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AN OVERVIEW OF METHODS

FOR

DISCONTINUUM ANALYSIS

by

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INTRODUCTION

Discontinuum analysis of rock masses has evolved to a state where several different numerical methods are presently available. It is difficult to distinguish one method from another, and the techniques tend to overlap. Nevertheless, there are distinguishing features of the various discontinuum analysis techniques, and it is important to recognize the distinctions when choosing a method for a particular analysis. A summary review of discontinuum analysis methods is presented herein to assist in defining the capabilities and limitations of a particular method for a specific problem.

The distinguishing feature of a discontinuum analysis method compared to a continuum analysis approach is that the discontinuum method explicitly represents the discontinuities of the rock mass in the numerical formulation. Continuum methods account for the presence of discontinuities by an "equivalent continuum" representation. At the present time, equivalent continuum models can only give a limited representation for the behavior of jointed rock (i.e., these models cannot fully account for the various displacements associated with jointed media, such as sliding, separation, and rotation along joints). Recent advancements in material models for equivalent continuum [e.g., Muhlhaus (1988) and Pariseau (1988)] suggest that the development of more representative equivalent continuum models for jointed rock is quite possible. This area of research shows great potential and may eventually provide a solution to the basic problem of discontinuum analysis—i.e., the difficulty with modeling every joint in the rock mass.

Discontinuum analysis methods fall into two general groups. The first group consists of continuum codes which have been modified to represent discontinuous features explicitly. Special algorithms, commonly called interface elements or slidelines, are incorporated in these codes to simulate the presence of the features. The second group consists of discontinuum codes which are numerical techniques designed specifically to analyze the behavior of discontinuous or particulate systems. Several different numerical schemes have been utilized in the development of the codes in each of these groups but, in general, computational procedures used in both groups have similar features. The similarity is primarily in the type of computational solution method incorporated in the code.

SOLUTION METHOD

Table 1 presents a categorization of the various techniques according to computational solution method. As shown in this table, two general methods are used in both groups to solve for the mechanical motion of discontinuum systems: the <u>implicit</u> mechanical solution approach, and the <u>explicit</u> mechanical solution approach. Each approach provides a different method to solve mechanical equilibrium equations for either static or dynamic analysis. These approaches are commonly used in continuum analysis, as well.

Table 1

NUMERICAL CODES FOR DISCONTINUUM ANALYSIS

CONTINUUM CODES WITH <u>INTERFACE ELEMENTS</u>		<u>DISCONTINUUM CODES</u>		
implicit mechanical solution	explicit mechanical solution	kinematic stability	implicit mechanical solution	explicit mechanical solution
<u>EXAMPLES</u>:		<u>EXAMPLES</u>:		
*finite element/finite difference method with slidelines		*key-block method	*discontinuum deformation analysis	*distinct element method
*boundary element (e.g., displacement discontinuity method)				

In the implicit approach, the equations describing the motion for all elements in the problem are solved simultaneously. For a linear-elastic, static analysis, the implicit solution is performed once but, for non-linear problems, several iterations of the complete set of equations may be required to converge to the equilibrium solution state. For non-linear dynamic analysis, the implicit scheme requires convergence to a solution state at each timestep. The timestep can be arbitrarily large with regard to numerical stability, but can still be restricted by the path dependency of the non-linear behavior of the system.

The system of equations in the implicit method are solved directly using a standard matrix method—for example, Gauss elimination or a similar method. The solution method accounts for non-linearities by using an iterative procedure, such as the modified or unmodified Newton-Raphson method. The implicit approach is not well suited for problems that involve frequent changes to the connectivity between elements from, for example, highly non-linear behavior or dynamic loading. This is because the stiffness matrix must be reformulated every time a change in connectivity occurs.

In the explicit approach, unknown values of the variables relating to each element in the problem are calculated from known values in that element and its immediate neighbors. The equations relating these values are solved locally for each timestep*—there is no need to solve a complete system of equations. The reason for this independence of equations can be understood by considering that, in a physical system, there is a maximum speed at which information can propagate. For example, in an elastic solid, this speed corresponds to the compressional wave speed. In the explicit approach, the calculational timestep is selected sufficiently small that information cannot propagate further than one element during one timestep. If this timestep restriction is always satisfied, then the dependence of each element on its immediate neighbors is fulfilled, and the equations for each element can be solved independently. The equations are solved in

*This "timestep" may be a physically realistic timestep for dynamic analysis or a calculational increment progressing to an equilibrium state for a static analysis.

the explicit approach by direct integration+ using a numerical differencing scheme. The central difference method is generally preferred over other differencing schemes because it is second-order accurate.

The timestep limitation in the explicit approach restricts the computation efficiency for solving linear problems because many calculational timesteps may be required to reach the equilibrium state. However, for non-linear analyses with an explicit program, there is little appreciable increase in computer time over the linear analysis, whereas an implicit program becomes much less efficient and may take several iterations to reach the solution, solving the complete system of equations at each step. The explicit approach, in this instance, proves more advantageous, particularly when the non-linear behavior is associated with a dynamic analysis.

The explicit solution approach is directly suitable for dynamic analysis because the explicit time-marching scheme provides a reliable and efficient means for performing transient calculations. The method has also been adapted for static and quasi-static calculations by the use of two techniques, dynamic relaxation (Otter et al., 1966) and the conjugate gradient method (Consus et al., 1965), which facilitate the convergence to a static equilibrium or steady-state failure (collapse). In dynamic relaxation, the nodes of each element are moved in accordance to Newton's law of motion while, in the conjugate gradient method, convergence is achieved on the basis of a numerical iterative technique which does not involve reformulation of a stiffness matrix.

+For dynamic calculations, an alternative approach to direct integration for either implicit or explicit solution is a method called modal superposition [see, for example, Bathe and Wilson (1976)]. In this method, the equations of motion are first transformed into a generalized system of linear displacement equations known as a generalized eigenproblem. The eigenproblem yields n eigensolutions or n modes of displacement. Once these modes are known, they can be incorporated directly into each element and represent the response to loading. When many timesteps are required in the dynamic analysis, this approach can be more effective than direct integration. However, the approach is limited to problems involving linear behavior of the elements.

Cundall (1987) has shown that dynamic relaxation is better suited to model non-linear problems near failure than are iterative methods and can model collapse problems in a more realistic and efficient manner. With the conjugate gradient method, convergence for non-linear problems, particularly at failure, is not always guaranteed. While convergence is more certain with dynamic relaxation, the damping of inertial motion in this approach can still cause difficulties with problems at the collapse state because the viscous damping can introduce body forces which retard steady-state collapse. Cundall (1982 and 1987) describes the use of adaptive damping as an effective method to overcome this difficulty in dynamic relaxation. Adaptive damping continuously adjusts the viscosity such that the power absorbed by damping is a constant proportion of the rate of change of kinetic energy in the system. Therefore, as the kinetic energy approaches a constant or zero, the damping power also tends to zero.

Finally, because the explicit approach effectively "freezes" the strain-state of each element at each timestep, the non-linear material behavior can be followed directly, in incremental form, without the need for iterations. It is generally recognized by developers of non-linear constitutive models for numerical codes that the explicit procedure is more tractable than the implicit approach for constitutive model implementation (St. John, 1988). A comparison between explicit and implicit solution approaches is given in more detail by Cundall et al. (1980). Table 2 summarizes the principal distinctions between the two approaches.

Several continuum codes with interface elements have been developed using both the implicit and explicit approaches. Appendices I and II, respectively, contain partial lists of references for currently available codes of each type. Most of these codes are based on finite element or finite difference formulations and a trend toward the explicit scheme is seen in those codes applied to the analysis of geologic systems because of the complexity of the non-linear constitutive models. Both implicit and explicit boundary element codes are also available to model the response of media with distinct discontinuities. Implicit boundary element programs are generally used for static analysis while, for dynamic calculations, an explicit approach is often taken wherein a limiting timestep is used.

Table 2

COMPARISON OF CHARACTERISTICS OF EXPLICIT AND IMPLICIT SCHEMES

[Cundall et al., 1980]

EXPLICIT	IMPLICIT
Time-step must be smaller than a critical value for stability.	Time-step can be arbitrarily large, with unconditionally stable schemes.
Small amount of computational effort per time-step.	Large amount of computational effort per time-step.
No significant numerical damping introduced.	Numerical damping dependent on time-step present with unconditionally stable schemes.
No iterations necessary to follow nonlinear constitutive law.	Iterative procedure necessary to follow nonlinear constitutive law.
Provided that the time-step criterion is always satisfied, nonlinear laws are always followed in the correct physical way.	Always necessary to demonstrate that the above mentioned iterative procedure is a) stable b) follows the physically correct path (for path-sensitive problems).
Matrices are never formed. Memory requirements are always at a minimum. No bandwidth limitations.	Stiffness matrices must be stored. Ways must be found to overcome associated problems such as bandwidth. Memory requirements tend to be large.
Since matrices are never formed, large displacements and strains are accommodated without additional computing effort.	Additional computing effort needed to follow large displacements and strains.

Although not as many discontinuum codes exist as do continuum codes, several are available or currently under development. Most of the discontinuum codes are based on a direct integration, explicit solution scheme and are described as, generically, the distinct element method. A partial list of references to explicit distinct element codes is provided in Appendix III. One alternative form of distinct element method based on a solution achieved by modal superposition has also been developed [see, for example, Williams et al. (1985) in Appendix III]. Implicit discontinuum codes have only recently been developed and are not at as advanced a state as are distinct element codes. References to implicit discontinuum codes are given in Appendix IV.

In addition to the explicit and implicit solution schemes for discontinuum analysis, another technique has been developed based on static limit equilibrium theory. This approach primarily uses vector analysis to establish whether it is kinematically possible for any block in a blocky system to move and become detached from the system. Appendix V lists references to kinematic stability codes. The key-block method [Goodman and Shi (1985) in Appendix V] is one common example. The key-block is defined as the most unstable block in the system, and the stability of this block is considered to control the stability of the block system.

DISTINGUISHING FEATURES

As the preceding discussion indicates, the distinction among discontinuum analysis techniques based on numerical solution method alone cannot be clearly made. Most of the techniques, whether derived from continuum codes by adding interface elements or developed specifically for analysis of particulate behavior, use similar solution algorithms. The only unique approach is the kinematic stability technique. Nevertheless, distinctions between the various techniques do exist and it is important to identify and categorize these distinctions as an aid in determining the appropriateness of a specific discontinuum analysis technique for a particular problem application.

Four elements of discontinuum analysis are identified to distinguish the various techniques: These are:

(1) representation of the geometry of the discontinuous structure;

(2) description for deformability and strength of intact material (i.e., the material surrounding the discontinuities);

(3) description for compatibility across discontinuities (i.e., across the discontinuity at contacts with the surrounding intact material); and

(4) algorithm used to monitor and update the discontinuity contacts.

The relevance of each element to discontinuum analysis and the descriptions for these elements in the various analysis techniques are provided in the following sections.

Representation of Geometry of Discontinuous Structure — A principal motivation for the development of discontinuum analysis techniques is the recognition that the orientation and location of the discontinuity structure has a significant effect on the mechanical behavior of a geologic system. The geometry of the discontinuous structure (e.g., the dip, strike and location of faults and joints) alone can be a sufficient kinematic mechanism to drive the response of a jointed rock mass. The description of this structure in the numerical technique is, therefore, an important element in the discontinuous analysis.

The development of a joint generator (or block generator depending on the perspective of the numerical technique) is a key input factor for the discontinuum analysis. Because parameters characterizing discontinuities, such as joint sets, vary throughout space, these parameters are usually described probablistically. Several statistical joint generators have been developed and are currently being evaluated for application to discontinuum analysis. The review presented by Heliot (1988) provides a thorough summary of current joint generators.

The primary question in the development of a statistical joint generator is the definition of input parameters. The general consensus among developers is to define parameters independently for each set of joints. Parameters commonly used define the orientation (dip and strike), extent or length, and location (often specified by the spacing between joints). Most of the joint models developed to date have been two-dimensional and thus assume joints are oriented normal to the plane of analysis. Three-dimensional generators have typically been based on spatial variability of joint set parameters as measured in the field. Alternatively, three-dimensional models have been developed which consider joint traces as being defined by lines of intersections of Poisson discs.

The incorporation of a joint generation model in the discontinuum analysis technique is complicated by the fact that the analysis technique requires a description for the connectivity between the discontinuum components. This is required in order to have a well-formed topological structure for use in the mechanical calculations. The discontinuous rock mass defined by joint geometrical data must then be translated into a systematically connected blocky model defined by its topological data.

The incorporation of the topological data structure in the discontinuous analysis technique provides a primary distinction between continuum codes with interface elements and discontinuum codes. The data structure for discontinuum codes, such as the distinct element method and the key-block method, are devised specifically to incorporate the topological data required for a blocky system. The distinct element codes UDEC (in two dimensions) and 3DEC (in three dimensions), for example, include automatic joint generators which provide the means for creating joints statistically in sets based on geological data. The derived joint geometry is then used to create automatically a topological data structure for both blocks and joints (or contacts between blocks). Key-block methods can formulate topological data based on either a disc generation model or a joint orientation and spacing generation model [e.g., see Chan (1986) and Goodman and Chan (1983) in Appendix V].

Continuum codes with interface elements, on the other hand, generally do not include this topological connectivity directly in the data structure for the code. As a consequence, these codes are typically used for problems involving only a few non-intersecting, or occasionally intersecting, discontinuities (e.g., major faults or bedding planes) and have difficulty with simulations of many multiple intersecting discontinuities.

<u>Deformability and Strength of Intact Material</u> — Discontinuum analysis methods focus primarily on the effect of the mechanical behavior of the discontinuities on the response of the medium. Additionally, though, the behavior of the material surrounding the discontinuities can have a significant influence on this response. The influence of the intact material changes as the applied loading conditions change. For example, near a free surface in jointed rock, movements arise predominantly from slip and opening of joints. In these regions, the deformation of the intact material is negligible. On moving away from the free surface, toward the interior of the rock mass, joint displacements diminish in comparison with deformations of the intact rock, and the stress distribution is determined largely by the elastic and

strength properties of the rock. For dynamic analysis, the correct boundary conditions are especially important so that incident waves are propagated toward a rock structure, and reflected waves are absorbed. In these instances, deformability of the intact materials is required near the boundary to provide the correct propagation velocity and the correct driving impedance for absorbing boundaries.

Continuum models, in general, are designed to simulate accurately the deformation and strength characteristics of deformable intact material. In finite element and finite difference formulations, the intact regions are divided into a mesh of interconnected elements. Elastic and elastoplastic constitutive models are then specified to describe the deformation and strength behavior of each element. The order of each element defines the accuracy of the element to represent various kinematic modes of deformation. For example, triangular or quadrilateral plane-strain elements are commonly used in finite element and finite difference methods for linear elastic calculations. These elements, however, are too stiff for plasticity applications and will give inaccurate answers. Mixed discretization of elements in the finite difference approach is shown by Marti and Cundall (1982) to be one procedure which ensures accurate solutions. In this approach, adjacent triangular elements are paired, and volumetric strains are averaged, while the deviatoric strains are unchanged. Zienkiewicz (1977) proposed a similar modified element for the finite element method. It is important to recognize the influence of the kinematic constraints of the element, particularly for problem conditions involving high applied stress relative to the strength of the rock. When the loading approaches the collapse condition for the intact medium, the element discretization for the material can produce unacceptable results if the correct discretization is not used.

Another consideration for the representation of deformability is the form of the algorithm used to describe material strain in the model. It is common practice in many codes to assume that strains, both elastic and plastic, are infinitesimal, and that the initial geometry of a deforming body is not appreciably altered during the deformation process. Most codes based on implicit solution methods assume infinitesimal (small) strain conditions because the assumption of finite strain generally requires continued reformulation of the stiffness matrix, which decreases computational efficiency. Explicit codes can incorporate finite strain logic more readily, with only a minor increase in computation time. The accuracy of small strain codes for mechanical analyses decreases with increased deformation. The limiting condition for a small-strain assumption should be recognized before performing the analysis.

Different degrees of deformability have been incorporated into discontinuum codes. Originally, many of these codes were developed assuming all deformation was concentrated at the discontinuities—i.e., the material surrounding the discontinuities (the blocks) were treated as rigid bodies. Rigid block codes work well for problems where stress levels within the blocks are low and displacements between blocks are much higher than the deformation within the blocks. Rigid block models based on the distinct element method have been used successfully to examine slope stability, explosive cratering and particle flow. [For examples, see, Cundall (1976), Butkovich et al. (1988), and Board and Markham (1987) in Appendix III.] The kinematic stability techniques, such as the key block approach, may also be considered as rigid block models.

Simple deformability of blocks was introduced into discontinuum codes by Cundall et al., 1978 (Appendix III). With simply-deformable blocks, the distinct element method can model problems involving higher stress regimes. Simple deformability is an extension of rigid blocks whereby each block can also deform about its centroid. Simply-deformable blocks have been used for projectile penetration, ice-structure interaction, and hopper flow. The distinct element methods based on modal superposition are generalized versions of the scheme for simply-deformable blocks [e.g., Hocking et al. (1985), Williams and Mustoe (1987) and Williams (1988) in Appendix III]. The implicit discontinuum code, known as discontinuum deformation analysis, also is derived on the basis of simply-deformable blocks [see Shi and Goodman (1988) in Appendix IV].

Full deformability in the distinct element method was introduced by Cundall et al. (1978, Appendix III) in order to provide a more accurate representation of the deformation modes of discontinuous materials subjected to high and transient stresses. Each block is subdivided into a mesh of finite difference zones (triangular zones in 2-D and tetrahedral zones in 3-D). Each individual block, then, is a separate continuum model identical to an explicit, finite difference, continuum formulation. UDEC and 3DEC incorporate fully-deformable (finite strain) block logic which has been validated for static and dynamic analysis of a variety of problems [for examples, see Appendix III and Hart et al. (1987)].

Rigid and simply-deformable block models are not well suited for dynamic analysis involving high frequency shock waves (say, greater than 50 Hz). Experience with numerical analysis has shown that a minimum of approximately ten elements per wavelength is normally required to obtain meaningful results from a dynamic calculation (i.e., the element size must be smaller than approximately one-tenth of the wavelength) [Kuhlmeyer and Lysmer, 1973]. In continuum codes, this requires refinement of the mesh to satisfy the restriction on element size for a specific frequency of input wave. For a discontinuum code with rigid or simply-deformable blocks, this requires reduction of the block size to meet the wavelength criterion. Thus, for high frequency input, the required block size could be much smaller than the actual joint spacing. With fully-deformable blocks, refinement is only necessary for the continuum mesh within each block.

A simple illustration of the influence of frequency on element size is given in Fig. 1. The model consists of a row of blocks for which a sinusoidal wave is applied at the left boundary, A, and monitored at the right boundary, B. The blocks can only move in the x-direction and have the following properties:

block size, s	1.0 unit
normal stiffness between blocks, k_n	10.0 units
block mass, ρ	1.0 unit

The blocks are rigid and cannot separate at contacts.

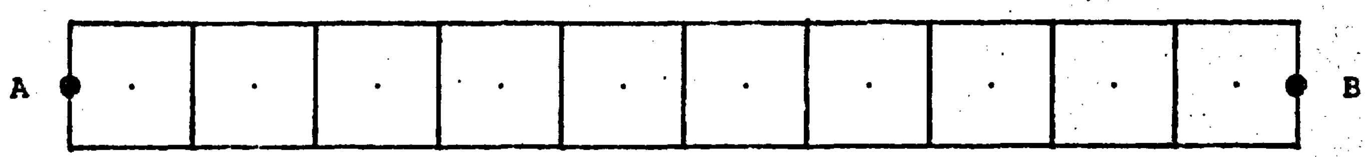

Fig. 1 Rigid Block Model with Sinusoidal Wave Applied at A and Monitored at B

The speed of the wave traveling through the blocks:

$$C_p = (k_n\ s/\rho)^{1/2} = 3.16 \text{ units}$$

For a base input frequency, f, of 0.05 units, the wavelength is $\lambda = C_p/f = 63.25$ units. The input wave at A and the monitored wave at B are shown in Fig. 2. If the frequency is increased by a factor of eight, to 0.4 units, a divergence between input and monitored waves occurs, as shown by Fig. 3. This discrepancy in the waves is caused by the high frequency of the input. The wavelength in this instance is only 7.9 units, which is less than ten times the element size.

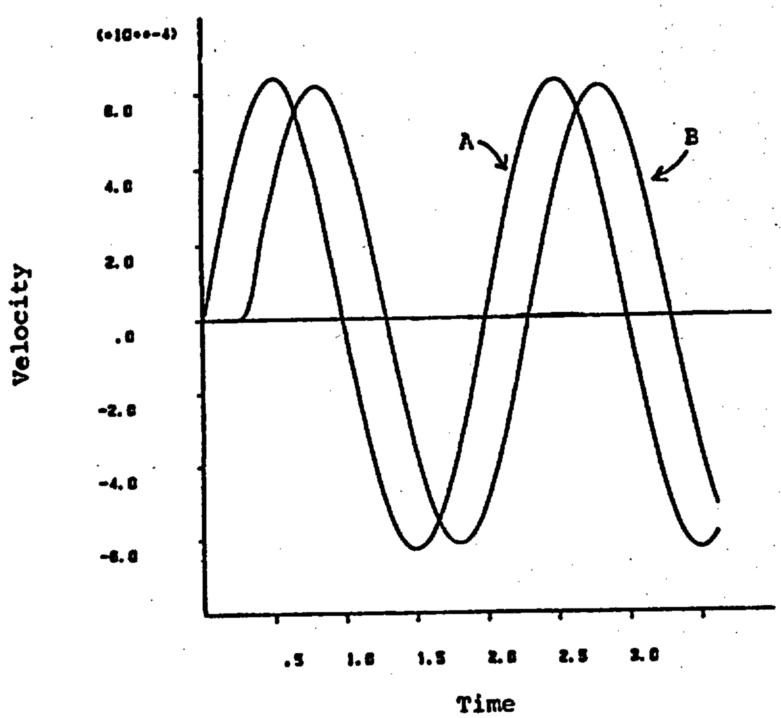

Fig. 2 Wave Transmission in Rigid Block Model for Input Frequency of 0.05 Units

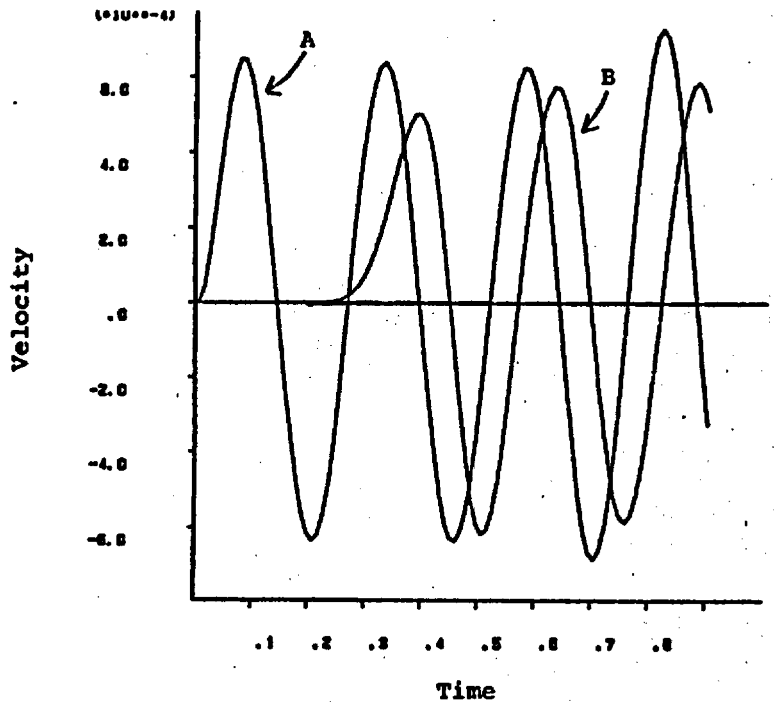

Fig. 3 Wave Transmission in Rigid Block Model for Input Frequency of 0.4 Units

The blocks can be made fully deformable by subdividing each block into four triangular zones, as shown in Fig. 4. The properties are adjusted such that the wave speed remains constant at 3.16 units, to match the rigid block simulation. This time divergence of the input and monitored waves does not occur for the input frequency of 0.4 units (see Fig. 5). In this instance, the ratio of wavelength to element size is 15.8/1.

Control over the refinement of the mesh in fully-deformable blocks permits the accurate solution of high frequency dynamic analysis in the same manner as in continuum codes. In discontinuum analysis, where the wavelength is small, relative to the joint spacing, fully-deformable blocks can give an accurate solution for the wave propagation without introducing fictitious joints (blocks) to satisfy the requirements for accurate wave attenuation.

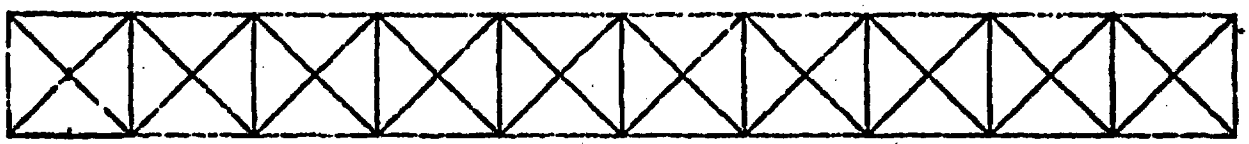

Fig. 4 Deformable Block Model (Four Zones per Block)

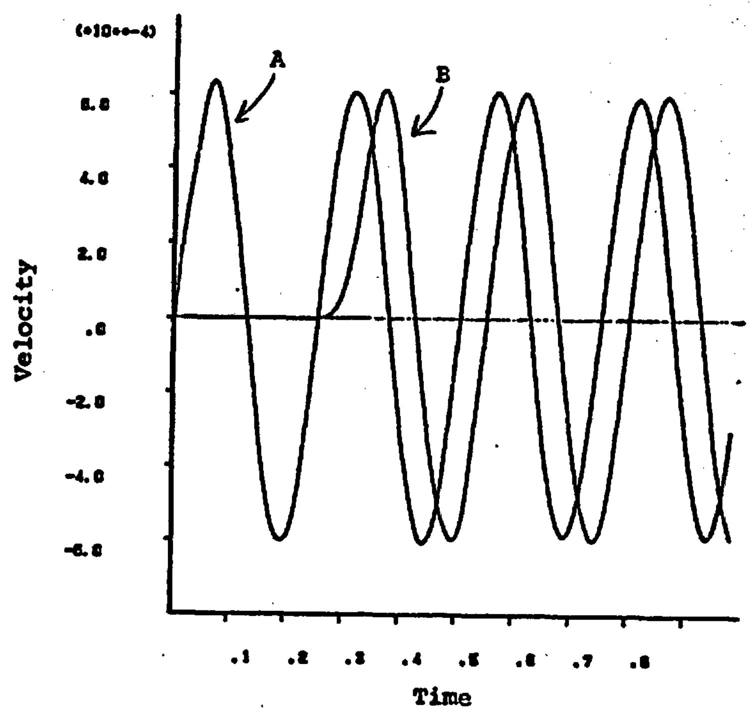

Fig. 5 Wave Transmission in Deformable Block Model for Input Frequency of 0.4 Units

<u>Compatibility Across Discontinuities</u> — A distinction can be made in the approach taken by a discontinuum analysis technique to satisfy compatibility conditions enforced between the interacting bodies in the model—i.e., compatibility across the discontinuity. Compatibility conditions must be satisfied for slideline interfaces in continuum codes, as well as for contacts between interacting particles in discontinuum codes. Slideline interfaces in continuum codes and contacts in discontinuum codes, in reality, are lists of surface elements or nodal points from one or more bodies arranged in some data structure. These elements are continually examined for the purpose of identifying if one body has penetrated the boundary of another body. Compatibility must be satisfied when contact is determined. The compatibility requirement is also important because it affects the representation for displacement and separation between contacts. Thus, the requirement impacts the description for joint material behavior used in the analysis technique.

Several different types of algorithms have been developed to enforce compatibility across discontinuities. Some of the more common methods are:

(1) special finite element joint elements;

(2) master-slave (or symmetric interaction) elements;

(3) Lagrange multipliers;

(4) penalty functions; and

(5) linear spring and damper algorithms.

Thin joint elements, developed specifically to model sliding interfaces, are customized finite elements designed for this purpose (e.g., see Goodman and Dubois, 1972). These elements are integral parts of a finite element mesh and thus satisfy compatibility as any other element. The elements have a special constitutive model to approximate gap and friction behavior. However, joint elements have limited applicability because of their fixed connectivity. They cannot properly model large tangential motion or rotation between two bodies. They are analogous to infinitesimal strain elements and become inaccurate for motion exceeding 10% of the dimension representing the contact area. Several implicit finite element codes with slideline elements have incorporated this logic [e.g., D'Appolonia (1981), Ewing and Rainey (1976) and Goodman (1976) in Appendix I].

Master-slave and symmetric interaction elements have been used in several continuum codes with slideline logic. In the master-slave algorithm, one of the two mating surfaces defines the master, and the other defines the slave. Local element stresses for the slave side are used to compute forces for the slave nodes which have penetrated the master side. These are resolved into normal and shear components. The shear component is limited by Coulomb slip, while the normal component is used to calculate a residual force required to move the slave node to the master surface. This force is controlled so that the velocity of the slave node will eventually be the same as the velocity of the master surface. Master-slave algorithms permit large slip but can introduce asymmetry to the movement. Symmetric interaction algorithms overcome this problem by duplicating and reversing master-slave elements and calculating the average motion. Master-slave slidelines have been used principally in explicit continuum codes [e.g., Stone et al. (1985), Biffle (1984) and Key (1986) in Appendix II].

Other methods developed to satisfy compatibility at contacts also allow arbitrary friction corresponding to Coulomb slip and a gap closure constraint. Lagrange multipliers have been used by specifying the multipliers to correspond to the normal and frictional forces required to impose the constraints on positions and velocities at the contacts. Relative displacements of nodes on either side of a contact or slideline are monitored and, if overlap of the two surface is indicated by the relative displacement, then the relative normal displacement is constrained to be a specified gap distance. The normal force imposed to satisfy the constraint is the Lagrange multiplier. Likewise, frictional constraints are applied if the gap is closed and the coefficient of friction is non-zero. The multipliers are introduced into a functional describing the displacement constraints. The functional is minimized to provide the equilibrium solution, as in classical mechanics. This approach has limitations in that the existence and uniqueness of the multipliers is not guaranteed for systems with friction (Lötstedt 1979). Also, the computation of the multipliers is very complicated and time-consuming for large systems. References by Bechtel National (1981) and Morgan (1981), in Appendix I, describe the implementation of Lagrange multipliers in implicit continuum codes with slideline logic.

The penalty function approach has been applied in several different discontinuum codes to enforce compatibility. This approach is used in a similar fashion to the Lagrange multiplier approach to minimize a functional which describes the displacement constraints at the contacts. Penalty coefficients act in a manner analogous to very stiff strings which optimize the minimizing of the functional. In physical terms, the coefficients push the penetrating block back to the surface along the shortest path. Coulomb's law is also used to limit shear stress along the contact. The penalty function approach has been used in implicit discontinuum codes [e.g., Shi and Goodman (1988) and Gussmann (1988) in Appendix IV] and is referenced in the modal superposition method for distinct elements [Williams et al. (1985) in Appendix III], although the procedure is not described.

Linear spring and damper algorithms are mathematically similar to penalty functions, but they equate the spring stiffness in physical terms to the normal and shear stiffnesses of the joint. This approach was proposed by Cundall (1971, Appendix III) for the distinct element method. The springs act as penalty functions which are added to the equations of motion to yield a solution where violation of the contact overlap constraints are kept small. The damper connected to each spring damps out high frequency oscillations. The damping can pose difficulties with approaching the correct solution, particularly for problems involving collapse. However, Cundall (1987) has developed techniques to minimize the adverse effects of damping. The spring and damper approach, used with an explicit solution algorithm, is well suited to simulate both the elastic and failure response of actual joints. The approach has been used extensively with direct integration distinct elements (e.g., see the distinct element references in Appendix III).

<u>Contact Monitoring and Update</u> — At each calculational step in a discontinuum analysis technique, discontinuities must be examined for the features of one body penetrating the boundary of another body. This examination process requires that the location of each surface contact point be tracked relative to the surface element with which it is likely to come in contact. The process used to monitor contacts will affect both the computational efficiency of the technique and the ability to represent large translational and rotational movements along the discontinuities.

Monitoring and updating locations of contacts between discontinuous bodies can be a very time-consuming process, especially in a multi-jointed system. The monitoring process requires a data structure which defines the existing connectivity and potential connectivity between boundaries of discontinuous bodies. Several different data structures have been developed for this purpose.

In many of the implicit continuum codes with interface elements, the connectivity between surface nodes of adjacent elements is fixed. This permits an efficient monitoring of contacts because only small displacements are permitted such that the contacts will not change. This type of data structure has limited application, though, because the fixed connectivity prevents the analysis of large tangential or rotational motion. It is noted that some distinct element codes have this restriction imposed on contact updating [e.g., see Dowding et al. (1983) in Appendix III].

The data structure typically used in explicit continuum codes with interface elements permits large translation and rotation, but requires the user to specify, in advance, the nodes along each element which have the potential for contact with an adjacent element [e.g., see Key (1986) and Itasca (1987) in Appendix II]. This restriction inhibits the modeling of multiple intersecting jointed systems because all the potential contacts may not be identified.

For this reason, the pre-processing of multi-jointed systems with a kinematic stability technique, such as the key block method, has been proposed to select the most unstable jointed region [i.e., key block(s)] for modeling with interface elements. Then, only a few key-block joints need be modeled.

It should be noted that kinematic stability techniques only analyze the conditions necessary for the onset of motion. It is assumed that this motion will continue indefinitely—i.e., that the forces do not change with motion. Methods such as key block theory, alone, cannot provide information on stresses and displacements of a rock mass and cannot calculate the combined response of the rock matrix and the discontinuities. The continuum code with interface elements, then, supplements the kinematic stability technique by simulating the change in forces after key block motion begins.

Automatic algorithms for monitoring and updating contacts have been developed for distinct element codes. The original rigid block codes employ a system of "cells" which are used for course classification of blocks and contact nodes. Only a limited search of cells in the vicinity of contact edges (interfaces) is required to locate potential contacts. The procedure is described by Cundall (1974) in Appendix III for two-dimensional analysis. This approach is used in 3DEC, and the three-dimensional algorithm is described by Cundall (1988), in Appendix III. The approach updates the contact information in each cell and thus allows blocks to come into and out of contact continually. This permits the simulation of large translation and rotation of blocks. The approach is well suited for problems involving particle flow, and several of the distinct element codes developed for this purpose use a form of this algorithm (see the references by Cundall, Walton and Taylor in Appendix III for examples).

Cundall (1980, Appendix III) proposed a different data structure in which the topological properties of the data structure correspond closely to the properties of the physical system of blocks that they represent. Thus, the connectivity of the physical system is built into the data structure, which consists of linked lists of voids between blocks as well as lists of blocks. It is then a simple matter to scan the local voids surrounding a block in order to obtain a list of all possible contacts. This scheme is considerably more efficient than the cell scheme, but it requires well-developed connectivity. The main application of this approach is for a system of initially connected blocks, such as a jointed rock mass. The scheme is used in UDEC and works extremely well for two-dimensional analysis of jointed rock. Unfortunately, the data structure does not translate to three dimensions.

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