RDB received 4/19/05

DST

COMPUTER SERVICES S.A.

2/15/05
70 FR 7777

81, Avenue Louis Casaï
CH - 1216 Geneva, Switzerland

☎ Country 41, City 22
Tel : 798 91 88
Fax : 798 32 49 email: dst@iprolink.ch

(3)

Geneva, April 8, 2005

U.S. Nuclear Regulatory Commission
Rules and Directives Branch
Office of Administration
Washington, DC 20555-0001

Dear Sirs:

I respectfully submit comments for proposed Revision 2 of Regulatory Guide 1.92 (DG-1127). I enclose a paper copy of my comments as well as a copy of a document entitled "The Mathematics of Dynamic Analysis of Piping". The document is referred to in my comments. I also enclose a diskette containing the comments and the document in the format of Word 2000.

Thank you for your attention in this matter.

Very truly yours,

Kent Gordis

Kent Gordis

SISP Review Complete
Template = ADM-013

E-RIDS = ADM-03
Add T.Y. Chang (TYC)

Comments Related to the "Draft Regulatory Guide DG-1127".

My comments concern "Appendix A: General Discussion of the Response Spectrum Method".

I find the formulation ambiguous and I suggest that a more rigorous mathematical approach can lead to improved engineering solutions.

In the following, I mention several specific points which I find ambiguous in the mathematical formulation of Appendix A. I also attach a document containing an outline of a rigorous mathematical formulation of the dynamic piping analysis problem.

My first comment refers to equations (A.1)

Equation (A.1) is given as: $M\ddot{X} + C\dot{X} + KX = -M\,U_b\ddot{u}_g$ and it is stated that "X=column vector of relative displacements ($mx1$)".

The first question is: relative to what?

We may compare this formulation to the formulation of the attached document in which the equations of motion for support movement loading are derived from the general case of the equations of motion, with applied load $f(t) = K_B x_B(t)$. K_B is the N by M boundary stiffness coupling matrix and $x_B(t)$ is the M by 1 vector of support movements. N is the number of dynamic degrees-of-freedom and M is the number of boundary degrees-of-freedom.

The transformation to relative coordinates is then defined in the attached document by expression (29): $x(t) = u(t) + v(t)$, where $v(t)$ is defined to be the instantaneous configuration of the structure which is in static equilibrium with the support movements. The equivalent of equation (A.1) is formulated in the attached document in the expression (32): $M\ddot{u}(t) + Ku(t) = -MK^{-1}K_B\ddot{x}_B(t)$, where the damping has been omitted.

Comparing equation (A.1) to expression (32) we observe that for the attached document, the mathematical definition of the relative coordinates is explicitly stated and the conditions satisfied by both $u(t)$ and $v(t)$ are explained. Further, the time dependence has been explicitly indicated.

The second question is: exactly what is the meaning of the term $\ddot{u}_g$, defined as the "ground acceleration"?

Since U_b is defined as an ($mx1$) vector, the implication is that $\ddot{u}_g$ is a scalar, for otherwise how could U_b be multiplied on the right by $\ddot{u}_g$? But if $\ddot{u}_g$ is a scalar ground acceleration, which acceleration is being used? Is it X or Y or Z? And how does equation (A.1) relate to problems where different support points have different accelerations?

In the attached document the ground acceleration $\ddot{x}_B(t)$ is an M by 1 vector, where M is the number of support degrees-of-freedom. In this notation, it is clear that each support degree-of-freedom can have an independent support acceleration time history and the time dependence is shown explicitly.

My next comments refer to equations (A.2) and (A.3)

Equation (A.2) is given as: $X = \Phi Y$, where Φ = normalized mode shape matrix, $\Phi^T M \Phi = I$, where I = (nxn) identity matrix and Y is ($nx1$) vector of generalized coordinates.

We note that X is defined in (A.1) as an ($mx1$) vector, where m = number of dynamic degree-of-freedom. For equation (A.2) to be coherent, Φ and Y must be (mxm) and ($mx1$) respectively, that is they must include all m dynamic degrees-of-freedom. They cannot be dimensioned as defined by "n = number of modes considered".

However, both the orthogonality principle, $\Phi^T M \Phi = I$, and the equation (A.3) are valid if Φ is defined as the rectangular matrix whose columns are a subset of the mode shape vectors. Then Φ will be (mxn) and Y will be ($nx1$), where m = number of dynamic degrees-of-freedom and n = number of modes considered.

From this notation we may infer that the components of Y of equation (A.2) are the scalar modal amplitudes defined in expression (4) of the attached document and that in equation (A.3) Y_j is the j^{th} component of Y which is equivalent to $y_j(t)$ in the attached document.

The right side of equation (A.3) is given as: $-\Gamma_j \ddot{u}_g$. As was previously mentioned in discussing equation (A.1), there is an ambiguity in the definition of $\ddot{u}_g$. Further, Γ_j, the "modal participation factor of mode j" is defined as $\Phi_j^T M U_b$, which appears to be the mass weighted method of calculating participation factors, possibly single support level factors. In the attached document, a simple form of multiple support level participation factors is given in expression (38). These factors are support stiffness weighted.

Comments Related to the "Draft Regulatory Guide DG-1127".

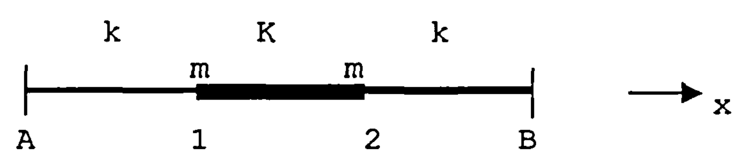
My next comment concerns what could be termed "the mathematical understanding" of the solution.

The key to understanding the solution is an understanding of the scalar modal amplitudes, which are designated by the parameters $y_j(t)$ in the attached document (and by Y_j in equation (A.3)). Very useful information about these parameters can be deduced from their representation as convolution integrals as, for example, expression (13) in the attached document. As we see from expression (13), the applied force acting in the j^{th} mode is given by $e_j^T f(t) = g_j(t)$.

This representation shows that the only modes which do not contribute to the solution are modes j which are orthogonal or nearly orthogonal to the applied load vector. This can occur when the significant components of a mode all correspond to degrees-of-freedom where the applied load is zero. This situation can arise for fluid hammer loads acting primarily in parts of a piping system which are remote from a mode shape vector. However, in the case of support movement loading, since supports are usually more rigid than the piping, many high frequency modes of a piping structure may be expected to excite support degrees-of-freedom.

There is another possibility for orthogonality. This is the case where although significant mode shape components correspond to degrees-of-freedom where non-zero applied load components are acting, their inner product with the mode shape components is zero or close to zero. This can occur when the structure and the load have related symmetries.

As an example, consider the extremely simple structure shown below.



Three springs connect two mass points, each with mass m, to the boundary as shown in the sketch. The springs connecting the boundary to the mass points, (A1) and (B2), each have rigidity k.

This system has two degrees-of-freedom and two mode shape vectors, e_1 and e_2, shown below:

Mode 1 : $e_1 = \begin{vmatrix} e \\ e \end{vmatrix}$ Mode 2 : $e_2 = \begin{vmatrix} e \\ -e \end{vmatrix}$

$\omega_1^2 = \dfrac{k}{m}$ $\omega_2^2 = \dfrac{k+2K}{m}$

The support movement loading vector is simply a *2x1* column vector with components $x_A(t)$ and $x_B(t)$. It follows that $g_1(t) = e\,(x_A(t) + x_B(t))$ and $g_2(t) = e\,(x_A(t) - x_B(t))$.

We see that if support points A and B are moving in parallel, which means that $x_A(t) = x_B(t)$ for all *t*, then $g_2(t) = 0$ for all *t* and mode 2 cannot be excited.

If support points A and B are moving in opposite directions, so that $x_A(t) = -x_B(t)$ for all *t*, then $g_1(t) = 0$ for all *t* and mode 1 cannot be excited.

For any other type of support movement, both modes will be excited.

In this example if the "included mass" of the two modes is calculated using the usual method, all the mass will be included in mode 1 and mode 2 will have zero mass. On the other hand, if the support points are moving in opposite directions, the entire response will be in a mode with zero mass. The explanation of this seeming paradox is that the definition of "included mass" is based on the hypothesis of the parallel movement of all supports. This calculation of "included mass" filters out the contribution of modes which are approximately anti-symmetric and emphasizes the contribution of modes which are approximately symmetric.

The same remarks apply to multi-dimensional structures. If all support points are moving in parallel, anti-symmetric modes cannot be excited. But these modes can be excited by other types of support movement and, indeed, by other dynamic loads such as fluid hammers.

The conclusion is that the concept of "included mass" has physical significance only for one type of loading: single support movement. The idea that there is a certain quantity of mass in each mode which according to its amplitude causes larger or smaller responses, is, quite simply, erroneous. There exist loads which can excite any mode. For example, when (on page 10) the DRG refers to "the residual rigid response of the missing mass modes" it isn't clear which modes are involved.

The related concepts of "included mass" and "missing mass" should not be used to determine the cut-off frequency (except, possibly for single support level problems) and should not be used to construct a rigid mode correction.

There is a general and rigorous procedure for making these decisions. By "general" I mean that it applies to all types of dynamic analysis of piping.

This procedure is as follows:

1. Use the method described in the section "Determining Cut-Off Frequency" of the attached document (page 8). This method is based on representing the scalar modal amplitudes, $y_j(t)$, as convolution integrals. To carry out this method, it is necessary to have access to the time history loading data.

2. Use the method described in the section "Left Out Force" of the attached document (page 7) to construct the rigid mode correction. This method can be carried out for all types of dynamic analysis of piping: general time history analysis, support acceleration time history analysis, single or multi-level floor response spectrum analysis.

To end my comments on a positive note, based on the formulation in the attached document, I derive an alternate form for the solution.

Substituting the right term of expression (19) into expression (17) and multiplying on the left by K and using (2), we obtain

$$Kx(t) = f(t) + \sum_{j=1}^{N} Me_j h_j(t)$$

where

$$h_j(t) = \frac{\omega_j}{\beta_j} \int_{\tau=0}^{t} G_j(t-\tau)\, e_j^T \dot{f}(\tau)\, d\tau$$

The scalars $h_j(t)$, which converge to zero as ω_j increases, determine by how much and in which modes the solution deviates from the rigid solution. These parameters, which depend on the derivative of the applied load, may furnish valuable information about the solution.

The Mathematics of Dynamic Analysis of Piping

Kent Gordis
April 2005

Topics

Equations of Motion

- Solution by Modal Superposition
- Properties of Mode Shape Vectors
- Uncoupling the Equations of Motion
- Convolution Integral

Rigid Mode Correction

- Limit Theorem
- Domain of validity
- Left Out Force
- Determining Cut-Off Frequency

Constructing Bounds

- Combining Modal Bounds
- Example of Expected Value

Support Movement Load

- Equation of Motion in Relative Coordinates
- Equality of Mass and Support Stiffness Weighted Participation Factors
- Rigid Mode Correction in Relative Coordinates
- Concept of Support Levels

Influence of Early Methods

Summary

Equations of Motion

The "equations of motion" of a multi-degree of freedom mechanical system of mass M, damping C and stiffness K loaded by a time-dependent force $f(t)$ is given by:

(1) $$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t)$$

where:		
	M	N by N mass matrix
	C	N by N damping matrix
	K	N by N stiffness matrix
	$f(t)$	N by 1 vector of force loads acting at mass points
	$x(t)$	N by 1 vector of displacements at mass points

The equations of motion represent a system of N simultaneous, linear second order differential equations. The problem is to solve these equations for $x(t)$. Once $x(t)$ is known, other solution parameters such as instantaneous forces, moments, stresses, accelerations, etc. may be calculated. In most practical cases, the objective is to construct an envelope for the solution parameters.

The solution $x(t)$ depends on the initial conditions. In the following it is assumed that the structure is initially at rest so that:

(1.1) $$x(0) = \dot{x}(0) = 0$$

By the method of "mode shapes" or "eigenvectors", it will be shown how to solve (1) and (1.1). This method, which was first developed in the 19th Century, transforms (1) into N uncoupled scalar linear second order differential equations.

Solution by Modal Superposition

The mode shapes (or eigenvectors) are N by 1 vectors denoted by e_j and the corresponding scalar natural circular frequencies ω_j satisfy:

(2) $$Ke_j = \omega_j^2 Me_j$$

Because K and M are symmetric and positive-definite, there are always N linearly independent mode shape vectors e_j which satisfy (2).

This means that the mode shape vectors, e_j, "span the space" or, in other words, that for any N-vector A there exist scalars $a_1, a_2, \ldots, a_N$ such that

(3) $$A = \sum_{j=1}^{N} e_j\, a_j$$

In particular, at any instant of time t there exist scalars $y_j(t)$ so that the instantaneous solution $x(t)$ satisfies

(4) $$x(t) = \sum_{j=1}^{N} e_j y_j(t)$$

This expression shows that once the mode shape vectors have been determined, to solve the equations of motion, it is sufficient to evaluate the scalars $y_j(t)$. Since the mode shape vectors, e_j, are time and load independent, all time dependent information about the solution is contained in the scalars $y_j(t)$, which are called the "scalar modal amplitudes".

Further, if the scalars $\hat{Y}_j$ are time-bounds for the $y_j(t)$, then (4) may be used to construct bounds for the solution.

In the following, it is shown how to solve for the scalar modal amplitudes $y_j(t)$ and how to construct bounds $\hat{Y}_j$.

<u>Properties of Mode Shape Vectors</u>

First, we discuss the properties of the mode shape vectors.

The mode shape vectors are orthogonal in the sense that

(5) $$e_{j_1}{}^T M\, e_{j_2} = 0 \quad \text{for} \quad j_1 \neq j_2$$

And since (2) is homogeneous in e_j, the mode shape vectors may be normalized so that

(6) $$e_j{}^T M\, e_j = 1$$

The N by N matrix E may be defined as the matrix whose j^{th} column is e_j. Then (5) and (6) imply that

(7) $E^T\, M\, E = I$ where I is the N by N identity matrix

From (7), $E^T = (ME)^{-1}$ and since inverses commute,

(8) $$M\,E\,E^T = I$$

and (8) may be written in the form

(9) $$\sum_{j=1}^{N} M\,e_j\,e_j^{\,T} = I$$

Expression (9) will be used to show how to accurately approximate the contribution of the rigid modes to the total system response without actually extracting these modes.

Uncoupling the Equations of Motion

We are now ready to solve for the scalar modal amplitudes, $y_j(t)$.

We assume that there exist scalars α_j such that the damping matrix C satisfies

(10) $$C e_j = 2\,\alpha_j M e_j$$

We then substitute (4) into (1) and multiply on the left by $e_{j_0}^{\,T}$ we obtain

(11) $$\sum_{j=1}^{N} e_{j_0}^{\,T}\left(Me_j\ddot{y}_j(t) + Ce_j\dot{y}_j(t) + Ke_jy_j(t)\right) = e_{j_0}^{\,T} f(t)$$

Using (2) and (10) and the orthogonality conditions (5) and (6), we obtain for each j_0 in the range $1 \le j_0 \le N$ a scalar second order linear differential equation:

(12) $$\ddot{y}_{j_0}(t) + 2\alpha_{j_0}\,\dot{y}_{j_0}(t) + \omega_{j_0}^{\,2} y_{j_0}(t) \;=\; e_{j_0}^{\,T} f(t) \;= g_{j_0}(t)$$

Note that both e_{j_0} and $f(t)$ are N by 1 vectors but their so-called "inner product", $e_{j_0}^{\,T} f(t)$, is a scalar which is defined to be $g_{j_0}(t)$.

From the linear independence of the mode shape vectors, e_j, it follows that

(12.1) $$y_{j_0}(0) = \dot{y}_{j_0}(0) = 0$$

Convolution Integral

In the following, we replace the subscript j_0 by j in (12) and (12.1). If the function $g_j(t)$ satisfies the condition $g_j(0) = 0$, then the solution to (12) is given by the so-called "convolution" integral

$$(13) \quad y_j(t) = \frac{1}{\beta_j} \int_{\tau=0}^{t} e^{-\alpha_j(t-\tau)} \sin \beta_j (t-\tau) \, g_j(\tau) \, d\tau$$

where

$$(13.1) \quad \alpha_j^2 + \beta_j^2 = \omega_j^2$$

We will use the properties of the convolution integral to investigate the solution to the equations of motion and, in particular, we will derive the "rigid mode correction".

Rigid Mode Correction

A defect of the method described above is that it involves calculating all the modes. For a large class of practical engineering problems, it is possible to calculate a highly accurate approximation of the solution to the equations of motion using only the lower modes, that is modes with natural frequencies below a cut-off frequency.

In the following, we will explain this method and also how to determine the cut-off frequency.

We start by rewriting (4) in a different form:

$$(15) \quad y_j(t) = \frac{1}{\omega_j^2} e_j^T r_j^f(t)$$

If $f(0) = O$ the we define the N by 1 vector $r_j^f(t)$ as the "force response" in mode j as follows:

$$(16) \quad r_j^f(t) = \frac{\omega_j^2}{\beta_j} \int_{\tau=0}^{t} e^{-\alpha_j(t-\tau)} \sin \beta_j (t-\tau) \, f(\tau) \, d\tau$$

$r_j^f(t)$ is an N by 1 vector whose components of have the same units as the corresponding components of $f(t)$.

It follows that if $f(0) = O$, we may write (4) in the form:

$$(17) \quad x(t) = \sum_{j=1}^{N} \frac{1}{\omega_j^2} e_j e_j^T r_j^f(t)$$

Limit Theorem

The following limit theorem is the key to the rigid mode correction:

We assume that the applied load, $f(t)$, satisfies the condition $f(0) = O$. Then:

$$(18) \quad \lim \omega_j \to \infty \quad r_j^f(t) = f(t)$$

The limit theorem (18) is based on the following identity:

$$(19) \quad r_j^f(t) = \frac{\omega_j^2}{\beta_j} \int_{\tau=0}^{t} e^{-\alpha_j(t-\tau)} \sin \beta_j(t-\tau) f(\tau)\, d\tau = f(t) - \frac{\omega_j}{\beta_j} \int_{\tau=0}^{t} G_j(t-\tau) \dot{f}(\tau)\, d\tau$$

where

$$(19.1) \quad G_j(t) = e^{-\alpha_j t} \left(\frac{\alpha_j}{\omega_j} \sin \beta_j t + \frac{\beta_j}{\omega_j} \cos \beta_j t \right)$$

Domain of Validity

Before using these results, we will show that these methods may be used to solve the equations of motion (1) and (1.1) for any piping structure initially at rest, even if $f(0) \neq O$.

Define the N by 1 vector $x'(t)$ by:

$$(19.2) \quad x(t) = x'(t) + K^{-1} f(0)$$

Substituting this expression for $x(t)$ into the equations of motion (1) we obtain:

$$(19.3) \quad M\ddot{x}'(t) + C\dot{x}'(t) + Kx'(t) = f(t) - f(0) = f'(t)$$

From this definition of $f'(t)$, we see that

(19.4) $f'(0) = 0$

If at time $t=0$ the piping structure is at rest, it follows that

(19.5) $Kx(0) = f(0)$ and $\dot{x}(0) = 0$

From (19.5) and (19.2) it follows that

(19.6) $x'(0) = \dot{x}'(0) = 0$

Using (19.6) and (19.4), we see that the methods of this paper may be may be used to solve (19.3) for $x'(t)$. Then, using (19.2), we may construct the solution, $x(t)$, to the equations of motion (1).

This argument shows that the methods used in this paper apply to any dynamic problem satisfying (1) and (1.1) which is initially at rest, even if $f(0) \neq 0$.

Left Out Force

Let ω_R be a cut-off frequency. Then using (17) we may rewrite (4) in the form:

(20) $$x(t) = \sum_{\omega_j < \omega_R} e_j\, y_j(t) + \sum_{\omega_j \geq \omega_R} \frac{1}{\omega_j^2} e_j\, e_j^T\, r_j^f(t)$$

Multiplying both sides of (20) on the left by K and using (2) we obtain:

(21) $$K\, x(t) = \sum_{\omega_j < \omega_R} K\, e_j\, y_j(t) + \sum_{\omega_j \geq \omega_R} M\, e_j\, e_j^T\, r_j^f(t)$$

Since we have assumed $f(0) = 0$ then, in light of the limit theorem (18), if we choose ω_R sufficiently large, then we can approximate the last term on the right of (21) by:

(22) $$\sum_{\omega_j \geq \omega_R} M\, e_j\, e_j^T\, r_j^f(t) \approx \sum_{\omega_j \geq \omega_R} M\, e_j\, e_j^T\, f(t)$$

Finally, we use (9) to rewrite (22) as:

$$(23) \quad \sum_{\omega_j \geq \omega_R} M\, e_j\, e_j^T\, f(t) = \sum_{\omega_j < \omega_R} \left(I - M\, e_j\, e_j^T\right) f(t)$$

where I is the N by N identity matrix.

The right side of (23), which involves only modes with frequencies $\omega_j < \omega_R$, defines an instantaneous load (with the units of force) which when applied as a static load approximates the response of the piping system in the modes with frequencies $\omega_j \geq \omega_R$. This load vector is called the "left out force" and the instantaneous solution it generates when applied to the structure is called the "rigid mode correction".

Determining Cut-Off Frequency

To complete this general discussion of how to solve the equations of motion, we describe a method for determining the cut-off frequency ω_R.

We start by choosing a set of test frequencies ω_i and a constant c such that for all i

$$(24) \quad c = \frac{\alpha_i}{\beta_i}$$

c is called the "fraction of critical damping".

For each i, use (16) to calculate the force response $r_i^f(t)$ for each non-zero component of the N by 1 applied force vector $f(t)$. Compare these responses to $f(t)$ and choose ω_R as the frequency at which the responses are equal to the applied force, within engineering accuracy, for all the components of $f(t)$.

An important point is that the cut-off frequency ω_R is independent of the structure being analyzed and depends only on the characteristics of the applied force vector $f(t)$. This point suggests that concepts such as "modal mass", which are load independent and depend only on the structure, are not useful for determining the cut-off frequency or for constructing a rigid mode correction.

Constructing Bounds

We next discuss the problem of constructing bounds for the time history solution.

One method is to calculate a solution $x(t)$ at each time step, and to use these values to calculate a complete solution at each time step, that is forces, moments, stresses, accelerations, etc. Once these instantaneous solutions have been calculated, the bound for each solution component is simply the largest absolute value that has been calculated.

Another class of methods is based on using modal bounds.

For these methods, the modal amplitudes defined in (4) are calculated at each time step or they are estimated by another method. Then, for each mode the values $\hat{Y}_j$ are:

$$(25) \quad \hat{Y}_j = \textit{bound for} \quad y_j(t) \quad \textit{in interval} \quad 0 \le t \le T$$

We may then define bounds $\hat{X}_j$ for each modal component on the right side of (4) as:

$$(26) \quad \hat{X}_j = e_j \hat{Y}_j$$

$\hat{X}_j$ is an N by 1 vector. Each of its N components is a bound for the corresponding component of the N by 1 vector $e_j y_j(t)$. Similarly, modal bounds, for any response component $R_j(t)$, may be calculated based on $\hat{Y}_j$. Denote the modal bound for a selected response component by $\hat{R}_j$

Combining Modal Bounds

We require a method to combine the modal bounds to construct a bound for the total system response.

If we denote by $\hat{R}$ the total system response for a solution component, then a class of methods for constructing these bounds is based on the following quadratic form:

$$(27) \quad \hat{R}^2 = \sum_{j=1}^{N} \hat{R}_j^{\,2} + 2 \sum_{i<j} \varepsilon_{ij} \hat{R}_i \hat{R}_j$$

The parameters ε_{ij} are called the "modal correlation coefficients" for modes *i* and *j*.

If $\varepsilon_{ij} = sign\left(\hat{R}_i\ \hat{R}_j\right)$, then $\hat{R}$ is the absolute sum of the modal responses.

If $\varepsilon_{ij} = 0$, then $\hat{R}$ is the SRSS sum of the modal responses.

There are two reasons why we expect the absolute sum method to be too conservative.

First, most piping systems have a large dimensionality N and therefore many mode shape vectors. The probability that the modal maxima will coincide is small.

Second, in many cases the true maximum of any response component will be of very short duration. To measure the effect of the load on the structure, it is plausible to seek not the actual maximum response but rather the most probable response, which is called the "expected value".

Example of Expected Value

To illustrate how the expected value may be calculated, we use a very simple example. Let A, B, C denote the magnitudes of three numbers whose signs are unknown. The problem is to derive the expected value of their sum S=A+B+C, based on the premise that each number has equal probability of being positive or negative. We investigate this problem by considering S^2.

There are $2^3 = 8$ sums with different sign configurations. These are:

$$S_1^2 = (+A+B+C)^2 = A^2+B^2+C^2+2AB+2AC+2BC$$

$$S_2^2 = (+A+B-C)^2 = A^2+B^2+C^2+2AB-2AC-2BC$$

$$S_3^2 = (+A-B+C)^2 = A^2+B^2+C^2-2AB+2AC-2BC$$

$$S_4^2 = (+A-B-C)^2 = A^2+B^2+C^2-2AB-2AC+2BC$$

$$S_5^2 = (-A+B+C)^2 = A^2+B^2+C^2-2AB-2AC+2BC$$

$$S_6^2 = (-A+B-C)^2 = A^2+B^2+C^2-2AB+2AC-2BC$$

$$S_7^2 = (-A-B+C)^2 = A^2+B^2+C^2+2AB-2AC-2BC$$

$$S_8^2 = (-A-B-C)^2 = A^2+B^2+C^2+2AB+2AC+2BC$$

If we multiply each of these values by its probability (=1/8) and add the resulting values, noting that all cross-product terms add up to zero, we arrive at:

$$S^2(\text{expected}) = A^2 + B^2 + C^2$$

It is clear that this result may be extended to any number of magnitudes A, B, C, D, ...

Various methods have been used for defining the modal correlation coefficients ε_{ij}. These include the NRC Reg. Guide 1.92 double sum method, the CQC method, the GAC methods developed by Westinghouse and others. These methods are outside the scope of this paper.

Support Movement Load

To conclude the discussion, we explain how to solve a special case of the equations of motion where the load is due entirely to support movements.

This type of load can be defined by:

(28) $f(t) = K_B \, x_B(t)$

where:

K_B	N by M boundary coupling stiffness matrix
$x_B(t)$	M by 1 vector of displacements at support points

Note that the boundary coupling stiffness matrix is rectangular (N by M) and the vector of applied support displacements $x_B(t)$ is M by 1. M is the number of support degrees of freedom.

Equations of Motion in Relative Coordinates

The solution of the equations of motion would be identical to the general case except for the fact that for both engineering and numerical analysis reasons, the equations are transformed into a "relative" coordinate system. The transformation is defined by:

(29) $x(t) = u(t) + v(t)$

where $v(t)$ is defined by:

(30) $K \, v(t) = K_B \, x_B(t)$

This condition defines $v(t)$ as the deformation which is in static equilibrium with the instantaneous support movements. For this reason, $v(t)$ is called the "secondary" or "pseudo-static" part of the solution.

$u(t)$ is the displacement with respect to $v(t)$ due to the effects of inertia. For this reason, $u(t)$ is called the "primary" or the "inertial" part of the solution.

Substituting (29) into (1) and rearranging the terms, we obtain the equations of motion in the relative coordinate system of $u(t)$ (without damping):

(31) $$M\,\ddot{u}(t) + K\,u(t) = -M\,\ddot{v}(t) - K\,v(t) + K_B\,x_B(t)$$

Using (30), this reduces to:

(32) $$M\,\ddot{u}(t) + K\,u(t) = -M\,K^{-1}\,K_B\,\ddot{x}_B(t)$$

And we assume that: $x(0) = \dot{x}(0) = u(0) = \dot{u}(0) = v(0) = \dot{v}(0) = 0$

In analogy to (16), we define the M by 1 vector $r_j^{\,a}(t)$ as the "acceleration response" in mode j, defined by:

(33) $$r_j^{\,a}(t) = \frac{\omega_j^{\,2}}{\beta_j} \int_{\tau=0}^{t} e^{-\alpha_j(t-\tau)} \sin\beta_j(t-\tau)\,\ddot{x}_B(\tau)\,d\tau$$

The components of $r_j^{\,a}(t)$ have the same units as the corresponding components of $\ddot{x}_B(t)$.

In analogy to (4) we define the scalar modal amplitudes in the u-space, $y_j^{\,u}(t)$, by

(34) $$u(t) = \sum_{j=1}^{N} e_j\,y_j^{\,u}(t)$$

The right side of (32) is the effective applied load in the u-space. Therefore, $r_j^{\,fu}(t)$, the force response in the u-space, satisfies:

(35) $$r_j^{\,fu}(t) = -M\,K^{-1}\,K_B\,r_j^{\,a}(t)$$

In analogy to (15) we write:

$$(36) \quad y_j^u(t) = \frac{1}{\omega_j^2} e_j^T \left(-M K^{-1} K_B r_j^a(t)\right)$$

Which, using the symmetry of M and K and the rules for forming the matrix transpose of a product, we may write in the form:

$$(37) \quad y_j^u(t) = \left(\frac{-1}{\omega_j^2} K_B^T K^{-1} M e_j\right)^T r_j^a(t)$$

Finally, using (2) we obtain:

$$(38) \quad y_j^u(t) = \left(\frac{-1}{\omega_j^4} K_B^T e_j\right)^T r_j^a(t) = \sum_{m=1}^{M} P_{mj} \left[r_j^a(t)\right]_m$$

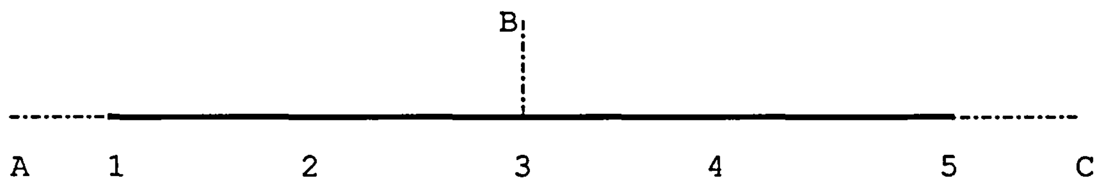
Where $\left[r_j^a(t)\right]_m$ is the instantaneous acceleration response in mode j at support degree-of-freedom m, and where P_{mj} is the "participation factor" in mode j at support degree-of-freedom m.

This formulation of the participation factors differs from the usual formulation, which is based on the so-called "single support level" case. This is the case which applies to a sub-class of support movement problems for which all support points are moving in parallel. For the single support level case, there are exactly three participation factors corresponding to the X, Y, Z directions. In this case, the participation factors are usually formulated as products of mass *times* mode shape component, a method I call "mass weighted". Note that the factors in (38) are support-stiffness weighted.

Equality of Mass and Support Stiffness Weighted Participation Factors

In the following, I will indicate why for the single support level case, the participation factors of (38) can be reduced to the mass weighted factors.

Consider the mechanical system shown below consisting of five mass points connected by four springs with supports (A1), (B3) and (C5) acting at three of the mass points.



Suppose e is a mode shape vector for this system with circular frequency ω so that $K\,e = \omega^2\,M\,e$. In expanded form, this condition may be written as:

$$\begin{vmatrix} K_{1A}+K_{12} & -K_{12} & 0 & 0 & 0 \\ -K_{21} & K_{21}+K_{23} & -K_{23} & 0 & 0 \\ 0 & -K_{32} & K_{32}+K_{34}+K_{3B} & -K_{34} & 0 \\ 0 & 0 & -K_{43} & K_{43}+K_{45} & -K_{45} \\ 0 & 0 & 0 & -K_{54} & K_{54}+K_{5C} \end{vmatrix} \begin{vmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{vmatrix} = \omega^2 \begin{vmatrix} m_1e_1 \\ m_2e_2 \\ m_3e_3 \\ m_4e_4 \\ m_5e_5 \end{vmatrix}$$

By symmetry, $K_{ij} = K_{ji}$, and we see that the non-boundary stiffness terms in the following sum add up to zero, so that:

$$m_1e_1 + m_2e_2 + m_3e_3 + m_4e_4 + m_5e_5 = \frac{1}{\omega^2}\left(K_{1A}e_1 + K_{3B}e_3 + K_{5C}e_5\right)$$

This argument can be generalized in the N-dimensional vector space.

Rigid Mode Correction in Relative Coordinates

To complete the general solution to the equations of motion in the u-space defined in expression (32), we show how to construct the rigid mode correction.

In the u-space, using (32) the right side of (23) becomes:

$$(39) \qquad \sum_{\omega_j<\omega_R} -(I - Me_j e_j^T)\, MK^{-1}K_B \ddot{x}_B(t)$$

The expression (39) defines an instantaneous left out force vector which when applied as a static load in the u-space approximates the response in the modes with frequencies $\omega_j \geq \omega_R$. This term can be rewritten in the form:

$$(40) \qquad \left(-MK^{-1}K_B \;\; + \sum_{\omega_j<\omega_R} Me_j \left(\frac{1}{\omega_j^2} K_B^T e_j \right)^T \right) \ddot{x}_B(t)$$

This form of the left-out-force is convenient for calculations since it involves terms directly related to the participation factors defined in (38).

The Concept of Support Levels

The preceding discussion involved the concept of "single support level". In the following, I explain the concept of "support level" and show that although this concept may be useful from an engineering point of view, it is not important in terms of solving the equations of motion.

The support movement loading, $x_B(t)$, is an M by 1 vector. Denote by $[x_B(t)]_m$ the m^{th} component of this vector. Suppose that for two components, *m1* and *m2*, there exists a scalar *c* such that:

$$(41) \qquad [x_B(t)]_{m2} = c\,[x_B(t)]_{m1}$$

Components *m1* and *m2* are called "dependent". If *m1* and *m2* are both components for the same global direction (X, Y or Z) and if $c = 1$, then the two components of the support load are moving in parallel.

From (41), it follows that:

$$(42) \qquad [\ddot{x}_B(t)]_{m2} = c\,[\ddot{x}_B(t)]_{m1}$$

and therefore from (33) that:

$$(43) \qquad \left[r_j^a(t)\right]_{m2} = c\,\left[r_j^a(t)\right]_{m1}$$

Then the contribution of the *m1* and *m2* support degrees-of-freedom to the last term in expression (38) is:

$$(44) \qquad P_{m1\,j}\left[r_j^{\,a}(t)\right]_{m1} + P_{m2\,j}\left[r_j^{\,a}(t)\right]_{m2} = (P_{m1\,j} + cP_{m2\,j})\left[r_j^{\,a}(t)\right]_{m1}$$

In this case we may eliminate *m2* from expression (38) as follows:

$$(45) \qquad y_j^{\,u}(t) = \sum_{m=1}^{M} P_{mj}\left[r_j^{\,a}(t)\right]_m = \sum_{m=1, m\neq m2}^{M} P'_{mj}\left[r_j^{\,a}(t)\right]_m$$

where

$$(46) \qquad P'_{m1\,j} = P_{m1\,j} + cP_{m2\,j} \quad \text{and} \quad P'_{m\,j} = P_{m\,j} \quad m \neq m1$$

Similarly, all dependent components can be removed so that (38) may be written in the form:

$$(47) \qquad y_j^{\,u}(t) = \sum_{m=1}^{M'} P'_{mj}\left[r_j^{\,a}(t)\right]_m$$

where M' is the number of independent groups of support points. P'_{mj} is the participation factor in mode j for the dependent support points in group m. This "group participation factor" is calculated as the algebraic sum of terms similar to (46) for all dependent support points in the group.

Two support points are said to be "at the same level" if the movements for all three global directions are parallel.

We have already defined a structure in which all support points are moving in parallel as a "single support level" structure. For such a structure, there are exactly three independent components of the support movement corresponding to the global directions X, Y and Z and we see from expression (47) that:

$$(48) \qquad y_j^{\,u}(t) = P_{xj}\left[r_j^{\,a}(t)\right]_x + P_{yj}\left[r_j^{\,a}(t)\right]_y + P_{zj}\left[r_j^{\,a}(t)\right]_z$$

Influence of Early Methods

Historically, the first dynamic analysis of piping systems subject to support movement loading was performed to construct bound solutions for earthquake loads.

Because it was necessary for code compliance purposes to separate primary and secondary terms, the dynamic problem was formulated in the u-space of relative coordinates.

Further, to simplify the numerical analysis for calculation by early computers, it was assumed that all support points were moving in parallel. The acceleration responses were not calculated but were taken from floor response spectra, so that the modal bounds in the u-space were calculated by:

$$(49) \qquad \hat{Y}_j^{\,u} = P_{xj}\, r_{xj}^{\,a} + P_{yj}\, r_{yj}^{\,a} + P_{zj}\, r_{zj}^{\,a}$$

where the P_{xj}, P_{yj}, P_{zj} are respectively X, Y and Z single level participation factors for mode j, calculated by the mass-weighted method previously discussed, and where

$r_{xj}^{\,a}$, $r_{yj}^{\,a}$, $r_{zj}^{\,a}$ are respectively X, Y and Z spectral accelerations corresponding to the natural frequency of mode j.

In the very early days, no rigid mode correction was made. Later, attempts were made to use the concept of "missing mass" to deal with this problem.

This early methodology influenced the future developments in the dynamic analysis of piping in the following ways:

- The major emphasis is on floor response spectrum methods. It is not obvious from many formulations of the dynamic problem that all important methods involved in solving dynamic problems are based on the formulation of the solution for the general time history problem.

- Some formulations do not explain the role of convolution integrals in solving the equations of motion.

- It is not obvious in some formulations that the mathematics underlying the rigid mode correction is identical for all types of dynamic analysis, including general and u-space time history analysis and single or multiple support level response spectrum analysis. There are still widely used piping analysis computer programs which do not include a valid rigid mode correction for some types of dynamic analysis. Few programs provide a method for determining the cut-off frequency ω_R and the limit theorem (18) is not well known.

- Although most piping analysis computer programs today provide a method for calculating multiple support level problems, the formulation in many cases is still based on single support level mathematics, using mass-weighted participation factors, using a definition of "included mass" which is based on single support level participation factors, and, in some cases, using a single support level rigid mode correction.

- It is not always apparent in the formulation of multiple support level u-space dynamic analysis problems that $u(t)$ is not the complete solution (or bound solution) for the equations of motion and that it is necessary to add a solution (or bound solution) for $v(t)$.

- There is a general lack of rigor in theoretical documentation, which is often ambiguous and sometimes wrong.

Summary

We may now summarize the steps described in this paper for solving the equations of motion by the method of modal superposition.

- Using the expression (4), the solution may be calculated once the mode shape vectors, e_j, and the scalar modal amplitudes, $y_j(t)$, have been determined.

- The $y_j(t)$ can be calculated by the convolution integral (13) provided that the applied load satisfies the condition $f(0) = 0$. It is shown how to use these methods even if $f(0) \neq 0$, provided the piping structure is initially at rest.

- The limit theorem (18) is derived from the convolution integral and this theorem together with the expression (9), permits the construction of an accurate approximation for the contribution to the system response of modes for which $\omega_j \geq \omega_R$, without actually extracting these modes. This approximation is constructed by applying the left out force defined in expression (23) as a static load. A method is given for determining ω_R. The concept of modal mass is not used.

- Two methods for generating bounds for the solution are discussed. One method consists of calculating a complete solution at each time step and retaining the maximum for each solution component. A second method is to calculate modal bounds by calculating or estimating bounds for the scalar modal amplitudes $y_j(t)$. There are several methods, which are outside the scope of this paper, for combining these modal bounds to construct a plausible bound for the total system response.

- A variation of these methods is used to calculate solutions for a special class of problems for which (1) the load consists of support movements and (2) the equations are solved in relative coordinates.

- For support movement loads, groups of supports which are moving in parallel are called "support levels". If all the support points are moving in parallel, the structure is said to be a "single support level" structure. The concept of a single support level structure, which comes directly from the original formulation of these problems more than 40 years ago, is probably no longer a useful engineering hypothesis. The general concept of support levels is not used in this paper.

- Finally, it is suggested that an increase in the level of mathematical rigor can lead to improved engineering solutions.