) W_{km} \quad (25)$$

where

$$(j|\delta L|l) = \sum_{z=1}^M \sum_{z'=1}^M a_j^{z'} \delta L_{z'z} a_l^z \quad (26)$$

and

$$W_{km} = \int \psi_k(r) W(r) \psi_m(r) dr \quad (27)$$

Since in the case of $k = m$, W_{km} corresponds to the statistical weight defined by Glasstone and Edlund,¹⁹ W_{km} is hereafter called "generalized statistical weight." Integration over r in Eq. (27) is performed on the whole core.

With the help of Eqs. (15), (16), and (25), the perturbed flux, $\Phi'(r)$, given by Eq. (24) can be rearranged as

$$\Phi'(r) = a_0 \psi_0(r) + (0|\delta L|0) a_0 \sum_{k=1}^{\infty} \frac{W_{0k}}{\delta B_z^2 + B_{z00}^2 - B_{z0k}^2} \psi_k(r) + \sum_{k=0}^{\infty} \left[\sum_{j=1}^{M-1} \frac{a_j (j|\delta L|0)}{\delta B_z^2 + B_{z00}^2 - B_{zjk}^2} \right] W_{0k} \psi_k(r) \quad (28)$$

The second term on the right side of Eq. (28) describes distortion in radial flux shape caused by insertion of the test lattice by the sum of the higher mode of $\psi_k(r)$ with its amplitude given by $W_{0k}/(\delta B_z^2 + B_{z00}^2 - B_{z0k}^2)$. In this term, it is conditioned that the fundamental mode of energy spectrum, a_0 , does not change by the insertion. Further (fine) distortion ascribable to the change in energy spectrum is expressed, in the third term of Eq. (28), by the combination of each $\psi_k(r)$ with the higher mode of a_j .

If the substitution is made up to the full core, the distortion expressed by the higher mode of $\psi_k(r)$ should vanish, and the spectrum inherent to the new lattice is formed. This situation is also expressed by Eq. (28) because in the fully substituted core, W_{0k} is unity or zero accordingly, as $k = 0$ or not because of the orthogonality of $\psi_k(r)$. Thus, in the case of the fully substituted core, the equation is reduced to

¹⁸P. M. MORSE and H. FESHBACH, *Methods of Theoretical Physics*, Part 2, p. 1001, McGraw-Hill Book Company, New York (1953).

¹⁹S. GLASSTONE and M. C. EDLUND, *The Elements of Nuclear Reactor Theory*, D. Van Nostrand Company, New York (1952).

$$\Phi'(\tau) = \left[a_0 + \sum_{j=1}^{M-1} \frac{a_j(j|\delta L|0)}{\delta B_z^2 + B_{z00}^2 - B_{zj0}^2} \right] \psi_0(\tau) \quad (29)$$

Substitution of $\Phi'(\tau)$ in Eq. (23) with Eq. (28) gives the expression for δB_z^2 approximated to the second order. This leads to

$$\begin{aligned} \delta B_z^2 = & (0|\delta L|0)W_{00} + (0|\delta L|0)^2 \sum_{k=1}^{\infty} \frac{W_{0k}^2}{\delta B_z^2 + B_{z00}^2 - B_{z0k}^2} \\ & + \sum_{j=1}^{M-1} \sum_{k=0}^{\infty} W_{0k}^2 \frac{(0|\delta L|j)(j|\delta L|0)}{\delta B_z^2 + B_{z00}^2 - B_{zjk}^2} \end{aligned} \quad (30)$$

As in Eq. (28), the effect of substitution on the axial critical buckling is also expressed in Eq. (30) in terms of the flux shape distortion and the spectrum change.

By using Eq. (14), the double sum in Eq. (30) can be rewritten as

$$\begin{aligned} & \sum_{j=1}^{M-1} \sum_{k=0}^{\infty} W_{0k}^2 \frac{(0|\delta L|j)(j|\delta L|0)}{\delta B_z^2 + B_{z00}^2 - B_{zjk}^2} \\ & = \sum_{j=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|0)}{B_0^2 - B_j^2} \\ & \times \left[\sum_{k=0}^{\infty} W_{0k}^2 - \sum_{k=0}^{\infty} \frac{(\delta B_z^2 + B_{rk}^2 - B_{r0}^2)W_{0k}^2}{\delta B_z^2 + B_{rk}^2 - B_{r0}^2 + B_0^2 - B_j^2} \right] \end{aligned} \quad (31)$$

To further rewrite the right side of Eq. (31), the following relation is utilized. Since the eigenfunctions $\psi_k(\tau)$ form a complete orthogonal set, it is possible to expand $W(\tau)\psi_0(\tau)$ in a series of $\psi_k(\tau)$. Thus,

$$W(\tau)\psi_0(\tau) = \sum_{k=0}^{\infty} W_{0k}\psi_k(\tau) \quad (32)$$

Both sides of Eq. (32) are multiplied by themselves and integrated over τ on the whole core; then it is obtained that

$$W_{00} = \sum_{k=0}^{\infty} W_{0k}^2, \quad (33)$$

because the cross terms vanish through integration due to the orthogonality of $\psi_k(\tau)$, and, furthermore, $W^2(\tau)$ equals $W(\tau)$.

Then, Eq. (31) can be written as

$$\begin{aligned} & \sum_{j=1}^{M-1} \sum_{k=0}^{\infty} W_{0k}^2 \frac{(0|\delta L|j)(j|\delta L|0)}{\delta B_z^2 + B_{z00}^2 - B_{zjk}^2} \\ & = W_{00} \sum_{j=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|0)}{B_0^2 - B_j^2} \\ & \times \left[1 - \sum_{k=0}^{\infty} \left(\frac{W_{0k}^2}{W_{00}} \right) \cdot \frac{\left(\frac{\delta B_z^2 + B_{rk}^2 - B_{r0}^2}{B_0^2 - B_j^2} \right)}{1 + \left(\frac{\delta B_z^2 + B_{rk}^2 - B_{r0}^2}{B_0^2 - B_j^2} \right)} \right] \end{aligned} \quad (34)$$

When k is a small number, the value of W_{0k}^2 is not as small as that of W_{00} . However, the value of

$$\left(\frac{\delta B_z^2 + B_{rk}^2 - B_{r0}^2}{B_0^2 - B_j^2} \right)$$

is very small for two reasons:

1. The B_j^2 usually has a large negative value for j other than zero.
2. The B_{rk}^2 remains in small value with the small number of k provided that the core radius is large enough.

As a result, the second term in the brackets can be ignored. As k becomes a large number,

$$\left(\frac{\delta B_z^2 + B_{rk}^2 - B_{r0}^2}{B_0^2 - B_j^2} \right)$$

cannot be neglected due to the considerably large value of B_{rk}^2 . The other factor, W_{0k}^2 , however, is small enough close to zero in the case of the larger k number. Therefore, the second term in the brackets of Eq. (34) can be omitted without any meaningful effect on the accuracy of δB_z^2 in Eq. (30).

Finally, Eq. (30) can be rewritten as

$$\delta B_z^2 \cong \alpha W_{00} + \beta \sum_{k=1}^{\infty} \frac{W_{0k}^2}{\delta B_z^2 + B_{rk}^2 - B_{r0}^2}, \quad (35)$$

with

$$\alpha = (0|\delta L|0) + \sum_{j=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|0)}{B_0^2 - B_j^2} \quad (36)$$

and

$$\beta = (0|\delta L|0)^2 \quad (37)$$

Equation (35) can be reduced to a simple formula with a convenient form for data analysis, as

$$Y = \alpha + \beta X, \quad (38)$$

with

$$X = W_{00}^{-1} \sum_{k=1}^{\infty} W_{0k}^2 / (\delta B_z^2 + B_{rk}^2 - B_{r0}^2) \quad (39)$$

and

$$Y = \delta B_z^2 / W_{00} \quad (40)$$

The X vanishes in a fully substituted core. The α gives the difference in material buckling of the two lattices. As shown by the definition of β given in Eq. (37), the linear equation given by Eq. (38) always has a positive or zero slope.

IV. DATA ANALYSIS AND RESULTS

In Sec. III, both reference and test lattices were regarded in the formulation as consisting

of homogeneous cells. However, when the test clusters are arranged around the core axis with the fourfold rotation symmetry in the substituted core, $W(r)$, defined by Eq. (19), also has the property of the fourfold rotation symmetry. Since $\psi_0(r)$ has the constant azimuthal part, W_{0k} vanishes unless the azimuthal part of $\psi_k(r)$ is equal to $\cos(4p\phi)$ ($p = 0, 1, 2, \dots$). Thus, $\psi_k(r)$, which are subject to this condition, are selected from the set of solutions given by Eq. (10). This selection yields

$$\psi_k(r) = \epsilon_p [\sqrt{\pi} J_{4p+1}(\lambda_{4p,q})]^{-1} J_{4p}(\lambda_{4p,q} r/R) \cos(4p\phi), \quad k = 0, 1, 2, \dots, \quad (41)$$

where p and q are zero or positive integers and where a particular pair of (p, q) is associated to each k . Here, (p, q) is rearranged with increasing k so that the value of $\lambda_{4p,q}$ increases with k . The factor $\epsilon_p [\sqrt{\pi} J_{4p+1}(\lambda_{4p,q})]^{-1}$ is obtained from the normalization condition given by Eq. (11), where

ϵ_p is unity or $\sqrt{2}$ depending on whether or not $p = 0$. Equation (41) yields

$$W_{0k} = \epsilon_p [\pi R^2 J_1(\lambda_{00}) J_{4p+1}(\lambda_{4p,q})]^{-1} \times \int_{-\pi}^{\pi} d\phi \int_0^R r dr J_0(\lambda_{00} r/R) \times W(r) \cos(4p\phi) J_{4p}(\lambda_{4p,q} r/R) \quad (42)$$

and

$$B_{rk}^2 = (\lambda_{4p,q}/R)^2 \quad (43)$$

To obtain W_{0k} , Eq. (42) was numerically integrated by the Simpson method, dividing a unit cell with 100 meshes.

The infinite sum over k in Eq. (39) was approximated by the finite sum; then, a value of X was calculated from W_{0k} and B_{rk}^2 given by Eqs. (42) and (43). With the use of X and Y defined by Eqs. (39) and (40), plots of the typical experimental results of buckling differences are shown in Figs. 3 and 4 together with the best-fit lines to Eq. (38). As predicted in Sec. III, it was realized that Y is in linear relation with X and that the slope is positive. The values of α and β were obtained by the least-squares method, using a weight either dependent or independent of each experimental error. However, no meaningful difference was found in the values of α and β .

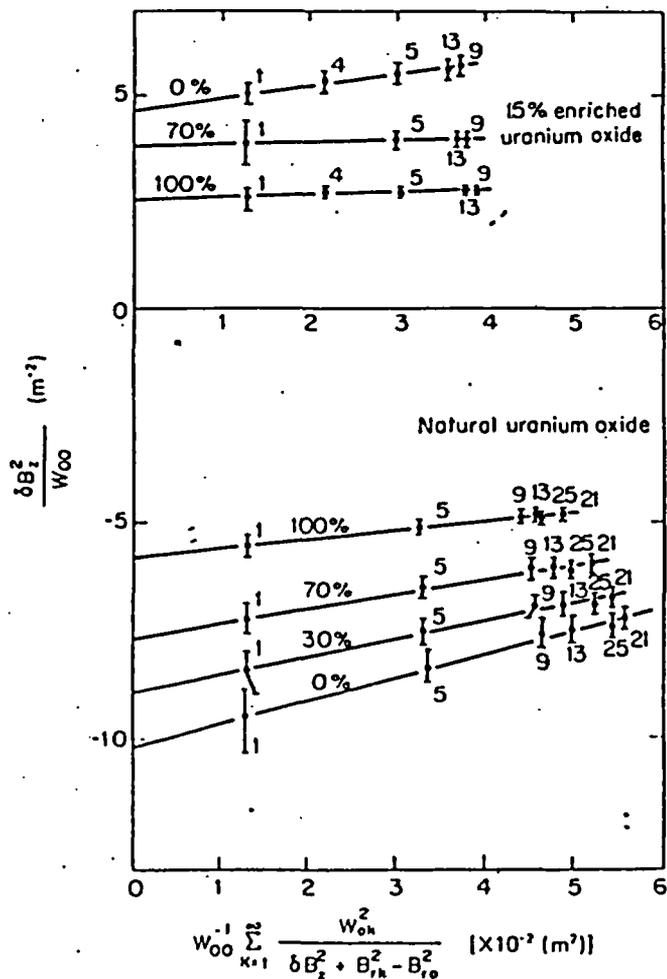


Fig. 3. Results of critical substitution measurements. The figures next to the points represent the number of fuel clusters in the test region. Lattice pitch = 22.5 cm.

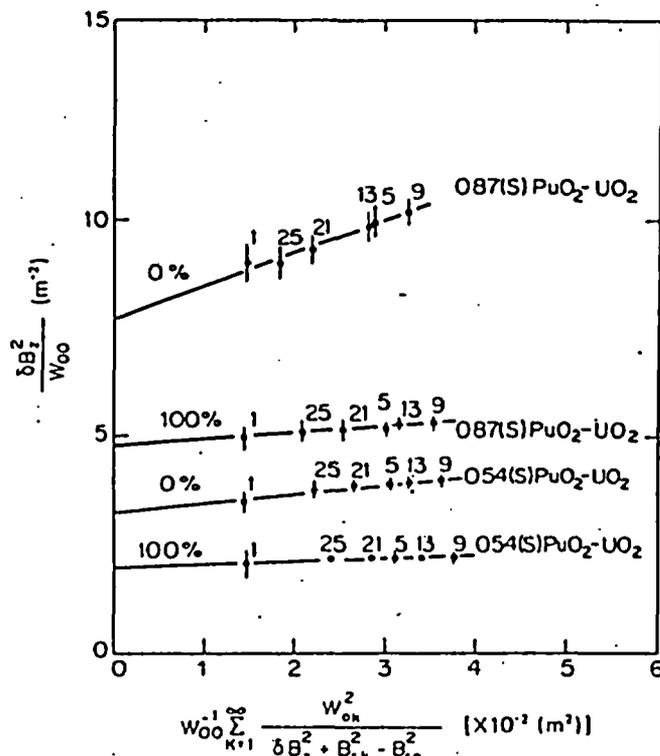


Fig. 4. Results of critical substitution measurements. The figures next to the points represent the number of fuel clusters in the test region. Lattice pitch = 25.0 cm.

To find out how the evaluation of α and β depends on the number of terms used in the calculation of X , α and β were evaluated as functions of the number after which the summation was broken off over k . As seen in Fig. 5, sufficient convergence of α and β is obtained by terms >30 . Since the summation was made up to the 64th term in the present analysis, the error resulted in X is $<0.02\%$.

To obtain material bucklings of test lattices, those of reference lattices are added to the resultant values of α , after being corrected to the measurement condition of the test lattice; the actual correction was at most 1%. The results are given in Tables V and VI, together with each experimental condition of temperature and of D_2O moderator purity. The material bucklings of the test lattices were also normalized to the condition of $22^\circ C$ and 99.45 D_2O mol% by means of temperature and purity coefficients calculated by the code METHUSELAH II (Ref. 17). The large difference between measured and normalized values occurs in the case of the natural uranium lattice of 22.5-cm pitch and 0% void fraction. The difference in this case is ascribed to the fairly

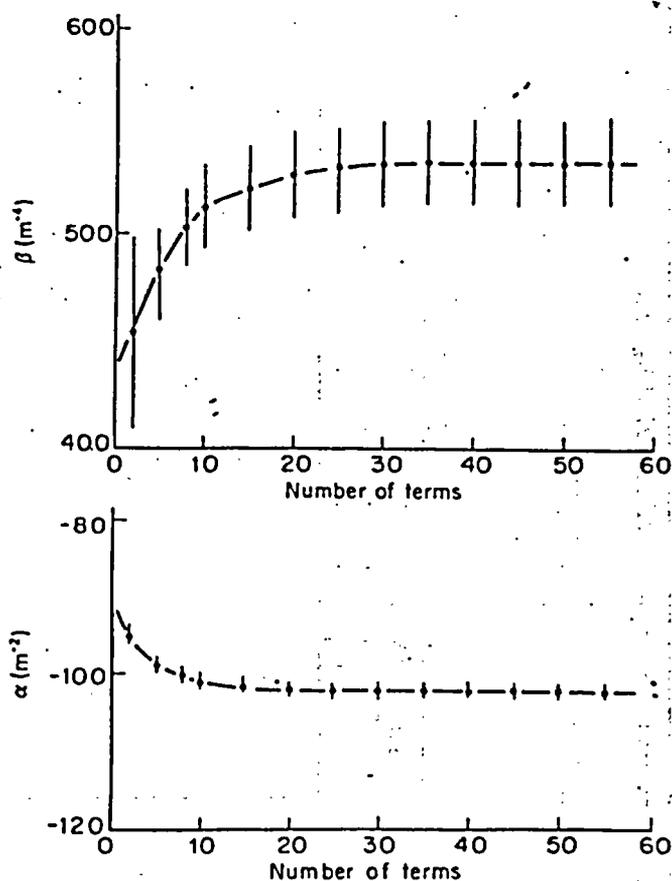


Fig. 5. Typical variation of α and β with number of terms. Test fuel = natural uranium, lattice pitch = 22.5 cm, void fraction = 0%.

large value of the purity coefficient calculated at 0.7 m^{-2}/D_2O mol%.

V. DISCUSSION AND CONCLUSION

The linear function with positive slope gave good descriptions of the behavior of the data. In a few cases where the slope is close to zero, however, the evaluation gives a negative value of β , as shown in Table V. It seems that these negative slopes originate from dispersion of measured data, because they are as small as the uncertainty in β .

The linear relation obtained by the present second-order perturbation method should also be examined in constancy of its intercept and linearity to the magnitude of perturbation. Thus, α and β were evaluated as a function of the maximum number of substituted test clusters. Table VII shows that α is practically insensitive to the choice of the maximum number but that β is somewhat sensitive to it. The sensitivity of β depends on the magnitude of α , the difference in material bucklings between two lattices; for smaller α , the change in β is small enough remaining in the fitting error. For a larger α value, however, β changes to some extent, accompanying the larger fitting error that increases with the maximum number.

This suggests that the linearity of the present method should be scrutinized in connection with the perturbation effect of the third order or more. The third-order term, F (Ref. 18), which is to be added to the right side of Eq. (30), is given by

$$F = \sum'_{j,k} \sum'_{l,m} \frac{(0|\delta L|j)(j|\delta L|l)(l|\delta L|0)W_{0k}W_{0m}W_{km}}{(\delta B_z^2 + B_{z00}^2 - B_{zjk}^2)(\delta B_z^2 + B_{z00}^2 - B_{zlm}^2)} \quad (44)$$

Here, the prime in symbols

$$\sum'_{j,k} \text{ and } \sum'_{l,m}$$

denotes elimination of cases where $j = k = 0$ or $l = m = 0$. Equation (44) can be approximated as (see the Appendix)

$$\begin{aligned} F \cong & \left[\sum_{j=1}^{M-1} \sum_{l=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|l)(l|\delta L|0)}{(B_0^2 - B_j^2)(B_0^2 - B_l^2)} \right] W_{00} \\ & + 2 \left[(0|\delta L|0) \sum_{j=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|0)}{B_0^2 - B_j^2} \right] \\ & \times \sum_{k=1}^{\infty} \frac{W_{0k}^2}{\delta B_z^2 + B_{rk}^2 - B_{r0}^2} + (0|\delta L|0)^2 \\ & \times \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{W_{0k}W_{0m}W_{km}}{(\delta B_z^2 + B_{rk}^2 - B_{r0}^2)(\delta B_z^2 + B_{rm}^2 - B_{r0}^2)} \quad (45) \end{aligned}$$

$\beta_T^2 = 2.41$

TABLE V

Evaluated Parameters and Material Bucklings of Test Lattices of 22.5-cm Pitch

β_T^2

Test Fuel	Simulated Void Fraction (%)	Purity of Heavy Water (mol%)	Temperature (°C)	α (m ⁻²)	β (m ⁻⁴)	Material Buckling of Test Lattice β_T^2 (m ⁻²)	Normalized Material Buckling of Test Lattice ^a (m ⁻²)
NU	0	99.77	22.5 ± 0.4	-10.21 ± 0.09	53.29 ± 2.09	0.47 ± 0.13	-1.94 0.25
	30	99.45	25.4 ± 0.6	-8.94 ± 0.08	40.14 ± 1.76	1.48 ± 0.13	-2.25 1.50
	70	99.45	21.1 ± 0.3	-7.66 ± 0.07	32.40 ± 1.53	2.13 ± 0.14	-2.28 2.12
	100	99.77	20.5 ± 0.3	-5.82 ± 0.04	20.35 ± 0.93	2.82 ± 0.09	0.41 2.79
1.5 EU	0	99.77	23.4 ± 0.3	4.70 ± 0.04	26.16 ± 1.31	15.37 ± 0.10	12.85 15.37
	30	99.45	24.5 ± 1.2	4.81 ± 0.10	6.73 ± 3.43	15.24 ± 0.14	12.85 15.26
	70	99.45	21.7 ± 0.4	3.81 ± 0.01	4.45 ± 0.36	13.61 ± 0.10	12.25 13.61
	100	99.77	20.7 ± 0.4	2.46 ± 0.04	9.35 ± 1.45	11.10 ± 0.09	8.59 11.24
0.54 (S) PuO ₂ -UO ₂	0	99.45	22.6 ± 1.3	4.27 ± 0.12	12.55 ± 3.75	14.89 ± 0.15	13.48 14.89
	30	99.45	25.7 ± 1.0	3.61 ± 0.06	6.22 ± 1.92	14.03 ± 0.12	11.25 14.05
	70	99.45	25.1 ± 0.7	2.90 ± 0.04	-2.24 ± 1.19	12.67 ± 0.11	11.25 12.69
	100	99.45	19.1 ± 0.8	1.76 ± 0.03	9.02 ± 0.96	10.52 ± 0.09	7.11 10.49
0.87 (S) PuO ₂ -UO ₂	0	99.45	18.8 ± 0.7	9.14 ± 0.07	89.57 ± 2.64	19.78 ± 0.11	13.37 19.76
	100	99.45	16.6 ± 0.9	4.77 ± 0.05	8.36 ± 1.57	13.56 ± 0.09	11.15 13.51
0.87 (R) PuO ₂ -UO ₂	0	99.45	23.0 ± 0.5	4.78 ± 0.08	17.44 ± 2.72	15.39 ± 0.12	12.98 15.39
	100	99.45	19.7 ± 0.5	2.10 ± 0.03	0.00 ± 1.00	10.85 ± 0.09	8.24 10.83

^a Values normalized to the condition of 22°C and 99.45 D₂O mol%.

FUEL ROD CLUSTERS IN D₂O

TABLE VI

Evaluated Parameters and Material Bucklings of Test Lattices of 25.0-cm Pitch

$\beta_T^2 = 2.57$

Test Fuel	Simulated Void Fraction (%)	Purity of Heavy Water (mol%)	Temperature (°C)	α (m ⁻²)	β (m ⁻⁴)	Material Buckling of Test Lattice β_T^2 (m ⁻²)	Normalized Material Buckling of Test Lattice ^a (m ⁻²)
NU	0	99.45	18.1 ± 0.6	-9.19 ± 0.08	54.37 ± 2.00	0.69 ± 0.13	-1.90 0.68
	30	99.45	20.9 ± 0.3	-7.86 ± 0.09	-33.91 ± 2.27	2.36 ± 0.13	-0.23 2.36
	70	99.45	20.1 ± 0.3	-6.38 ± 0.10	15.37 ± 2.50	3.92 ± 0.14	1.33 3.91
	100	99.45	20.0 ± 0.3	-5.43 ± 0.07	13.24 ± 2.02	4.49 ± 0.11	1.90 4.47
1.5 EU	0	99.45	18.9 ± 0.9	3.29 ± 0.12	43.25 ± 4.28	13.16 ± 0.16	10.87 13.14
	30	99.45	20.1 ± 0.3	4.16 ± 0.04	4.47 ± 1.45	14.39 ± 0.11	11.80 14.37
	70	99.45	20.2 ± 0.3	3.61 ± 0.03	0.46 ± 1.15	13.91 ± 0.10	11.32 13.89
	100	99.45	19.7 ± 0.5	2.59 ± 0.01	5.28 ± 0.48	12.51 ± 0.09	9.92 12.48
0.54 (S) PuO ₂ -UO ₂	0	99.45	25.0 ± 0.3	3.20 ± 0.07	21.18 ± 2.43	13.03 ± 0.12	13.44 13.04
	100	99.45	24.4 ± 0.9	1.93 ± 0.04	7.66 ± 1.45	11.80 ± 0.10	9.21 11.82
0.87 (S) PuO ₂ -UO ₂	0	99.45	20.9 ± 0.8	7.65 ± 0.14	77.46 ± 5.62	17.51 ± 0.17	14.92 17.51
	100	99.45	22.7 ± 0.5	4.76 ± 0.03	14.94 ± 1.18	14.65 ± 0.09	12.06 14.66

^a Values normalized to the condition of 22°C and 99.45 D₂O mol%.

TABLE VII
Variation of α and β with Maximum Number of Test Fuel Clusters in Test Region
(Test Fuel = natural uranium, lattice pitch = 22.5 cm)

Maximum Number of Test Fuel Clusters	W_{00} of Test Region	Void Fraction			
		0%		100%	
		α (m^{-2})	β (m^{-4})	α (m^{-2})	β (m^{-4})
9	0.204682	-10.35 ± 0.01	59.23 ± 0.33	-5.86 ± 0.04	22.23 ± 1.16
13	0.281479	-10.32 ± 0.04	57.75 ± 1.07	-5.84 ± 0.03	21.56 ± 0.87
21	0.425378	-10.25 ± 0.08	54.98 ± 1.88	-5.83 ± 0.03	20.91 ± 0.80
25	0.484135	-10.21 ± 0.09	53.29 ± 2.09	-5.82 ± 0.04	20.35 ± 0.93

As far as the first two terms are concerned, the third-order perturbation effect has already been implicitly contained in the expression for δB_z^2 given by Eq. (35), since these terms are proportionate to the two terms of Eq. (35), respectively. Therefore, the first two terms cannot be any obstacle to the linearity in question. On the other hand, the third term of Eq. (45) cannot be included in any term of Eq. (35). Consequently, it is possible for the third term to cause the break in linearity.

As the substituted area increases, the values of W_{0k} , W_{0m} , and W_{km} in Eq. (45) increase. The denominator of the third term, however, also increases due to the increase in δB_z^2 if its value is positive, thus preventing the value of the third term in Eq. (45) from increasing. Where δB_z^2 is negative, the denominator should accelerate, to some extent, the increase of the third term with the increase in the substituted area. The absolute value of this third term also depends on the other factor, $(0|\delta L|0)^3$, which increases with the value of α , as shown by Eq. (36). These tendencies, originating from the third-order perturbation effect, well explain the characteristics of β (see Table VII) that the fitting error in β increases with the maximum number of substituted clusters only when the value of α is large with a negative sign.

Therefore, where α is large with a negative sign, the substitution of test clusters should be limited within the suitable maximum number such as 9, to retain a small enough effect in the third term of Eq. (45). In such cases, improvement in statistical accuracy should be attained by increasing the number of substitution patterns, instead of increasing substituted clusters.

For the α with a positive sign, approximate values of the third term of Eq. (45) was numerically estimated for the present experimental lattices. The value obtained was confirmed to be less than the experimental error in δB_z^2 of each substitution.

It is desirable that an amount of fissile material needed in substitution measurements be as small as possible. Thus, it is very interesting to determine to what extent the test region can be reduced. Figures 3 and 4 give a guide to reduction of the test region. The figures show that the value of X , defined by Eq. (39), increases at first with expansion of the test region and then decreases after it attains maximum. Since X is thought to be a kind of parameter designating how radial flux shape deviates from a fundamental mode, the above-mentioned behavior of X corresponds to the fact that the radial flux shape deviates at first from the J_0 function as the test region expands and finally returns to the original form of it. If the presence of the third term of Eq. (45) is negligible, a linear relation is evidently established between X and Y , which indicates that plots of measured $\delta B_z^2/W_{00}$ should retrace the same straight line. Consequently, the material buckling of the test lattice can be determined only through a one-way range of progressive substitution: from zero to the maximum point attained by X .

As seen in Figs. 3 and 4, X attains the maximum point by substituting ~10% of the whole core in most cases. Even for other cases where buckling differences are negative, the necessary test region does not exceed 20%. In such cases, as discussed before, the test region, comprising >9 test clusters, does not contribute to the improvement of experimental accuracy in α . Therefore, it is concluded that by using 10% of the whole core for substitution, it is possible to determine the difference in material bucklings between two lattices in cases of such slightly enriched fuel clusters.

In the present second-order analysis, the iterative perturbation method was adopted. That is, denominators of Eq. (35) contain δB_z^2 . To ascertain the effectiveness of the iterative perturbation method, measured values of $\delta B_z^2/W_{00}$ were also plotted in Fig. 6 against

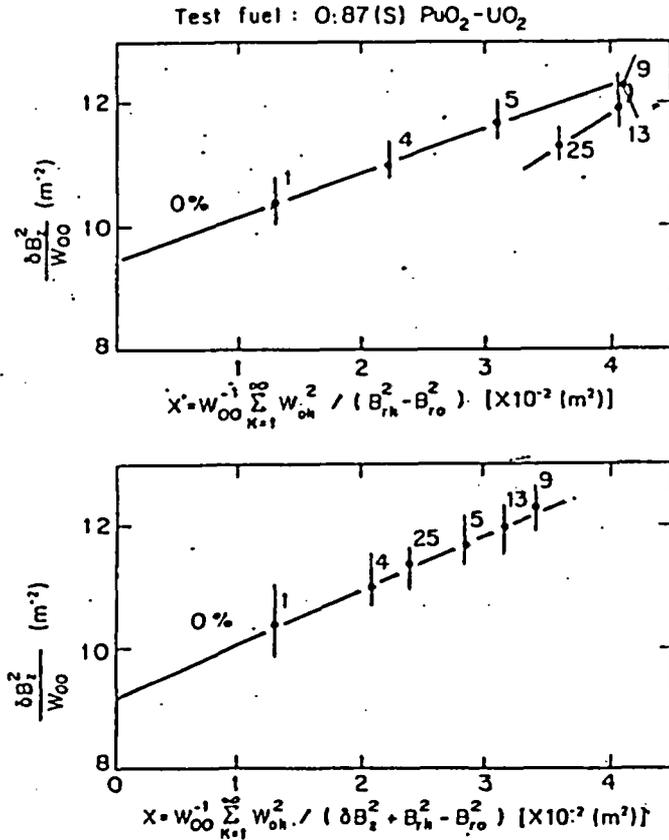


Fig. 6. Effect of δB_z^2 in Eq. (35). The upper figure shows that there exists no linear relation between $\delta B_z^2/W_{00}$ and X' . Lattice pitch = 22.5 cm, void fraction = 0%.

$$X' = W_{00}^{-1} \sum_{k=1}^{\infty} W_{0k}^2 / (B_{rk}^2 - B_{r0}^2) ,$$

by omitting δB_z^2 from the denominators of Eq. (39). Figure 6 shows that there is now no linear relation between X' and $\delta B_z^2/W_{00}$, despite good linearity seen between X and $\delta B_z^2/W_{00}$. Therefore, δB_z^2 cannot be omitted from the denominators of Eq. (39), unless $|\delta B_z^2|$ is extremely small compared to $(B_{rk}^2 - B_{r0}^2)$ ($k \neq 0$). Thus, the iterative perturbation method is effective in the present second-order analysis for the substitution measurements.

The code METHUSELAH II (Ref. 17) predicted that the diffusion coefficient of the thermal neutron differs by 1% or less between the test and reference lattices. The simple analysis based on first-order perturbation theory¹⁴ showed that evaluated differences in material buckling varied by not more than 1%, although the difference in diffusion coefficients was included in the analysis. Therefore, neglect of the difference in diffusion coefficients does not contribute significantly to the error in material buckling differences given in Tables V and VI.

Also, according to the prediction by the code METHUSELAH II (Ref. 17), neutron diffusion in the present lattices is anisotropic up to 5 or 6%. The effect of the anisotropy can easily be treated on the assumption that the ratio of radial-to-axial diffusion coefficients is independent of neutron energy. This assumption leads to the expression that $B_{zjk}^2 = B_j^2 - (1 + \eta)B_{rk}^2$ instead of B_{zjk}^2 given by Eq. (14), where the quantity η accounts for the anisotropy. Use of the relation $B_{zjk}^2 = B_j^2 - (1 + \eta)B_{rk}^2$ modifies Eq. (35) into

$$\delta B_z^2 = \alpha W_{00} + \beta(1 - \eta) \sum_{k=1}^{\infty} W_{0k}^2 / (\delta B_z^2 + B_{rk}^2 - B_{r0}^2) \quad (46)$$

Here, $\eta \delta B_z^2$ is neglected in the denominators because it is small compared to $(B_{rk}^2 - B_{r0}^2)$ ($k \neq 0$). Equation (46) suggests that the effect of the anisotropy has no influence on α evaluated by means of Eq. (35), while β so evaluated contains the effect of the anisotropy. Consequently, the β given in Tables V and VI has an uncertainty of 5 or 6% in its meaning defined by Eq. (37), since η was not determined as a separate quantity in the present experiment.

The present study is summarized as follows. On the assumptions that neutron diffusion is isotropic and that there exists no difference in diffusion coefficients between the two lattices, the second-order iterative perturbation method has been applied to the analysis of changes in critical condition due to substitutions. This application led to a simple formula that connected measured buckling differences with generalized statistical weights of substituted regions. In the analysis using this formula, it is possible to take into account the effect of distortion in radial neutron flux distribution in the substituted core without any iterative correction procedure that is usually adopted in the first-order perturbation method. Also, it is not necessary in the present analysis to introduce any usual intermediate region for taking into account the effect of spectrum mismatch between the two lattices.

Less than 10% of substitution of the whole core enables us, with the use of the present formula, to determine the material buckling differences with sufficient accuracy for the cases where both positive and negative buckling differences are caused by substitutions.

The critical substitution measurements on 28-fuel-rod clusters in heavy water were successfully analyzed by means of the formula, and the material bucklings of lattices with this type of fuel clusters could be determined over the wide range of fuel composition.

APPENDIX

With the help of Eq. (14), the third-order term, F , given by Eq. (44) can be rewritten as

$$\begin{aligned}
 F = & (0|\delta L|0)^3 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{W_{0k}W_{0m}W_{km}}{(\delta B_z^2 + B_{rk}^2 - B_{r0}^2)(\delta B_z^2 + B_{rm}^2 - B_{r0}^2)} + 2(0|\delta L|0) \\
 & \times \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{l=1}^{M-1} \frac{(0|\delta L|l)(l|\delta L|0)W_{0k}W_{0m}W_{km}}{(\delta B_z^2 + B_{rk}^2 - B_{r0}^2)(\delta B_z^2 + B_{z00}^2 - B_{zlm}^2)} \\
 & + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=1}^{M-1} \sum_{l=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|l)(l|\delta L|0)W_{0k}W_{0m}W_{km}}{(\delta B_z^2 + B_{z00}^2 - B_{zjk}^2)(\delta B_z^2 + B_{z00}^2 - B_{zlm}^2)} . \quad (A.1)
 \end{aligned}$$

To further rewrite the right side of Eq. (A.1), the following relations are utilized. Since $\psi_m(r)$ forms a complete orthogonal set, it is possible to expand $W(r)\psi_k(r)$ and $W(r)\psi_0(r)$ in a series of $\psi_m(r)$. Thus,

$$W(r)\psi_k(r) = \sum_{m=0}^{\infty} W_{km}\psi_m(r) \quad (A.2)$$

and

$$W(r)\psi_0(r) = \sum_{m=0}^{\infty} W_{0m}\psi_m(r) . \quad (A.3)$$

Multiplication of both sides of Eqs. (A.2) and (A.3) by each other yields

$$\psi_0(r)W^2(r)\psi_k(r) = \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} W_{km}W_{0m'}\psi_m(r)\psi_{m'}(r) . \quad (A.4)$$

Integration of Eq. (A.4) over r yields

$$W_{0k} = \sum_{m=0}^{\infty} W_{0m}W_{km} , \quad (A.5)$$

because $W^2(r)$ equals $W(r)$ and the terms where $m \neq m'$ vanish due to the orthogonality of $\psi_m(r)$. The use of Eqs. (A.5) and (33) leads to

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} W_{0k}W_{0m}W_{km} = \sum_{k=0}^{\infty} W_{0k}^2 = W_{00} . \quad (A.6)$$

The second term of the right side of Eq. (A.1) can be rearranged as

$$\text{the second term} = 2(0|\delta L|0) \sum_{k=1}^{\infty} \left[\sum_{m=0}^{\infty} \sum_{l=1}^{M-1} \frac{(0|\delta L|l)(l|\delta L|0)W_{0m}W_{km}}{\delta B_z^2 + B_{z00}^2 - B_{zlm}^2} \right] \frac{W_{0k}}{\delta B_z^2 + B_{rk}^2 - B_{r0}^2} . \quad (A.7)$$

With the use of Eqs. (14) and (A.5), the sum over m and l in Eq. (A.7) can be rewritten as

$$\sum_{m=0}^{\infty} \sum_{l=1}^{M-1} \frac{(0|\delta L|l)(l|\delta L|0)W_{0m}W_{km}}{\delta B_z^2 + B_{z00}^2 - B_{zlm}^2} = W_{0k} \sum_{l=1}^{M-1} \frac{(0|\delta L|l)(l|\delta L|0)}{B_0^2 - B_l^2} \left[1 - \sum_{m=0}^{\infty} \left(\frac{W_{0m}W_{km}}{W_{0k}} \right) \frac{f_{lm}}{1 + f_{lm}} \right] , \quad (A.8)$$

with

$$f_{lm} = (\delta B_z^2 + B_{rm}^2 - B_{r0}^2)/(B_0^2 - B_l^2) \quad (l \neq 0) . \quad (A.9)$$

On the basis of the same argument made in Eq. (34), the second term in the brackets of Eq. (A.8) can be omitted. Consequently, the second term of Eq. (A.1) is approximated to

$$\text{the second term} \approx \left[2(0|\delta L|0) \sum_{l=1}^{M-1} \frac{(0|\delta L|l)(l|\delta L|0)}{B_0^2 - B_l^2} \right] \sum_{k=1}^{\infty} \frac{W_{0k}^2}{\delta B_z^2 + B_{rk}^2 - B_{r0}^2} . \quad (A.10)$$

With the use of Eqs. (14), (A.5), and (A.6), the third term of the right side of Eq. (A.1) can be rearranged as

$$\begin{aligned} \text{the third term} &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=1}^{M-1} \sum_{l=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|l)(l|\delta L|0)W_{0k}W_{0m}W_{km}}{(B_0^2 - B_j^2)(B_0^2 - B_l^2)} \left[1 - \frac{f_{jk}}{1 + f_{jk}} \right] \cdot \left[1 - \frac{f_{lm}}{1 + f_{lm}} \right] \\ &= W_{00} \sum_{j=1}^{M-1} \sum_{l=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|l)(l|\delta L|0)}{(B_0^2 - B_j^2)(B_0^2 - B_l^2)} \left[1 - \sum_{k=0}^{\infty} \left(\frac{W_{0k}^2}{W_{00}} \right) \frac{f_{jk}}{1 + f_{jk}} - \sum_{m=0}^{\infty} \left(\frac{W_{0m}^2}{W_{00}} \right) \frac{f_{lm}}{1 + f_{lm}} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{W_{0k}W_{0m}W_{km}}{W_{00}} \right) \frac{f_{jk}f_{lm}}{(1 + f_{jk})(1 + f_{lm})} \right], \end{aligned} \quad (\text{A.11})$$

with

$$f_{jk} = (\delta B_x^2 + B_{jk}^2 - B_{r0}^2)/(B_0^2 - B_j^2) \quad (j \neq 0). \quad (\text{A.12})$$

In the same way as in Eq. (A.8), the second, third, and fourth terms in the brackets of Eq. (A.11) can be omitted. Thus, the third term of Eq. (A.1) is approximated to

$$\text{the third term} \approx \left[\sum_{j=1}^{M-1} \sum_{l=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|l)(l|\delta L|0)}{(B_0^2 - B_j^2)(B_0^2 - B_l^2)} \right] W_{00}. \quad (\text{A.13})$$

With the use of Eqs. (A.10) and (A.13), therefore, the third-order term can be finally approximated to

$$\begin{aligned} F &\approx \left[\sum_{j=1}^{M-1} \sum_{l=1}^{M-1} \frac{(0|\delta L|j)(j|\delta L|l)(l|\delta L|0)}{(B_0^2 - B_j^2)(B_0^2 - B_l^2)} \right] W_{00} + 2 \left[(0|\delta L|0) \sum_{l=1}^{M-1} \frac{(0|\delta L|l)(l|\delta L|0)}{B_0^2 - B_l^2} \right] \\ &\quad \times \sum_{k=1}^{\infty} \frac{W_{0k}^2}{\delta B_x^2 + B_{jk}^2 - B_{r0}^2} + (0|\delta L|0)^3 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{W_{0k}W_{0m}W_{km}}{(\delta B_x^2 + B_{jk}^2 - B_{r0}^2)(\delta B_x^2 + B_{rm}^2 - B_{r0}^2)}. \end{aligned} \quad (\text{A.14})$$

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