Tennessee Valley Authority
Division of Water Management
Water Systems Development Branch

TWO TECHNIQUES FOR FLOW ROUTING WITH APPLICATION TO WHEELER RESERVOIR

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ABSTRACT

Implicit and explicit numerical flow routing models are developed. Computational stability and efficiency of the techniques are discussed. Each model possesses relative advantages which requires that model selection be contingent upon the application. A case study is made using Wheeler Reservoir of the Tennessee River. Results of flow computations are presented and compared with field measurements.

INTRODUCTION

Flow routing in rivers and reservoirs has been accomplished in recent years using a wide variety of numerical techniques. Explicit and implicit finite difference, method of characteristics and finite element models have been developed for this purpose. In this report two finite-difference models will be discussed and applied to Wheeler Reservoir of the Tennessee River situated in north Alabama as shown in Figure 1. These techniques were explored in an effort to improve the accuracy and computational economy of the flow predictions used in the computerized technique which predicts the optimum cooling system operation for the TVA Browns Ferry Nuclear Plant (Reference 1).

The first model presented uses an explicit finite difference method which, like all explicit methods, is subject to stability constraints. However, this technique has proven to be stable at time steps approaching the Courant stability limit. Compared to other flow routing techniques, the model is computationally efficient.

The second model discussed uses an implicit finite difference solution technique. The time step of the computation with implicit techniques is not subject to Courant stability limitations. Time steps that are many times that of an explicit technique can be used. This model, as is typical of those using implicit methods, involves a simultaneous solution of the finite difference approximations to the conservation equations throughout the entire length of channel. The solution procedure is iterative, hence, each time step requires more computation than an explicit method. This disadvantage must be weighed against

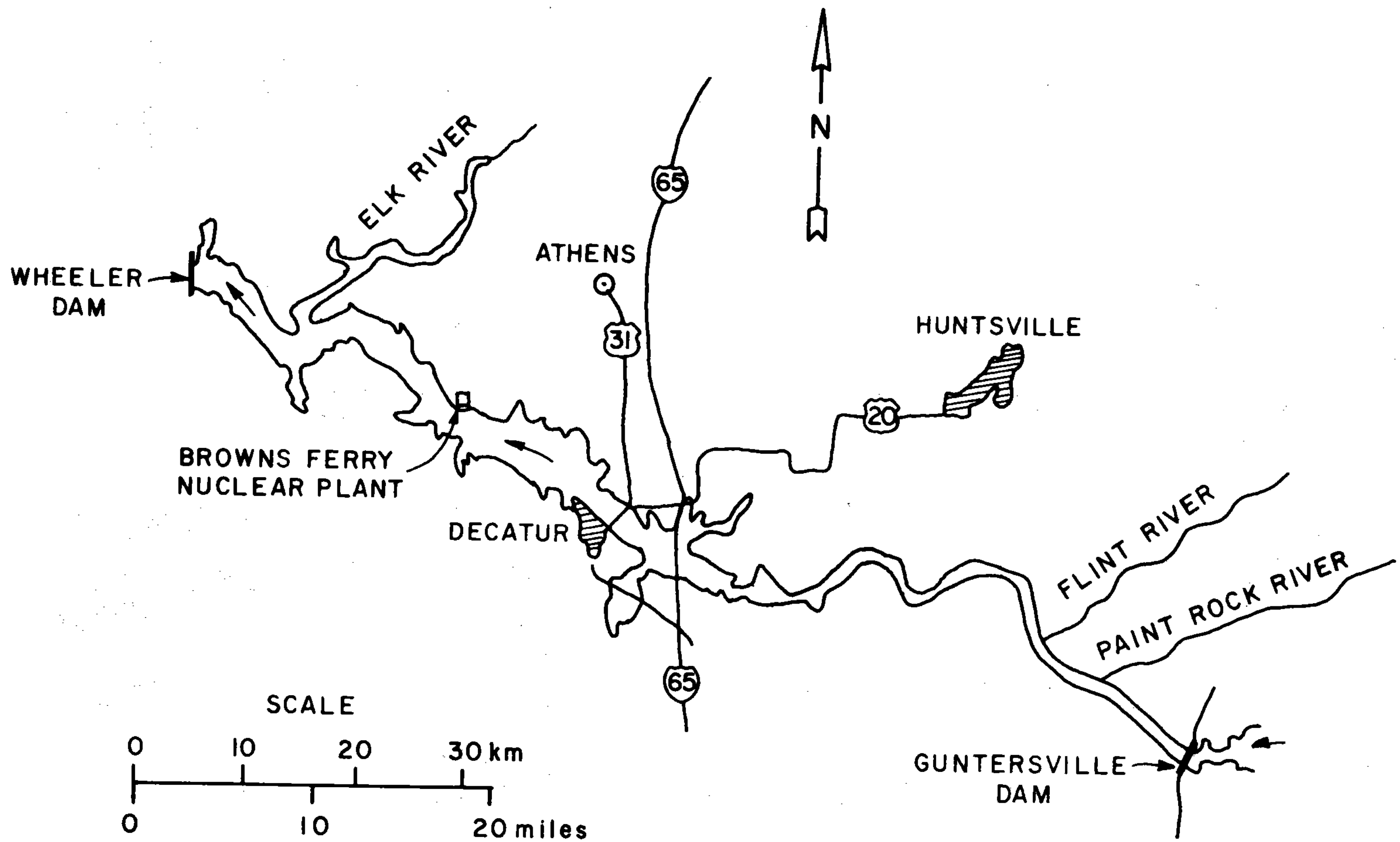

Figure 1: Location Map of Wheeler Reservoir

the obvious advantages of larger time steps, more flexibility in situating grid points and relatively little concern for computational stability.

Each of these numerical techniques for solving the conservation equations describing one-dimensional flow in rivers and reservoirs is discussed. Numerical stability considerations of each method are noted. Finally, computed flows and water surface elevations in Wheeler Reservoir are presented and compared with field measurements.

ONE-DIMENSIONAL CONSERVATION EQUATIONS

The equations to be solved in the two models are the conservation of mass and conservation of momentum equations for unsteady, gradually varying flow (Reference 2):

$$B\frac{\partial H}{\partial t}+\frac{\partial Q}{\partial x}-q=0 \qquad (1)$$

$$\frac{1}{A}\frac{\partial Q}{\partial t}+\frac{1}{A}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)+\frac{qV_x}{A}+g\left(\frac{\partial H}{\partial x}+S_f\right)=0 \qquad (2)$$

where

B = width of channel at the water surface

H = elevation of the water surface above a datum

A = cross-sectional area at a section

q = local volume inflow per unit time per unit length of channel

Q = volume flow rate

S_f = slope of the energy grade line

g = acceleration due to gravity

t = time

x = distance along the channel

V_x = x-component of velocity of the local inflow

R = hydraulic radius = A/wetted perimeter

Equations (1) and (2) were developed subject to the following assumptions:

a) One-dimensional incompressible flow

b) Uniform velocity throughout each cross-section

c) Free surface is level at each cross-section

d) Course of the river can be analyzed as a straight line; no river bends

e) Vertical pressure distribution is hydrostatic

f) Slope of the riverbed is small

g) Effects of friction and turbulence can be included as a resistance force which is a function of the square of the velocity and the depth of the stream (Manning equation)

The Manning equation used in each model to describe the slope of the energy grade line S_f is given as

$$S_f = \frac{Q|Q|n^2}{(1.486AR^{2/3})^2} \tag{3}$$

The inflows of tributary streams are distributed evenly over the reach in which they enter the reservoir.

SOLUTION OF THE CONSERVATION EQUATIONS

As discussed, two numerical techniques are presented for solving the conservation equations--an explicit and an implicit method. The explicit technique uses flows and water surface heights at a grid point and adjacent grid points along with a finite difference approximation to the governing equations to compute the flow and surface height at that point at a time Δt later. The implicit technique uses known (t_0) and unknown ($t_0 + \Delta t$) flows and surface heights at grid locations and a similar representation of the governing equations to determine the unknown flows and surface heights. The solution process differs from an explicit method in that the finite difference equations for all of the grid points at ($t_0 + \Delta t$) must be solved simultaneously. Both of the methods described in this report are accurate to second order in time and space, truncating terms of order Δt^3 (i.e., $0(\Delta t^3)$, and order of Δx^3 (i.e., $0(\Delta x)^3$).

Explicit Method

Computational Procedure

The explicit method presented in this report is an adaptation of a method developed for gas dynamics (References 3 and 4). This technique solves a finite-difference approximation to Equations 1 and 2 to determine H and Q at a time Δt later than when all conditions are known (t_o). This is accomplished by a two-step process for each location in the river where computations are performed.

The technique may be understood by first considering that Q and H will be computed at imax grid points (i.e., stations or river miles) along the reach of river or reservoir under investigation. Because two of these grid points must be positioned at upstream and downstream boundaries, then (imax -2) interior grid points are available. For this demonstration, the upstream and downstream boundaries correspond to locations of dams whose hourly average discharges, Q, are known. Also, the grid points are assumed to be evenly spaced by Δx, but this is merely for convenience in this demonstration.

At time t_o, all values of the dependent variables, Q(x,t) and H(x,t), must be known at each grid point i along the reach of the river. For the initial time step, Q(x,o) and H(x,o) must be assumed, but all subsequent values can be computed. These values of the dependent variables together with the boundary conditions are used to compute new values at a later time

$$t_1 = t_o + \Delta t$$

The procedure for computing new values of $Q(x,t_1)$ and $H(x,t_1)$ (which in subscript notation is Q_i^{t1} and H_i^{t1}) is best explained by considering the second order Taylor Series expansion of H (or Q),

$$H_i^{t_1} = H_i^{t_0} + \left(\frac{\partial H}{\partial t}\right)_i^{t_0} \Delta t + \left(\frac{\partial^2 H}{\partial t^2}\right)_i^{t_0} \frac{(\Delta t)^2}{2} + O(\Delta t)^3 \tag{4}$$

which can be approximated to the same order of accuracy as

$$H_i^{t_1} = H_i^{t_0} + \left(\frac{\partial H}{\partial t}\right)_i^{t_0} \Delta t + \left[\frac{\left(\frac{\partial H}{\partial t}\right)_i^{t_1} - \left(\frac{\partial H}{\partial t}\right)_i^{t_0}}{\Delta t}\right] \frac{(\Delta t)^2}{2} \tag{5}$$

Regrouping terms and simplifying, gives

$$H_i^{t_1} = H_i^{t_0} + \left[\left(\frac{\partial H}{\partial t}\right)_i^{t_0} + \left(\frac{\partial H}{\partial t}\right)_i^{t_1}\right] \frac{\Delta t}{2} \tag{6}$$

The $(\partial H/\partial t)_i^{t_0}$ can easily be computed from a finite differencing of Equation (1), but $(\partial H/\partial t)_i^{t_1}$ is unknown because conditions at t_1 are required.

A two-step procedure is used to evaluate Equation (6) for H_i^{t1} and an analogus expression for Q_i^{t1}. In the first step, provisional values of the dependent variables, denoted with a bar, $\bar{H}_i^{t1}$, are computed from the first order Taylor Series

$$\bar{H}_i^{t_1} = H_i^{t_0} + \left(\frac{\partial H}{\partial t}\right)_i^{t_0} \Delta t + O(\Delta t)^2 \tag{7}$$

and a forward finite difference approximation to Equation (1),

$$\left(\frac{\partial H}{\partial t}\right)_i^{t_0} = -\frac{1}{B}\left[\frac{\partial Q}{\partial x} + q\right] \approx -\frac{1}{B_i^{t_0}}\left[\frac{Q_{i+1}^{t_0} - Q_i^{t_0}}{\Delta x} - q_i^{t_0}\right] \tag{8}$$

Once provisional values, $\bar{H}_i^{t1}$ and $\bar{Q}_i^{t1}$, have been computed for all grid points, new values of H_i^{t1} and Q_i^{t1} are computed as the second step in the procedure. The provisional values are used to approximate $\left(\partial H/\partial t\right)_i^{t_1}$ in Equation (6), or

$$H_i^{t_1} = H_i^{t_0} + \left[\left(\frac{\partial H}{\partial t}\right)_i^{t_0} + \left(\frac{\partial \bar{H}}{\partial t}\right)_i^{t_1}\right]\frac{\Delta t}{2} \tag{9}$$

which, with Equation (7), can be rearranged for computational economy as

$$H_i^{t_1} = \frac{1}{2}\left[H_i^{t_0} + \bar{H}_i^{t_1} + \left(\frac{\partial \bar{H}}{\partial t}\right)_i^{t_1}\Delta t\right] \tag{10}$$

For the second step, a backward finite difference approximation to Equation (1) is used;

$$\left(\frac{\partial \bar{H}}{\partial t}\right)_i^{t_1} \approx -\frac{1}{\bar{B}_i^{t_1}}\left[\frac{\bar{Q}_i^{t_1} - \bar{Q}_{i-1}^{t_1}}{\Delta x} - q_i^{t_1}\right] \tag{11}$$

This backward difference combined with the forward difference in the first step provides the equivalent of second order spatial accuracy (i.e., terms of order $(\Delta x)^3$ neglected). This may be seen by substituting Equations (8) and (11) into Equation (9), the result of which is a pseudo central difference in the spatial direction. The sequence of forward and backward differences for the first and second steps is reversed for odd time steps as an additional suppression of numerical instability. The process is continued with values at t_1 used as t_0 for the next time step.

Stability Considerations

Any explicit finite difference technique used to solve Equations (1) and (2) will be limited in time step size by the theory of characteristics. As depicted in Figure 2, the explicit methods use information at grid points A, B, and C to determine conditions at P. Inside the triangle formed by the C+ and C- characteristics, values of the dependent variables rely upon conditions at points A, B, and C. This region is known as the "domain of dependence" of A, B, and C. Outside of the triangle, this relationship does not hold and the finite difference equations will produce erroneous results.

The C+ and C- characteristics trace surface wave motion on the x-t (distance-time) plane (Reference 5). In subcritical flow, disturbances from a given point travel upstream and downstream. Characteristics can be constructed from any point in the plane and will map out the propagation of a disturbance from that point both up (C-) and down (C+) the channel with time.

The grid spacing ratio $\Delta t/\Delta x$ must be small enough to insure that P is within the triangle denoted in Figure 2. This Courant stability criteria governing the computation of explicit techniques can be derived from the theory of characteristics as

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{V+C} \tag{12}$$

where v is the speed at which the water is flowing and c is the free surface wave speed defined as

$$C = \sqrt{g y_m} \tag{13}$$

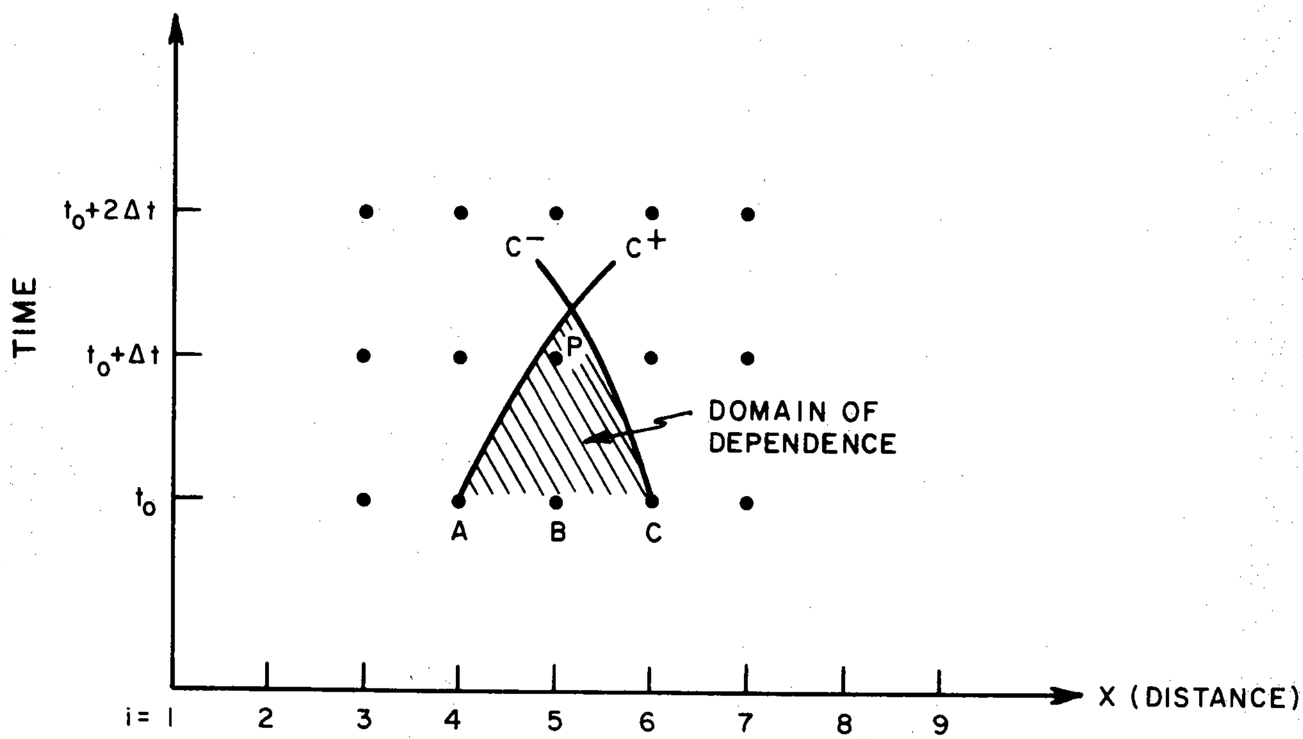

Figure 2: Stability Considerations for Explicit Finite Difference Methods (X-t Plane)

where y_m is the mean depth of the river in the critical reach (i.e. having the greatest depth).

It is important to use as large a value of $\Delta t/\Delta x$ as possible in an unsteady flow computation to ensure that the numerical wave speeds are good approximations to the physical wave speeds. A stable numerical technique will permit the use of a time step very close to the stability limit to promote accuracy and economy of the computation. With the exception of time step size, identical cases were solved with the explicit model, and the flow results at the Browns Ferry Nuclear Plant are given in Figure 3. The calculation which used the three-minute time step exhibited numerical dispersion. This dispersion introduced a time lag and had an attenuating effect on the flow extremes. The stability limit for the sectioning scheme used was calculated as 21.4 minutes. Time steps of 20 minutes produced stable results and were used in all subsequent computation.

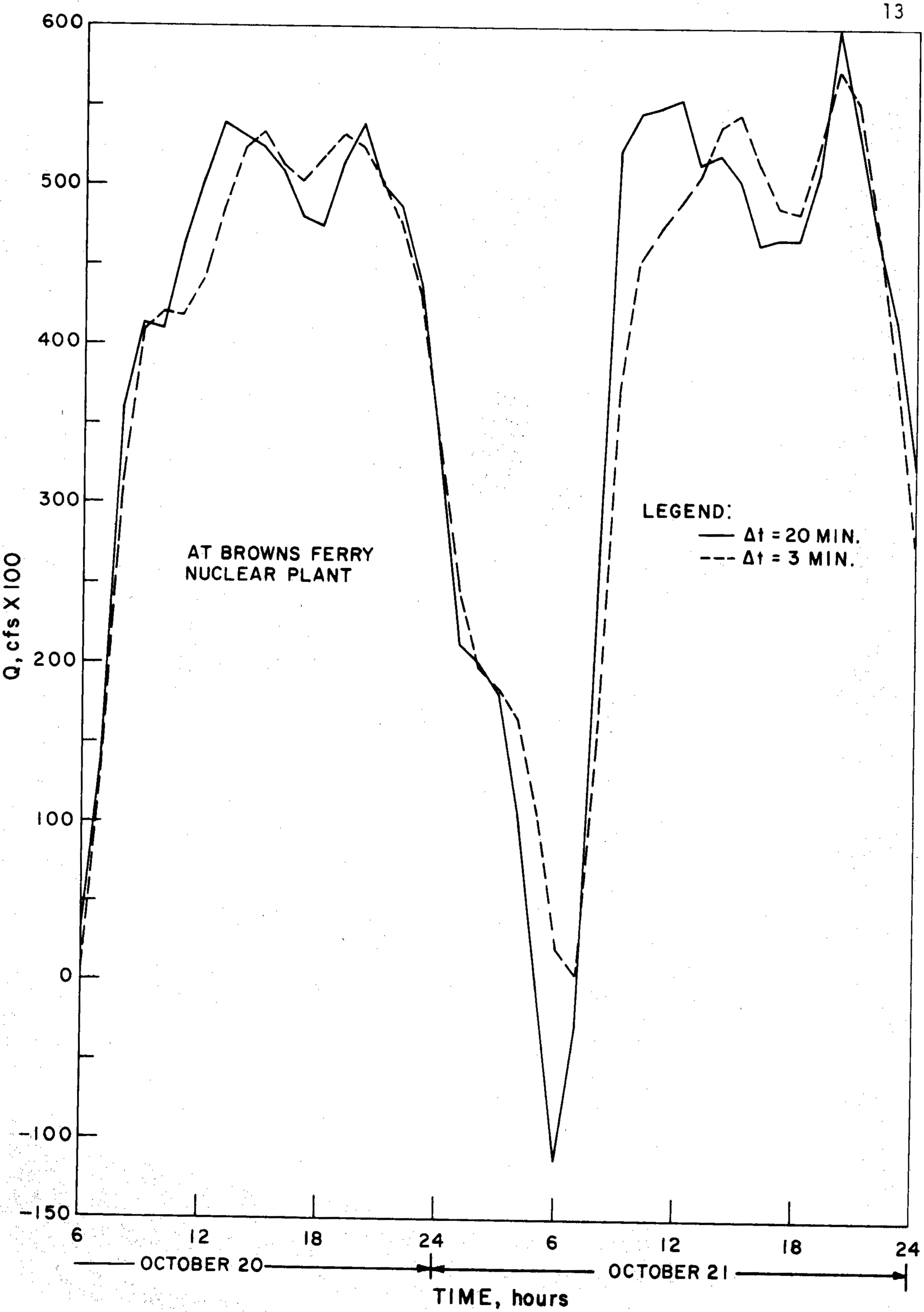

Figure 3: Effect of Time Step Size on Explicit Computation

Implicit Method

Computational Procedure

The finite difference form of Equations (1) and (2) in the implicit procedure is associated with a reach instead of a node as it was in the explicit technique. Using subscript and superscript notation, the H (or Q) derivatives are approximated by:

$$\frac{\partial H}{\partial t} = \left[\frac{\left(H_{i+1}^{t1} + H_{i}^{t1}\right)}{2} + \frac{\left(H_{i+1}^{t0} + H_{i}^{t0}\right)}{2}\right]\frac{1}{\Delta t} \tag{14}$$

$$\frac{\partial H}{\partial x} = \left[\left(1-\theta\right)\left(H_{i+1}^{t0} - H_{i}^{t0}\right) + \theta\left(H_{i+1}^{t1} - H_{i}^{t1}\right)\right]\frac{1}{\Delta x} \tag{15}$$

with θ being a weighting factor. The allowable range of θ's will be discussed in the section dealing with stability of this procedure. In this application, Δx was a variable, but if advantageous, Δt could also vary.

As previously mentioned, the implicit technique requires a simultaneous solution of the unknown discharges and water surface elevations throughout the study reach at a given time. The reason for this is that in an individual reach four unknowns appear (H_i^{t1}, Q_i^{t1}, H_{i+1}^{t1}, Q_{i+1}^{t1}), but only two equations, the conservation of mass and momentum, are available. Overlapping unknowns between reaches and boundary conditions bring the number of equations into balance with the unknowns, and the system is determinate by simultaneous solution. A further complication which enters is that the system of equations is nonlinear, and an iterative technique such as the Newton-Raphson method must be invoked. Written for the case of known discharges at the boundaries, the system of equations can be expressed as

$$f_1\left(H_1^{t_1}, H_2^{t_1}, Q_2^{t_1}, \ldots H_{imax-1}^{t_1}, Q_{imax-1}^{t_1}, H_{imax}^{t_1}\right) = 0$$

$$f_2\left(H_1^{t_1}, H_2^{t_1}, Q_2^{t_1}, \ldots H_{imax-1}^{t_1}, Q_{imax-1}^{t_1}, H_{imax}^{t_1}\right) = 0$$

$$f_{2(imax-1)}\left(H_1^{t_1}, H_2^{t_1}, Q_2^{t_1}, \ldots H_{imax-1}^{t_1}, Q_{imax-1}^{t_1}, H_{imax}^{t_1}\right) = 0 \tag{16}$$

or

$$f_i(X) = 0, \; i = 1, 2, \ldots, 2(imax-1)$$

The object of the Newton-Raphson procedure is to find the values of the unknowns at each new time step so that the nonlinear equations (16) are satisfied. Suppose that at the <u>kth</u> iteration the approximate solution, X_k, does not satisfy Equation 16 within an allowable tolerance. A differential change in f_i is

$$df_i = \frac{\partial f_i}{\partial X_1} dx_1 + \frac{\partial f_i}{\partial X_2} dx_2 + \ldots + \frac{\partial f_i}{\partial X_{2(imax-1)}} dx_{2(imax-1)} \tag{17}$$

Finite changes Δf_i can be approximated by changing dx_i to Δx_i in Equation 17. Hence, the approximate changes to be made to the solution, X_i, can be found by solving the following linear system

$$\begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \cdots & \frac{\partial f_1}{\partial X_{2(imax-1)}} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \cdots & \frac{\partial f_2}{\partial X_{2(imax-1)}} \\ \vdots & & & \\ \frac{\partial f_{2(imax-1)}}{\partial X_1} & \frac{\partial f_{2(imax-1)}}{\partial X_2} & \cdots & \frac{\partial f_{2(imax-1)}}{\partial X_{2(imax-1)}} \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \vdots \\ \Delta X_{2(imax-1)} \end{bmatrix} = \begin{bmatrix} -f_1(\bar{X}_K) \\ -f_2(\bar{X}_K) \\ \vdots \\ -f_{2(imax)}(\bar{X}_K) \end{bmatrix} \tag{18}$$

The Taylor series expansion of f_i (X_{k+1}) has been truncated after the linear correction term to arrive at Equations (18). For a single equation having only one unknown this is equivalent to solving

$$f\left(X_{K+1}\right) = f\left(X_K\right) + \frac{\partial f}{\partial X} \Delta X_K + O\left(\Delta X_K\right)^2 \qquad X_{K+1} = X_K + \Delta X_K \tag{19}$$

When the proper X has been found

$$f(X) = 0 \tag{20}$$

Applying the ΔX_i correction factors to the previous solution X_k, a test of convergence is made. If the required tolerance is not met, the procedure is repeated with X_{k+1} as known values and X_{k+2} to be determined. Convergence is usually achieved rapidly, but a drawback of the procedure is that the initial guess, X, must be reasonable. If care is not taken, the solution may converge to an improper result or diverge. Values of the unknowns at the previous time have been used in this capacity, and the problem has been avoided.

Formulation of the implicit model using the Newton-Raphson procedure results in a coefficient matrix having a band structure. The IBM subroutine GELB was used for solving this matrix. When convergence is achieved, the solution is advanced in time and the process is repeated. Variations in the friction term, S_f, with water surface elevation were not considered when forming the matrix in Equation (18).

Stability Considerations

Setting the weighting factor, θ in Equation (15), equal to zero simplifies the expression but removes the implicitness of the procedure. An explicit procedure having the time step limitations discussed above is the result. The intuitive value of $\theta = 0.5$ causes equal weighting to be given the X-derivatives at the new and old time steps. Stability problems often arise, however, when this θ is used (References 6 and 7). With $\theta = 1$, the equations and computations are also simplified and the solution is termed fully implicit.

The range of θ available to an implicit technique is $0.5<\theta\leq1.0$. Reference 6, 7 and 8 present consequences of various choices within this range for a number of case studies. Large values of θ result in a more stable but smoothed solution. Computational simplicity and stability achieved when using the fully implicit formulation must be balanced against the ability of less implicit schemes to simulate more sharply defined transients. A θ value of about 0.55 results in an optimal balance for many cases that have been reported.

When frictional resistance suddenly becomes an important term in the momentum equation, a larger θ may be required to maintain solution stability. For example, if at Guntersville Dam, the upstream boundary of the modeled reach, the flow release is rapidly increased, a numerical instability occurs at the adjacent section when $\theta = 0.55$ is used. If θ is increased temporarily, stability is maintained. In the present model θ is increased automatically when a large positive discharge fluctuation occurs at the upstream boundary. After passage of the initial wave, θ is reset to 0.55 and the computation proceeds. A comparison of computed flows in Wheeler Reservoir between the variation of θ technique described and the fully implicit procedure, $\theta = 1.0$, at the Browns Ferry Nuclear Plant are given in Figure 4.

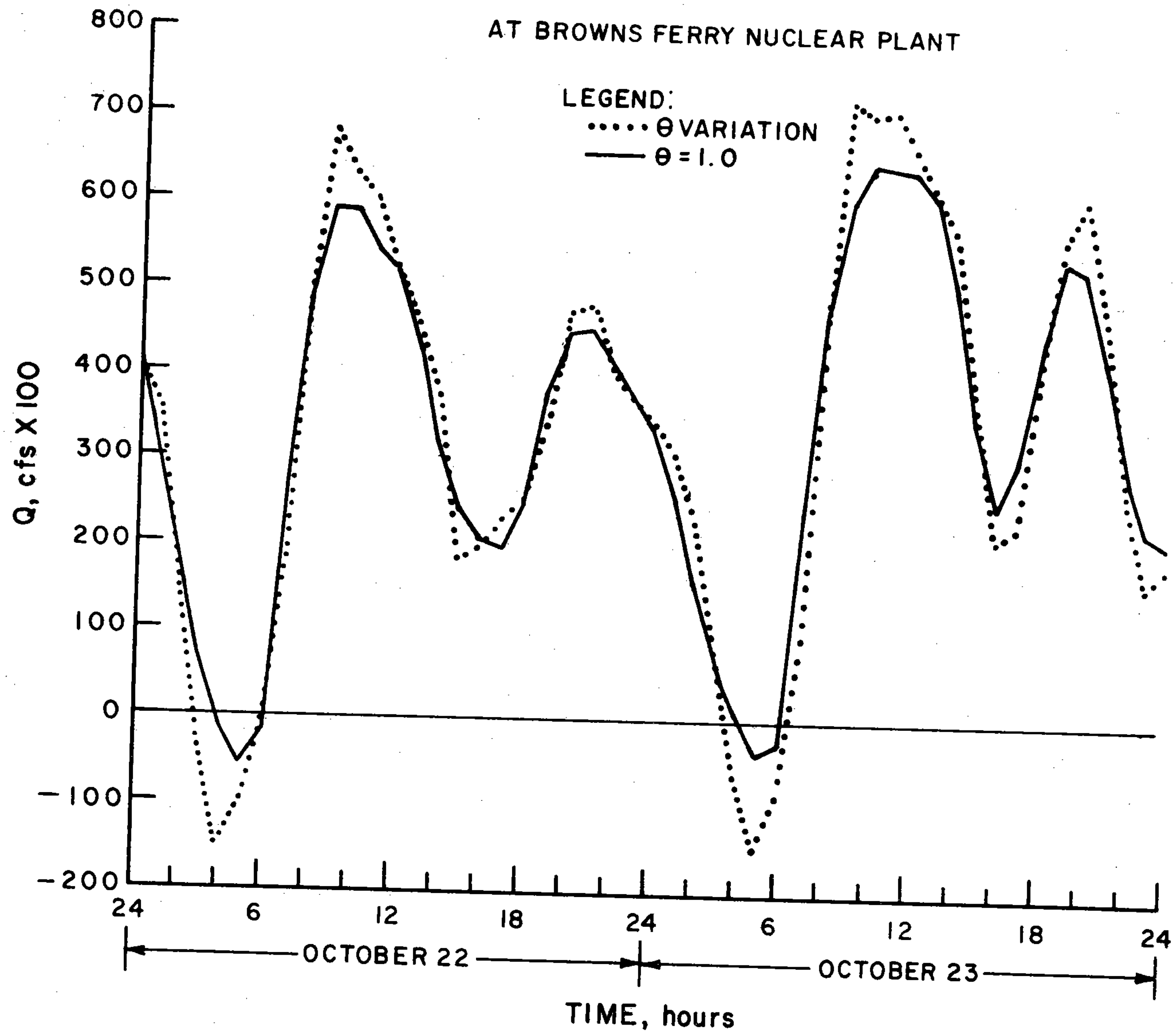

Figure 4 : Effect of Θ on Implicit Computation

RESULTS

Wheeler Reservoir flow simulations for the period of November 20-24, 1976, were conducted using the explicit and implicit models. The 74-mile (119-kilometer) long reservoir was modeled using eight reaches. In each case, information was desired at and downstream of the Browns Ferry Nuclear Plant. Numerical stability and dispersion are additional considerations when sectioning and selecting the computational time step to be used in the explicit model. The maximum computational time step is limited by the fast wave transmission in the deep, lower portion of the reservoir. A compromise between these considerations resulted in the equally spaced sectioning scheme shown in Figure 5a. Sectioning for the implicit model is not constrained by Courant stability limitations. Information requirements can be conveniently treated (Figure 5b).

Boundary conditions used in each model were the hourly average flow releases at Guntersville (Tennessee River Mile [TRM] 349.0) and Wheeler (TRM 275.0) Dams (Figure 6). The computation was begun on October 20, starting from a fictitious zero-flow-flat-pool initial condition. The initial twenty-four hours of computation were discounted to eliminate the effect of the initial condition. Results are given for the period of October 21-23.

The measured and computed water surface elevations at the Guntersville Dam tailwater are shown in Figure 7. Reservoir stage comparisons are given in Figure 8 at the Browns Ferry Nuclear Plant and the Wheeler Dam headwater. The agreement with measured values achieved by these models is excellent. A slight tendency toward in-

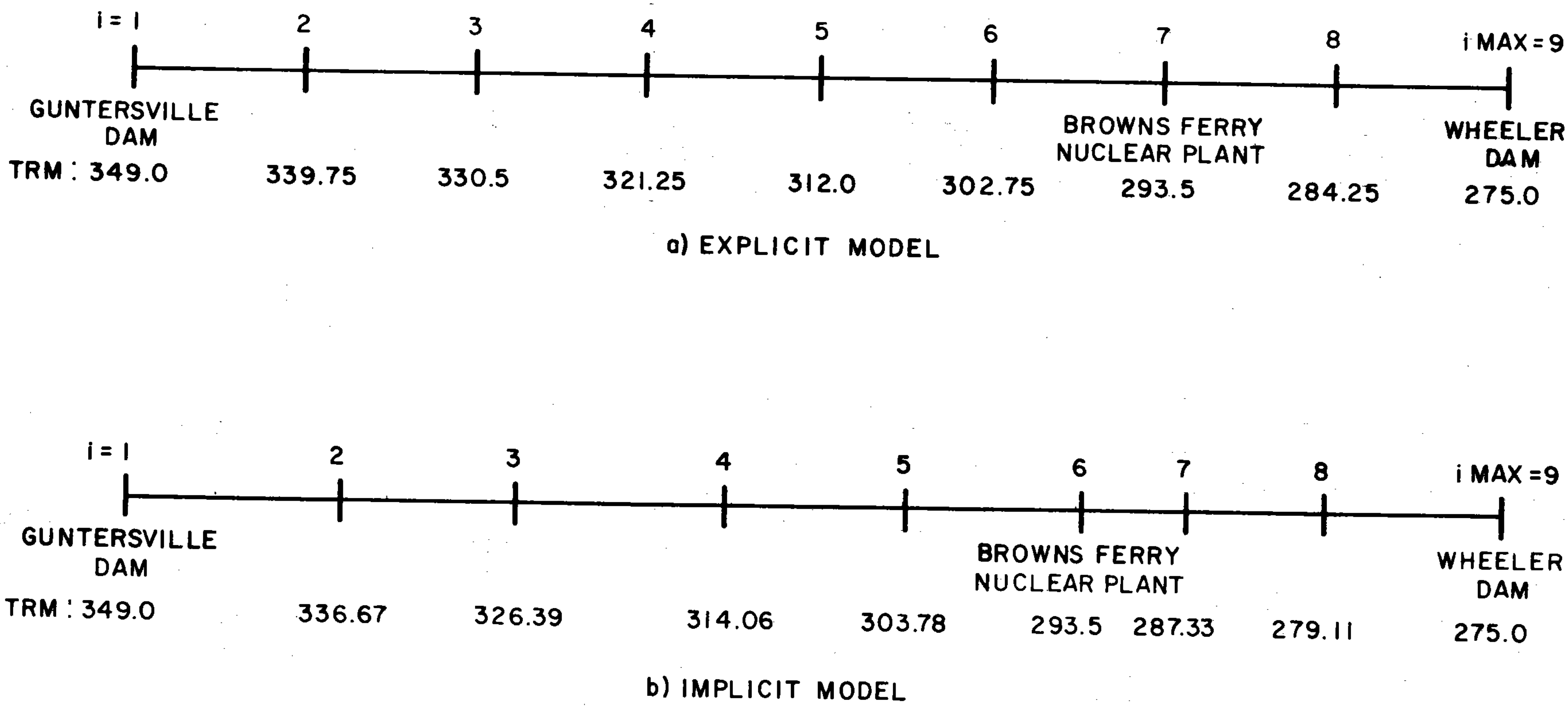

Figure 5 : Sectioning of Wheeler Reservoir Used in Finite Difference Models

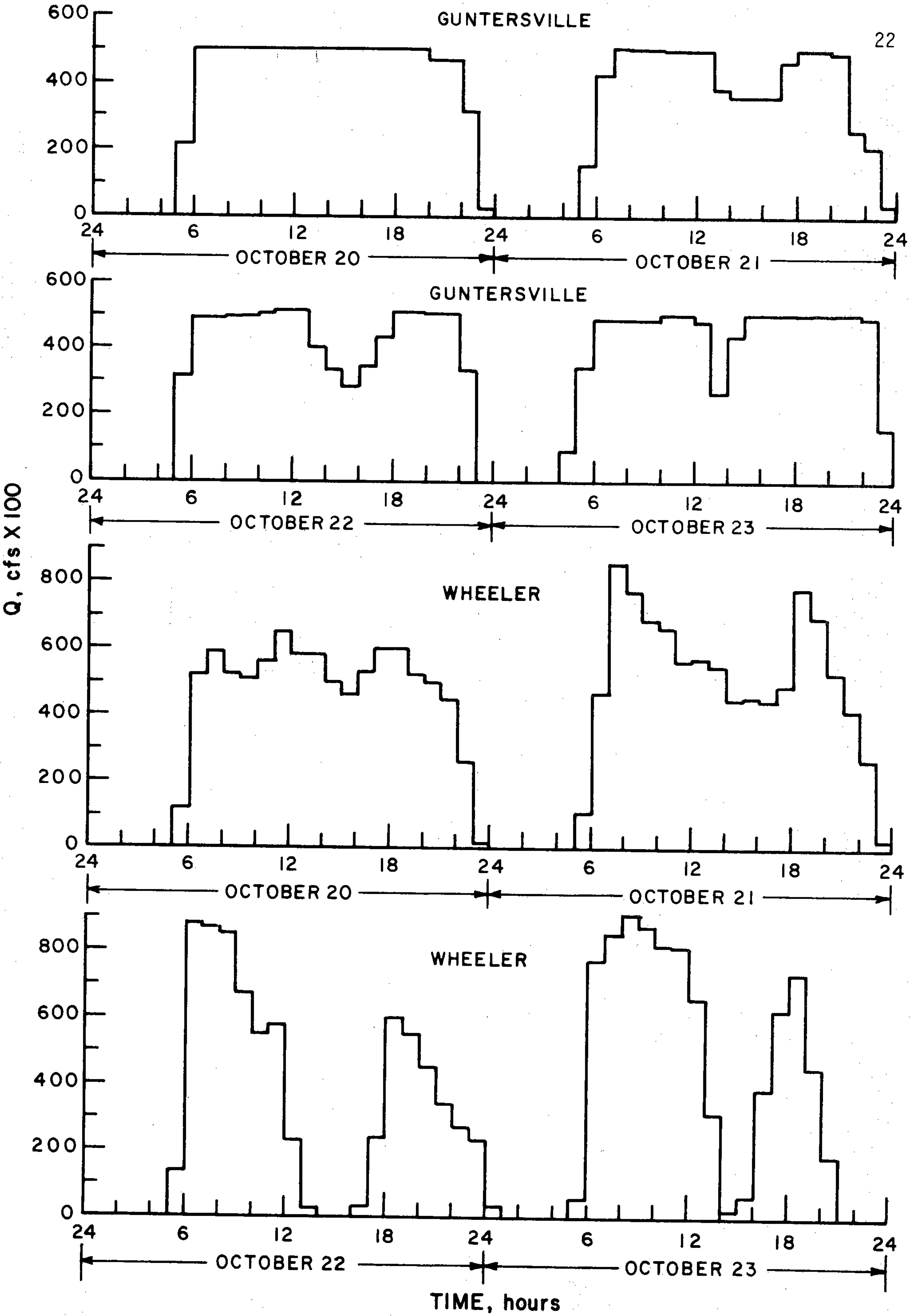

Figure 6 : Flow Releases at Guntersville and Wheeler Dams
October 20–23, 1976

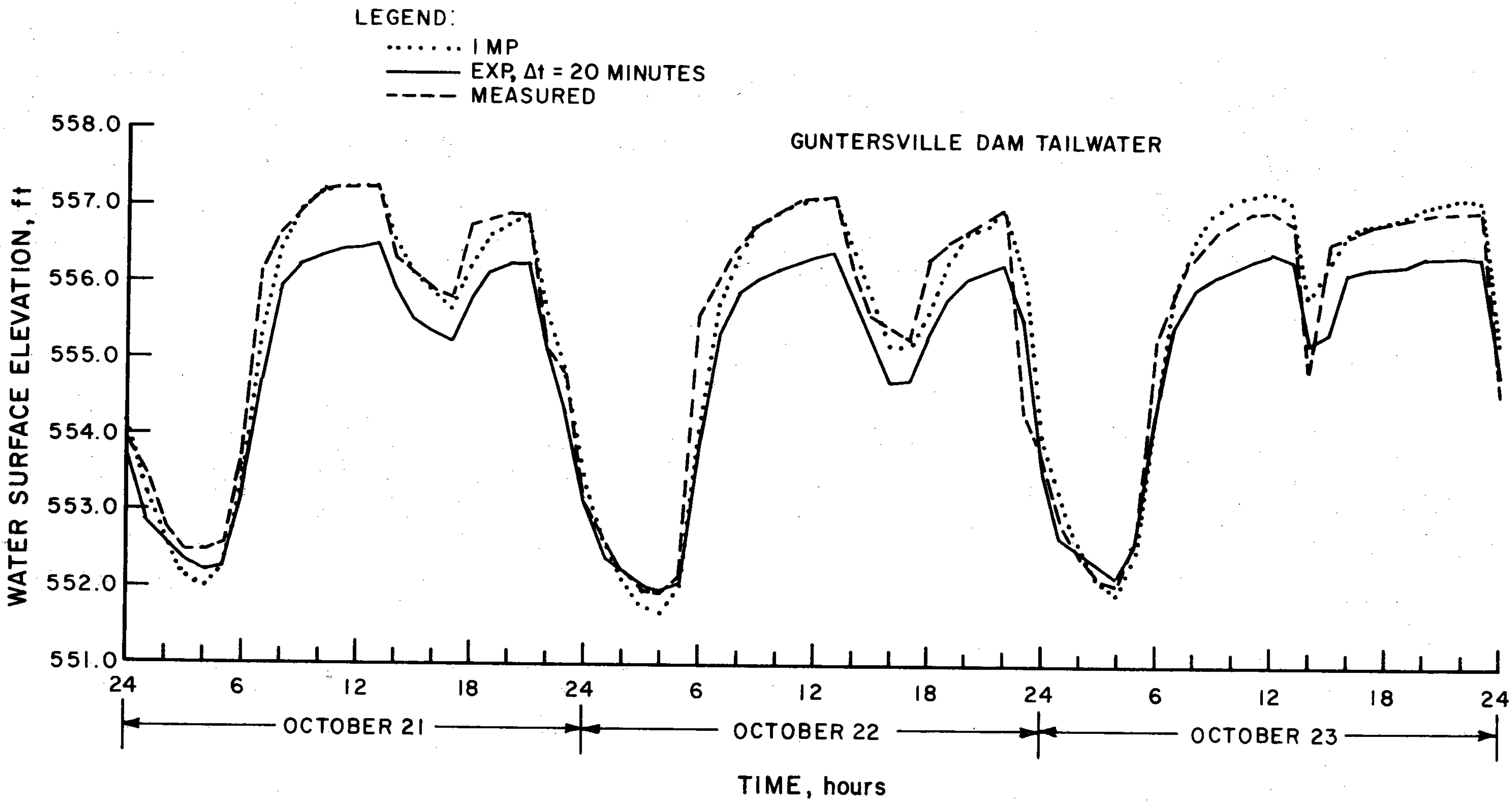

Figure 7 : Comparison of Measured and Computed Water Surface Elevations

stability can be noted in the Wheeler Dam headwater computation by the explicit method. Resectioning locally or reduction of the computational time step are methods which could be used to stabilize the results.

Computed flow in the reservoir at the Browns Ferry Nuclear Plant is given in Figure 9. Flow data for the study period were not available at any location in the reservoir. Information on flow direction at Browns Ferry is available for the period of October 22-28, 1976, and revealed that on October 22, upstream flow was present between 0400 and 0700 hours. Another flow reversal began at 0330 and remained until 0700 hours on October 23. On each of these days, downstream flow persisted at all other times. These data were most closely simulated with the implicit model.

Generally, the computed flows at the plant using each model are in agreement. Magnitude of the flow peaks and timing of the flow troughs are points of discrepancy. Computations in upstream reaches of the explicit model are being made at a small percentage of the stability limit. As was shown in Figure 3, lagging and rounding of the computed hydrograph can result. Experiments with the models revealed that the implicit model propagated waves more quickly in the upper portion of the reservoir than did the explicit model. An hour lag at Browns Ferry was fairly typical of the explicit model in response to a flow change at Guntersville Dam. The error in timing of the trough as computed by the explicit model can be attributed to this dispersion affect. Improved model performance could be expected if more sections were placed in the upper portion of the reservoir. The three major flow peaks were the result of large and sudden flow increases at Wheeler Dam. Under these conditions the implicit model exhibited a

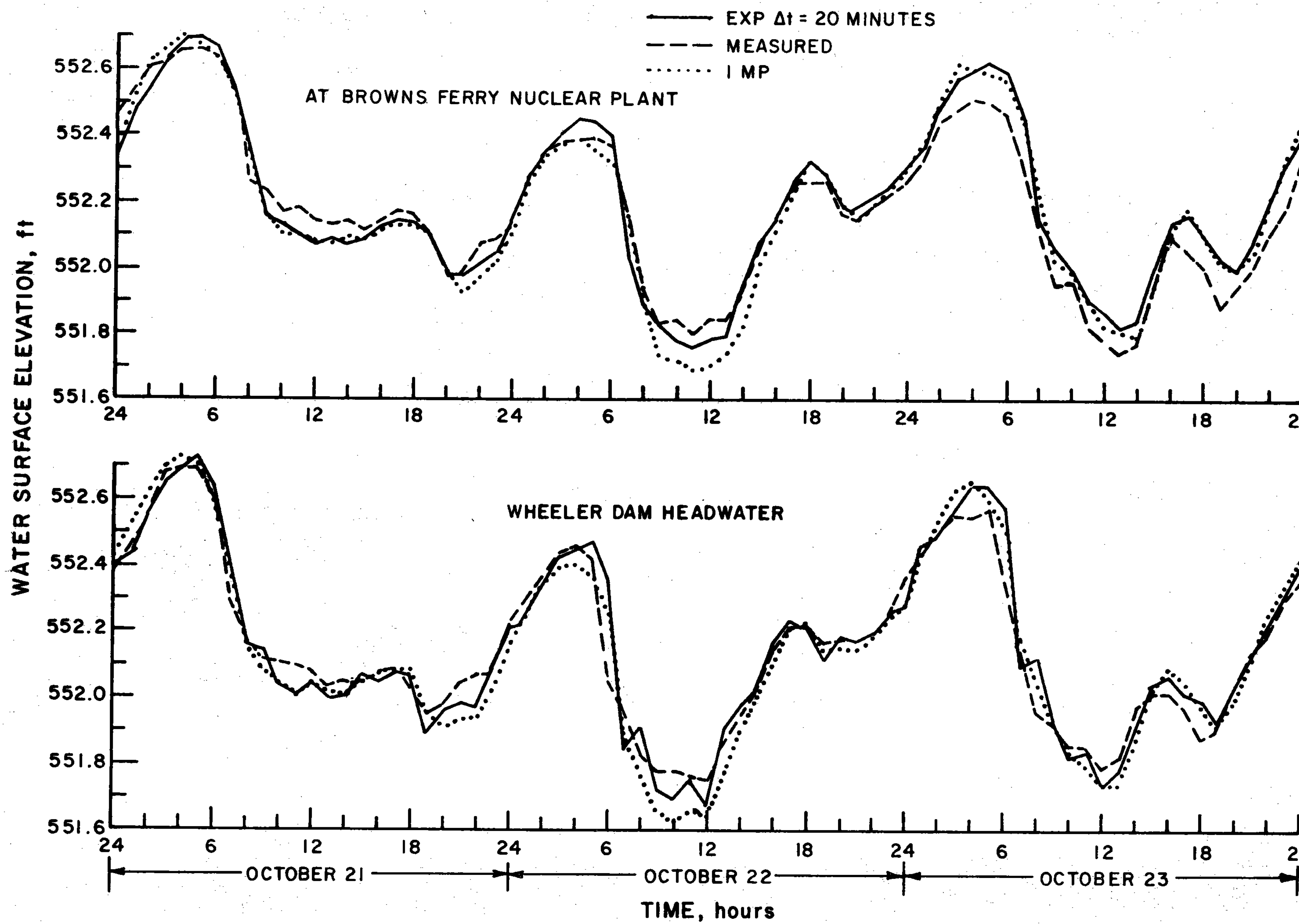

Figure 8: Comparison of Measured and Computed Water Surface Elevations

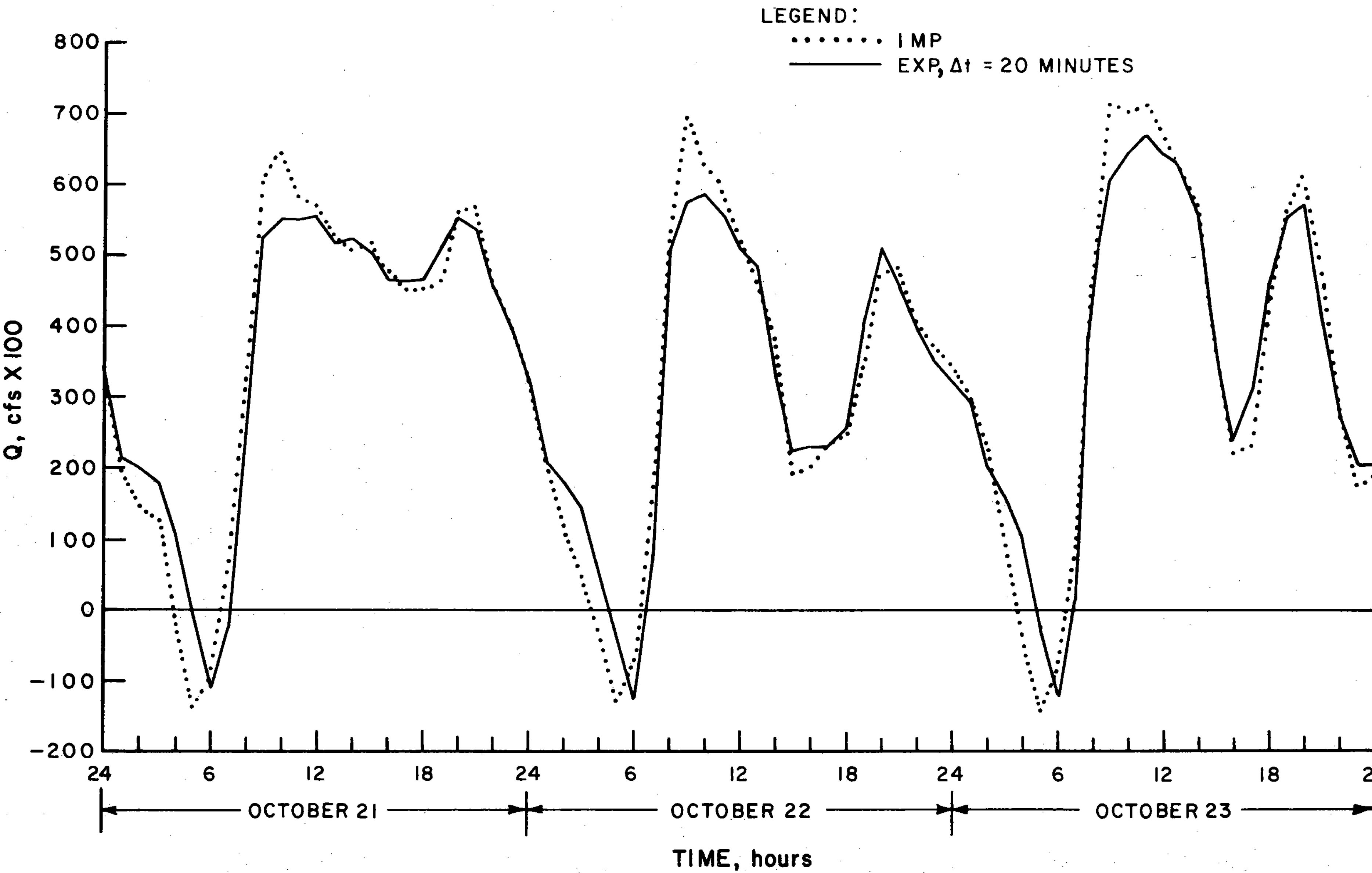

Figure 9 : Comparison of Computed Flow at Browns Ferry Nuclear Plant

slight tendency toward instability in the computed flows. It is suspected that these peaks are being overpredicted as a result. A temporary increase in Θ would likely resolve this problem.

CONCLUSIONS

Each of the flow routing techniques described in this report have advantages which are problem-dependent. The primary advantages of the explicit model are its small core requirements and computational simplicity. If computer limitations exist or if the problem being modeled is large (i.e., simulations for a large system or an extended time period), the size and economy of the model become important model selection considerations. Hydroelectic operations for which flow releases are typically rapidly varying and known on an hourly time scale, is a case where the explicit model is computationally more efficient relative to other routing techniques.

Advantages of the implicit model include flexibility in selection of the computational time step and placement of sections. Detailed information in a certain reservoir reach can be obtained by concentrating a large number of sections in the reach. Consideration of time step and associated computational stability is not a limitation. The implicit model achieves relative computational economy for cases where the prescribed boundary conditions are not rapidly varying. Flood movement in natural streams is an example of such a case.

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