BASIC CONCEPTS IN THE THEORY OF SEEPAGE OF HOMOGENEOUS LIQUIDS IN FISSURED ROCKS [STRATA]

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The modern theory of seepage [infiltration] is based on the concept of a porous medium consisting of impermeable grains separated by pore spaces. Comparison of the results of theoretical and laboratory investigations of non-steady-state flow of liquids with data for strata under natural conditions leads to the conclusion that current concepts of a porous medium are inadequate. In all natural strata, the development of some degree of fissuring is a characteristic feature. The description of non-steady-state flow of liquids in fissured strata by means of the usual equations of infiltration theory can lead, in some cases, to conflicting conclusions of qualitative nature.

At first glance, it appears that non-steady-state seepage in fissured rocks can be studied by assuming a system of fissures, which are regular to some extent, in the stratum. Apparently, for studying seepage in fissured rocks, this method is not promising. Even if it were possible to overcome the enormous mathematical difficulties involved in solving problems of non-steady-state flow in strata with a system of fissures of a sufficiently general type, it is not possible to determine the configuration of this system with any degree of reliability. Information obtained in the analysis of cores – specimens of the rock obtained by drilling from the surface – gives very incomplete data on the fissure system. The position is to some extent similar to that which occurs in investigating the flow of a liquid in an ordinary porous medium – even if it were possible to overcome all the difficulties involved in the integration of the equations of motion of a viscous liquid in the pore spaces, the method would not be suitable for investigating seepage, since the pore configuration remains unknown. Various models of a porous medium, which are based on one or another type of arrangement of the system of pores and grains and on the study of the motion of the liquid in such systems (ideal soil, fictitious soil, etc. [1]), proved suitable only for the qualitative investigation of seepage phenomena. Seepage theory has followed the trend which is characteristic of the mechanics of continuous media generally, namely, the introduction of mean characteristics of the media and flow (porosity, permeability, pressure, seepage velocity, etc.) and the formulation of basic laws in terms of these mean characteristics.

Such an approach, applied irrespective of whether or not the system of fissures is regular in the natural stratum, also proved most advantageous in investigating seepage in fissured rocks.

In this paper, the basic concepts of the motion of liquids in fissured rocks are presented. Mean-characteristics are introduced, whereby the averaging is carried out on a scale which is large compared to the dimensions of the individual blocks. The difference between the present scheme and the more usual scheme of seepage in a porous medium consists in the introduction at each point in space of two liquid pressures — liquid pressure in the pores and pressure of the liquid in the fissures a, and in taking into consideration the transfer of liquid between the fissures and the pores. Under certain assumptions, an expression is obtained for the intensity of this transfer. The basic equation of the seepage of a liquid in a fissured rock and the same general equations of the seepage of liquid in a porous medium with a double porosity are derived. These equations will obviously contain, as a particular case, the equations for the seepage of a liquid in an ordinary porous medium; in the paper an evaluation is made which indicates for which cases the latter equations are valid and when the more accurate expressions given in this paper have to be used. The formulation of the basic boundary-value problems for seepage equations in fissured rocks is considered. Some characteristic features of non-steady-state seepage in fissured rocks are discussed, particularly the possibility of the occurrence, under certain conditions, of a pressure jump [discontinuity] within the system and at the boundaries, similar to the "infiltration gap" in non-pressure seepage [2]. Conditions at jumps are derived, and the features pertaining to the formulation of boundary-value problems in the presence of jumps are pointed out. Solutions are given of certain specific problems of non-steady-state seepage in fissured rocks.

1. Basic physical concepts. A fissured rock consists of pores and permeable blocks, generally speaking blocks separated from each other by a system of fissures (Fig. 1). The dimensions of the blocks will vary for the various rocks within wide limits, depending on the extent to which fissures are developed in the rock. The widths of the fissures are considerably greater than the characteristic dimensions of the pores, so that the permeability of the fissure system considerably exceeds the permeability of the system of pores in the individual blocks. At the same time, it is a characteristic feature of fissured rocks that the fissures occupy a much smaller volume than the pores, so that the coefficient of fissuring of the rock m_1 – the ratio of the volume of the cavity space occupied by the fissures to the total volume of the rock – is considerably smaller than the porosity of the individual blocks m_2. Much factual data on fissured rocks has been published in [3-9]; the paper by Pirson [4] is of particular interest, since it gives a qualitative description of the structure of a porous medium with double porosity, which is close to that considered in this paper.

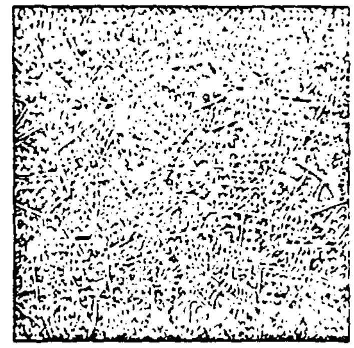

Fig. 1.

If the system of fissures is sufficiently well developed, the motion of the liquid in fissured rocks can be investigated by the following method. Unlike the classical seepage theory, for each point in space, not one liquid pressure but two, p_1 and p_2 are introduced. The pressure p_1 represents the average pressure of the liquid in the *fissures* in the neighborhood of the given point, while the pressure p_2 is the average pressure of the liquid in the *pores* in the neighborhood of the given point. For obtaining reliable averages, the scale of averaging should include a sufficiently large number of blocks. Therefore, it is necessary to take into consideration that any infinitely small volume includes not only a larger number of pores, as is assumed in the classical theory of seepage, but also that it contains a large number of blocks. This condition permits the use of the method of analysis of infinitesimals in investigating fissured rocks.

In a similar manner, two velocities of seepage of the liquid can be defined at each point in space: $\mathbf{V}_1$ and $\mathbf{V}_2$. Vector $\mathbf{V}_1$ of the seepage velocity of the liquid along the fissures is determined as follows: the projection of this vector in some particular direction is equal to the flow of the liquid through the cross-section of the fissures of a small zone passing through the given point in a direction perpendicular to the given direction, divided by the density of the liquid and the *total* area of this zone. In the same way, the projection of vector $\mathbf{V}_2$, the seepage velocity of the liquid through the pores in a given direction, will equal the flow of the liquid through the cross-section of the blocks of the small zone mentioned, also divided by the density of the liquid and the total area of the zone.

It is a characteristic of fissured rocks that the flow of the liquid proceeds essentially along the fissures, so that the flow velocity of the liquid through the blocks is negligibly small as compared to seepage of liquid along the fissures.

If the boundary between the fissures and the blocks is imagined impermeable, the fissured rock can be considered as being a coarse-grained porous medium in which the fissures play the role of pores and the blocks play the role of grains. If, furthermore, the fissures are sufficiently narrow and the velocity of the liquid is sufficiently small, the motion of the liquid along the fissures will be inertialess and Darcy's Law is fulfilled:

$$\mathbf{V}_1 = -\frac{k_1}{\mu}\,\mathrm{grad}\,p_1 \tag{1.1}$$

where k_1 is the permeability of the system of fissures and μ is the viscosity of the liquid. Application of Darcy's Law to seepage along the system of fissures is not of principal importance; if desired, inertia of the motion can be taken into account, using thereby a more complicated nonlinear law.

A characteristic feature of the non-steady-state motion of a liquid in fissured rocks is the transfer of liquid between the blocks and the fissures. Therefore, in investigating the seepage of liquids in fissured rocks it is necessary, in contrast to the classical theory of seepage, to take into consideration the outflow of liquid from the 'grains' – blocks into the 'pores' – of the fissures.

The process of transfer of liquid from the pores and the blocks takes, place essentially under a sufficiently smooth change of pressure, and, therefore, it can be assumed that this pressure is quasi-stationary, i.e. it is independent of time explicitly. It is obvious in such a case that during motion of a homogeneous liquid in the fissures of the rock, the volume of the liquid v, which flows from the blocks into the fissures per unit of time and unit of volume of the rock, depends on the following: (1) viscosity of the liquid μ; (2) pressure drop between the pores and the fissures $p_2 - p_1$; and (3) on certain characteristics of the rock, which can only be geometrical ones, i.e. they may have the dimension of length, area, volume, etc., or even be dimensionless. On the basis of dimensional analysis [10], we obtain for v an expression of the type

$$v = \frac{\alpha}{\mu}(p_2 - p_1) \tag{1.2}$$

where α is some new dimensionless characteristic of the fissured rock.

Thus, for the mass q of the liquid which flows from the pores into the fissures per unit of time, per unit of rock volume, the following equation is valid:

$$q = \frac{\rho\alpha}{\mu}(p_2 - p_1) \qquad (1.3)$$

where ρ is the density of the liquid.

It should be pointed out that in a somewhat different form relation (1.3) was applied for the integral estimate of the flow along the stratum as a whole [11].

2. Equation of motion of a uniform liquid in fissured rocks. In accordance with what has been said above, the law of conservation of mass of liquid in the presence of fissures can be written as follows:

$$\frac{\partial m_1\rho}{\partial t} + \operatorname{div}\rho \mathbf{V}_1 - q = 0 \qquad (2.1)$$

In view of the smallness of the volume of the fissures, the first term, which expresses the change in mass of the liquid due to compression in the fissures and changes in the volume of the fissures in some element of the rock, is small as compared to the second term, which expresses changes in the mass of the liquid caused by the inflow of the liquid along the fissures through the boundary of this element. Therefore, relation (2.1) can be disregarded. Inserting Equation (1.1) (Darcy's Law) into Equation (2.1), taking into consideration the fact that the liquid is slightly compressible so that

$$\rho = \rho_0 + \beta\delta p \qquad (2.2)$$

(ρ_0 is the density of the liquid at some standard pressure, for instance, the initial pressure in the stratum, β is the coefficient of compressibility of the liquid, δp is the change in the pressure relative to the standard pressure), assuming that the medium is homogeneous and neglecting the small higher-order terms, we obtain

$$k_1 \Delta p_1 + \alpha(p_2 - p_1) = 0 \qquad (\Delta \text{ is Laplace operator}) \qquad (2.3)$$

Further, the equations of conservation of mass of the liquid which is present in the pores can be written thus*:

* Strictly speaking, in Equation (2.4), m_2 will not represent the porosity of the blocks but the ratio of the volume of the pores to

$$\frac{\partial m_2 \rho}{\partial t} + \operatorname{div} \rho V_2 + q = 0 \qquad (2.4)$$

so that the quantity of liquid which flows into the fissures equals the quantity of the liquid which flows out of the blocks.

In view of the low permeability of the blocks, the second term of Equation (2.4), which expresses changes in the mass of the liquid within the pores in some element of the rock, due to the inflow of liquid along the pores through the boundaries of the element, can be disregarded as compared to the first term which represents changes in the mass of the liquid in the pores due to its expansion, and also to changes in the volume of the pores. Therefore, Equation (2.4) can be re-written as

$$\frac{\partial m_2 \rho}{\partial t} + q = 0 \qquad (2.5)$$

Furthermore, the porosity of the blocks m_2 in the case of a constant pressure of the upper strata of the rocks on the roof of the stratum depends, generally speaking, on the pressure of the liquid in the fissures p_1 and the pressure of the liquid in the pores p_2. However, the volume of the fissures in the rock is considerably smaller than the volume of the pores. It can be assumed that, in contrast to the liquid located in the pores, the liquid located in the fissures does not participate in supporting the upper strata of the rock formations. Therefore, the influence of the pressure of the liquid in the fissures p_1 on the porosity of the blocks can be disregarded as compared to the influence of the pressures of the liquid in the pores p_2, and it can be assumed that

$$dm_2 = \beta_{c2} dp_2 \qquad (2.6)$$

where β_{c2} is the coefficient of compressibility of the blocks. Taking into consideration, also, relations (1.3) and (2.2) and neglecting small terms of higher order, we obtain

$$(\beta_{c2} + m_0 \beta) \frac{\partial p_2}{\partial t} + \frac{\alpha}{\mu} (p_2 - p_1) = 0 \qquad (2.7)$$

where m_0 is the magnitude of the porosity of the blocks at standard pressure. Equations (2.3) and (2.7) describe the motion of the liquid in

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the entire volume of the rock, including the volume of the fissures. However, in view of the small relative volume of the fissures compared to the relative volume of the pores, m_2 can be considered as representing the porosity of the individual blocks.

fissured rocks. Eliminating from these equations p_2, we obtain for the pressure of the liquid in the fissures p_1 the equation

$$\frac{\partial p_1}{\partial t} - \eta \frac{\partial \Delta p_1}{\partial t} = \varkappa \Delta p_1 \qquad \left(\varkappa = \frac{k_1}{\mu(\beta_{c2} + m_0\beta)}, \quad \eta = \frac{k_1}{\alpha}\right) \tag{2.8}$$

The coefficient $\varkappa$ represents the coefficient of piezo-conductivity of the fissured rock; it is interesting that this does not correspond to the permeability of the system of fissures k_1 but to the porosity and compressibility of the blocks. The coefficient η represents a new specific characteristic of fissured rocks. If η tends to zero, it corresponds to a reduction of the block dimensions and an increase in the degree of fissuring, and Equation (2.8) will obviously tend to coincide with the ordinary equation for seepage of a liquid under elastic conditions.

An approximate estimate of the possible magnitudes of the coefficient η will be made. The dimensionless coefficient a, characterizing the intensity of the liquid transfer between the blocks and fissures, depends on the permeability of the blocks k_2 and the degree of fissuring of the rock, as a measure of which it is obvious to take the specific surface of the fissures σ, i.e. the surface of the fissures per unit of volume of the rock. The quantity σ has the dimension of the reciprocal of length. On the basis of dimensional analysis we obtain

$$\alpha \sim k_2\sigma^2 \tag{2.9}$$

From this and Equation (2.8) we obtain

$$\eta \sim \frac{k_1}{k_2\sigma^2} \sim \frac{k_1}{k_2} l^2$$

where l is the average dimension of a single block (the specific surface of the fissures is inversely proportional to the average dimension of a single block). Evaluations show that for various rocks the parameter η will assume values within wide limits - from a few cm^2 to values of the order of 10^{10} cm^2.

Determination of the parameter η should be carried out by means of data for the steady-state flow of liquids in fissured rocks. Thus, where natural strata are involved, determination of this parameter should be carried out only on the basis of investigations of the behavior of the stratum under non-steady-state conditions and not on the basis of tests carried out on rock specimens brought to the surface.

3. Equations of motion of a homogeneous liquid in a medium with double porosity. The system of equations (2.3), (2.7) represents a particular case of the system of equations of motion of a homogeneous liquid in a medium with double porosity. In some cases the latter equations may be of interest and, therefore, we will deal briefly with their derivation.

The motion of a uniform liquid in a "double" porous medium will be considered: the first porous medium consists of relatively wide pores of the first order – fissures and blocks; the relative volume of the pores of the first order, the porosity of the first order, equals m_1. The blocks in themselves are porous, consisting of grains which are separated by fine pores of the second order; together they form the second porous medium. The porosity of this medium – the porosity of the second order – is designated by m_2. It is pointed out that, generally speaking, m_2 cannot be considered equal to the porosity of the blocks, since m_2 represents the ratio of the volume of the second-order pores to the total volume of the elements of the rock in which a known space is occupied by the fissures – pores of the first order. In the case where the upper strata exert a constant pressure on the roof of the stratum, both porosities, m_1 and m_2, will depend on the pressures of the liquid in the pores of the first and second order, p_1 and p_2, so that

$$dm_1 = \beta_{c1} dp_1 - \beta_{*} dp_2, \qquad dm_2 = \beta_{c2} dp_2 - \beta_{**} dp_1 \tag{3.1}$$

where β_{c1}, β_{c2}, β_{*}, β_{**} are positive constant coefficients.

The equations for conservation of mass of the liquid for both media are of the form (2.1) and (2.4), respectively. Assuming that the flow of the liquid in the first medium (and thus also in the second medium) is inertialess, the Darcy law for both media can be written as

$$\mathbf{V}_1 = -\frac{k_1}{\mu} \operatorname{grad} p_1, \qquad \mathbf{V}_2 = -\frac{k_2}{\mu} \operatorname{grad} p_2 \tag{3.2}$$

where k_1 is the porosity of the system of pores of the first order and k_2 the porosity of the system of pores of the second order.

By inserting into Equations (2.1) and (2.4) relations (3.2), Expression (1.3) for the liquid flow from one medium to the other (which will obviously remain valid even in this more general case), relation (2.2) for the density of the liquid and relation (3.1) for the (differentials of the porosity), and discarding small quantities of higher order for the pressures of the liquid in both media p_1 and p_2, the following system of equations is obtained:

$$\begin{aligned} \frac{k_1}{\mu} \Delta p_1 &= (\beta_{c1} + m_{10}\beta) \frac{\partial p_1}{\partial t} - \beta_{*} \frac{\partial p_2}{\partial t} - \frac{\alpha}{\mu}(p_2 - p_1) \\ \frac{k_2}{\mu} \Delta p_2 &= (\beta_{c2} + m_{20}\beta) \frac{\partial p_2}{\partial t} - \beta_{**} \frac{\partial p_1}{\partial t} + \frac{\alpha}{\mu}(p_2 - p_1) \end{aligned} \tag{3.3}$$

where m_1 and m_2 are the values of the first and second order porosity at standard pressure.

If the pressure p_2 changes, say decreases, at a constant pressure of the upper strata on the roof, the porosity of the first order will increase, on the one hand, as a result of the compression of the blocks and, on the other hand, it will decrease as a result of compression by the overlying strata. These effects will apparently compensate each other to some extent. The situation is similar for the second-order porosity m_2 in the case of a change in the pressure p_1. It is, therefore, advisable to consider the model of the double porosity of the medium for which the porosity of each order depends only on the appropriate pressure, so that the coefficient β_* and β_{**} in Equation (3.1) can be considered small and the appropriate terms in Equation (3.1) can be disregarded.

Equations (3.3) for such a model of a porous medium with double porosity will be of the form similar to the equations for heat transfer in a heterogeneous medium considered by Rubinshtein [12]:

$$\frac{k_1}{\mu}\Delta p_1 = (\beta_{c1} + m_{10}\beta)\frac{\partial p_1}{\partial t} - \frac{\alpha}{\mu}(p_2 - p_1)$$

$$\frac{k_2}{\mu}\Delta p_2 = (\beta_{c2} + m_{20}\beta)\frac{\partial p_2}{\partial t} + \frac{\alpha}{\mu}(p_2 - p_1) \qquad (3.4)$$

Disregarding in Equations (3.3) the terms representing a change in the mass of the liquid due to the compressibility of the first medium and the compression of the liquid in the pores of the first order, and the changes in the mass of the liquid as a result of the seepage inflow along the pores of the second order, we again obtain the equations of motion of a liquid in a fissured porous medium (2.3) and (2.7).

4. Basic boundary-value problems of the theory of non-steady-state seepage in fissured rocks. Equation (2.8), to which corresponds the pressure distribution of the liquid in the pores p_1, can be written as

$$\beta_0\frac{\partial p_1}{\partial t} - \operatorname{div}\left[\frac{k_1}{\mu}\operatorname{grad} p_1 + \eta\beta_0\frac{\partial}{\partial t}\operatorname{grad} p_1\right] = 0 \qquad (4.1)$$

where β_0 is the total effect of compressibility equalling $\beta_{c2} + m_0\beta$. This form of writing the basic equation indicates that motion in the system of fissures can be considered as the motion of a liquid in a porous medium with a total compressibility coefficient β_0, and the expression for the velocity of seepage of the liquid can be written as

$$\mathbf{V} = -\frac{k_1}{\mu}\operatorname{grad} p_1 - \eta\beta_0\frac{\partial}{\partial t}\operatorname{grad} p_1 \qquad (4.2)$$

The initial and the boundary conditions have to be added to Equation (2.8). As in the theory of seepage in a porous medium, the steady-state initial conditions are of greatest interest in the given case (i.e. the harmonic initial distributions p_1, which satisfy Equation (4.1)). Among the possible types of boundary conditions the most important are the following:

1) The pressures p_1 at the boundary of the rock of volume s under consideration are given (first boundary-value problem):

$$p_1|_S = f(S, t) \tag{4.3}$$

2) At the boundary s the flow of the liquid is given (second boundary-value problem), the following quantity being given, in accordance with what was stated above, at the boundary of the surface S:

$$-\left\{\frac{k_1}{\mu}\frac{\partial p_1}{\partial n} + \eta\beta_0\frac{\partial}{\partial t}\left(\frac{\partial p_1}{\partial n}\right)\right\}\Big|_S = f(S, t) \tag{4.4}$$

($\partial/\partial n$ is the derivative along the normal to the surface S), and, finally,

3) At the boundary a linear combination of the pressure and the flow of the liquid, generally speaking with variable coefficients A and B, is given (mixed problem):

$$\left\{Ap_1 + B\left[\frac{k_1}{\mu}\frac{\partial p_1}{\partial n} + \eta\beta_0\frac{\partial}{\partial t}\left(\frac{\partial p_1}{\partial n}\right)\right]\right\}\Big|_S = f(S, t) \tag{4.5}$$

If the initial pressure-distribution is continuous and the boundary conditions are consistent with the initial ones (i.e. the boundary values of the initial distribution on approaching the boundary points equal to the boundary values of the corresponding functions at the initial instant of time), the solutions of the above-stated boundary-value problems will be the ordinary classical solutions (4.1). However, if the initial pressure distribution is discontinuous or if the initial and the boundary conditions are unrelated, then the derived distributions will also be discontinuous and there is no classical solution for the boundary-value problems formulated above; it is necessary to seek a generalized solution in the sense of Sobolev [13]. To proceed further it is necessary to derive the conditions at the discontinuities. It is sufficient to consider the one-dimensional case, since in the neighborhood of the given point the surface of discontinuity can be considered as being plane. Thus, it is assumed that within a sufficiently small vicinity on both sides of the isolated discontinuity surface $x = 0$ (x is the direction of the normal to the surface of the discontinuity), the function p_1 is continuous, has appropriate continuous derivatives and satisfies the equation

$$Lp_1 = \beta_0 \frac{\partial p_1}{\partial t} - \eta\beta_0 \frac{\partial^3 p_1}{\partial x^2 \partial t} - \frac{k_1}{\mu}\frac{\partial^2 p_1}{\partial x^2} = 0 \tag{4.6}$$

In the region $G(-h < x < h,\ 0 < t < T)$, where h is a small number, the terms in the expression Lp_1 are piece-wise continuous. By means of term-by-term integration of Lp_1 along the region G we obtain

$$\int_G Lp_1\,dx\,dt = \beta_0 \int_{-h}^{h} \left\{p_1(x,T) - p_1(x,0)\right\} dx - \int_0^T \left\{\eta\beta_0 \frac{\partial^2 p_1}{\partial x\,\partial t} + \frac{k_1}{\mu}\frac{\partial p_1}{\partial x}\right\}\Bigg|_{x=-h}^{x=h} dt = 0 \tag{4.7}$$

For $h \to 0$ the first integral tends to zero and the preceding equality yields

$$\int_0^T \left[\eta\beta_0 \frac{\partial^2 p_1}{\partial x\,\partial t} + \frac{k_1}{\mu}\frac{\partial p_1}{\partial x}\right] dt = 0 \tag{4.8}$$

where as usual the sign [ ] designates the difference between the values of the function on both sides of the discontinuity surface. Since T is arbitrary and the expression under the integral sign is a continuous function of time, it follows that the expression under the integral sign equals zero:

$$\left[\eta\beta_0 \frac{\partial}{\partial t}\left(\frac{\partial p_1}{\partial n}\right) + \frac{k_1}{\mu}\frac{\partial p_1}{\partial n}\right] = \eta\beta_0 \frac{\partial}{\partial t}\left[\frac{\partial p_1}{\partial n}\right] + \frac{k_1}{\mu}\left[\frac{\partial p_1}{\partial n}\right] = 0 \tag{4.9}$$

i.e. the condition of continuity at the surface of discontinuity of the total flow of the liquid ($\partial/\partial x$ was replaced by $\partial/\partial n$). To obtain the second condition, Equation (4.6) is multiplied by x and integration is carried out over the same region G:

$$\int_G xLp_1\,dx\,dt = \beta_0 \int_{-h}^{h} \left\{p_1(x,T) - p_1(x,0)\right\} x\,dx -$$

$$-\int_0^T \left\{\eta\beta_0 x \frac{\partial^2 p_1}{\partial x \partial t} + \frac{k_1}{\mu} x \frac{\partial p_1}{\partial x}\right\}\Bigg|_{-h}^{h} dt - \int_0^T \left\{\eta\beta_0 \frac{\partial p_1}{\partial t} + \frac{k_1}{\mu} p_1\right\}\Bigg|_{-h}^{h} dt = 0$$

As $h \to 0$, the first and second integrals will become zero, and thus we obtain

$$\int_0^T \left[\eta\beta_0 \frac{\partial p_1}{\partial t} + \frac{k_1}{\mu} p_1\right] dt = 0 \tag{4.10}$$

so that the second condition at the surface of discontinuity is obtained in the form

$$\left[\eta\beta_0 \frac{\partial p_1}{\partial t} + \frac{k_1}{\mu} p_1\right] = \eta\beta_0 \frac{\partial [p_1]}{\partial t} + \frac{k_1}{\mu} [p_1] = 0 \qquad (4.11)$$

For $\eta = 0$ the basic conditions at the surface of discontinuity (4.9) and (4.11) will change into the condition of continuity of the function and its derivative along the normal to any surface, i.e. the condition of absence of discontinuities*, which is well known in the theory of heat conduction and the theory of seepage in a porous medium.

Integrating (4.9) and (4.11), we obtain the conditions at the discontinuities in the form

$$[p_1] = [p_1]_{t=0}\, e^{-\varkappa t/\eta}, \qquad \left[\frac{\partial p_1}{\partial n}\right] = \left[\frac{\partial p_1}{\partial n}\right]_{t=0} e^{-\varkappa t/\eta}. \qquad (4.12)$$

so that the pressure jumps and normal derivative of pressure which occur, due to the discontinuity or due to inconsistent initial conditions, will not be eliminated instantaneously as in a porous medium (and as is the case for jumps in temperature and heat flow in the theory of heat conduction) but will decrease in accordance with the law $e^{-\kappa t/\eta}$. This property is a characteristic qualitative feature of the mathematical description of non-steady-state flow in fissured rocks, which is comparable with flow through a porous medium.

5. Some specific problems of non-steady-state flow in fissured rocks. General qualitative conclusions. *1. Non-steady-state flow of liquid in a gallery.* From the literature the importance of the study of non-steady-state seepage in drainage galleries is well known. This problem is formulated as follows: at the initial instant the pressure of the liquid in a semi-infinite stratum $(0 \leqslant x < \infty)$ P_0 is constant; the pressure at the boundary $x = 0$ suddenly assumes the value P_1, differing from P_0, and then remains constant. The problem of determining seepage flow requires the solution of the equation

* In obtaining the second law of conservation in a medium with a variable permeability coefficient k_1 it is necessary to multiply both parts of Equation (4.6) by

$$\int \frac{1}{k_1}\, dx$$

$$\frac{\partial p_1}{\partial t} - \eta \frac{\partial^3 p_1}{\partial x^2 \partial t} = \varkappa \frac{\partial^2 p_1}{\partial x^2} \qquad (5.1)$$

for the uncoordinated initial and boundary conditions

$$p_1(x, 0) = P_0, \qquad p_1(-0, t) = P_1 \qquad (5.2)$$

(the boundary pressure is given immediately to the left of the boundary $x = 0$). At the initial instant there will be a pressure jump at the boundary equal to $(P_0 - P_1)$; according to (4.12), at time t this jump will equal $(P_0 - P_1)e^{-\varkappa t/\eta}$, so that the pressure of the liquid immediately to the right of the boundary will equal

$$p_1(+0, t) = P_1 + (P_0 - P_1)e^{-\varkappa t/\eta} \qquad (5.3)$$

To find the pressure distribution at any desired instant of time t, the Laplace transform with respect to time t is applied. We set

$$p_1(x, t) = P_0 - (P_0 - P_1)\,u(x, t) \qquad (5.4)$$

Then, for determining $u(x, t)$ $(t \geqslant 0,\ 0 \leqslant x < \infty)$, we obtain the boundary-value problem

$$\frac{\partial u}{\partial t} - \eta \frac{\partial^3 u}{\partial x^2 \partial t} = \varkappa \frac{\partial^2 u}{\partial x^2}, \qquad u(+0, t) = 1 - e^{-\varkappa t/\eta}, \quad u(x, 0) = 0 \qquad (5.5)$$

Let

$$U(x, \lambda) = \int_0^\infty e^{-\lambda t} u(x, t)\, dt$$

Applying the Laplace transform to Equation (5.5) we obtain

$$\frac{d^2 U}{dx^2} - \frac{\lambda}{\varkappa + \lambda\eta} U = 0, \qquad U(0, \lambda) = \frac{1}{\lambda(\varkappa + \lambda\eta)}, \qquad U(\infty, \lambda) = 0 \qquad (5.6)$$

$$U = \frac{1}{\lambda(\varkappa + \lambda\eta)} \exp\left(-\sqrt{\frac{\lambda}{\varkappa + \lambda\eta}}\, x\right)$$

whence, on the basis of the known rule of inversion [14], we obtain

$$u(x, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \exp\left(-\sqrt{\frac{\lambda}{\varkappa + \lambda\eta}}\, x\right) \frac{d\lambda}{\lambda(\varkappa + \lambda\eta)} \qquad (5.7)$$

The evolution of the integral on the right-hand side of the previous equation yields

$$2\pi i - 2i \int_0^1 e^{-\sigma t} \sin\left(\sqrt{\frac{\sigma}{1-\sigma}}\, x\right) \frac{d\sigma}{\sigma(1-\sigma)} \tag{5.8}$$

Inserting this expression into Equation (5.7) and substituting the variables $\sigma/(1-\sigma) = \nu^2$, we obtain

$$u(x,t) = 1 - \frac{2}{\pi}\int_0^\infty \frac{1}{\nu} \sin \nu x \exp\left(-\frac{\nu^2 \varkappa t}{1+\nu^2\eta}\right) d\nu = 1 - \exp\left(-\frac{\varkappa t}{\eta}\right) -$$
$$- \frac{2}{\pi}\int_0^\infty \frac{1}{\nu} \sin \nu x \left\{\exp\left(-\frac{\nu^2 \varkappa t}{1+\nu^2\eta}\right) - \exp\left(-\frac{\varkappa t}{\eta}\right)\right\} d\nu \tag{5.9}$$

From this and from Equation (5.4) we finally obtain

$$p_1(x,t) = P_1 + \frac{2(P_0-P_1)}{\pi}\int_0^\infty \frac{\sin \nu x}{\nu}\left\{\exp\left(-\frac{\nu^2 \varkappa t}{1+\nu^2\eta}\right) - \exp\left(-\frac{\varkappa t}{\eta}\right)\right\} d\nu + \tag{5.10}$$
$$+ (P_0 - P_1)\exp\left(-\frac{\varkappa t}{\eta}\right)$$

Since the integral in Equation (5.10) and the integral obtained after differentiation with respect to x under the integral sign are both uniformly convergent, the expression given by Equation (5.10) can be differentiated with respect to x. Hence, from Equation (5.10) we obtain the expression for the flow of the liquid through the boundary of the stratum $x = 0$

$$q = -\frac{k_1}{\mu}\left(\frac{\partial p_1}{\partial x}\right)_{x=+0} = -\frac{2(P_0-P_1)k_1}{\pi\mu}\int_0^\infty \left\{\exp\left(-\frac{\nu^2 \varkappa t}{1+\eta\nu^2}\right) -\right.$$
$$\left. - \exp\left(-\frac{\varkappa t}{\eta}\right)\right\} d\nu = -\frac{2(P_0-P_1)k_1}{\pi\mu\sqrt{\eta}}\exp\left(-\frac{\varkappa t}{\eta}\right) \times$$
$$\times \int_0^\infty \left\{\exp\left(\frac{\varkappa t}{\eta(1+\zeta^2)}\right) - 1\right\} d\zeta \tag{5.11}$$

It is interesting to compare the derived solutions with the appropriate solution of the problem of the theory of seepage in a porous medium. This well-known self-similar solution is obtained from Equation (5.10) for the case when $\eta = 0$

$$p_1(x,t) = P_1 + \frac{2}{\pi}(P_0 - P_1)\int_0^{\frac{1}{2}\xi} e^{-\beta^2} d\beta = P_1 + (P_0 - P_1)\,\Phi\left(\frac{1}{2}\xi\right)$$
$$\left(\xi = \frac{x}{\sqrt{\varkappa t}}\right) \tag{5.12}$$

where Φ is the symbol of the Kramp function. For comparing the pressures, Equations (5.10) and (5.12) which correspond to fissured and ordinary porous media, respectively, the distribution quantities $u(x,\ t) = (p_1 - P_1)/(P_0 - P_1)$ for various values of the parameter $\varkappa t/\eta$ have been plotted as a function of the self-similar variable ξ in Fig. 2. (The calculations were carried out by A. L. Dyshko at the Computing Center of the Academy of Sciences, USSR). It can be seen that with increasing $\varkappa t/\eta$ the pressure distribution in a fissured rock tends to the self-similar distribution which is obtained in an ordinary porous me medium.

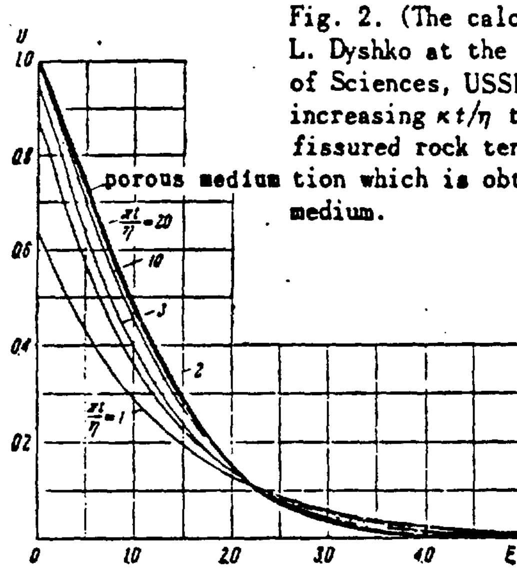

Fig. 2.

2. *Non-steady-state infiltration of a liquid from a well discharging at a constant rate.* In addition to the problem considered previously, the problem of non-steady-state infiltration of a liquid from a well of infinitely small radius with a constant yield is of considerable interest. It is formulated as follows: an infinite horizontal stratum of constant thickness h is penetrated by a vertical well of negligibly small radius. At the initial instant, the pressure of the liquid in the stratum is constant and equal to P. Then, a liquid begins to flow in or out at a constant volume rate Q.

The pressure of the liquid in the fissures $p_1(r,\ t)$ (r is the distance from the axis of the well) satisfies the equation

$$\frac{\partial p_1}{\partial t} - \eta \frac{\partial}{\partial t} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial p_1}{\partial r} = \varkappa \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial p_1}{\partial r} \tag{5.13}$$

with the initial conditions

$$p_1(r, 0) = P \tag{5.14}$$

In accordance with Equation (4.4) the boundary conditions can be expressed as

$$Q = -2\pi h \left\{\frac{k_1}{\mu}\left(r \frac{\partial p_1}{\partial r}\right) + \eta\beta_0 \frac{\partial}{\partial t}\left(r \frac{\partial p_1}{\partial r}\right)\right\}_{r=0}$$

We obtain therefrom

$$-\frac{Q\mu}{2\pi k_1 h} = \left(r\frac{\partial p_1}{\partial r}\right)_{r=+0} + \frac{\eta}{\varkappa}\frac{\partial}{\partial t}\left(r\frac{\partial p_1}{\partial r}\right)_{r=+0}$$

Integrating the last relation and applying the condition

$$(r\partial p_1/\partial r)_{r=+0} = 0 \qquad \text{for } t=0$$

we obtain the final formula for the boundary conditions

$$\left(r\frac{\partial p_1}{\partial r}\right)_{r=+0} = -\frac{Q\mu}{2\pi k_1 h}\left(1-e^{-\varkappa t/\eta}\right) \tag{5.15}$$

Setting

$$p_1(r,t) = P + \frac{Q\mu}{2\pi k_1 h}u(r,t) \tag{5.16}$$

we obtain the following boundary-value problem for determining the function $u(r, t)$:

$$\frac{\partial u}{\partial t} - \eta\frac{\partial}{\partial t}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} = \varkappa\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}$$

$$u(r,0)=0, \qquad \left(r\frac{\partial u}{\partial r}\right)_{r=+0} = -\left(1-e^{-\varkappa t/\eta}\right) \tag{5.17}$$

To solve the boundary-value problem (5.17) we again revert to the Laplace transform, the relations (5.17) being reduced to the following form:

$$\frac{1}{r}\frac{d}{dr}r\frac{dU}{dr} - \frac{\lambda}{\varkappa+\lambda\eta}U = 0, \quad \left(r\frac{dU}{dr}\right)_{r=0} = -\frac{\varkappa}{\lambda(\varkappa+\lambda\eta)} \tag{5.18}$$

Taking into consideration, also, the condition $U(\infty, \lambda) = 0$, we obtain

$$U(r,\lambda) = \frac{\varkappa}{\lambda(\varkappa+\lambda\eta)}K_0\left(\sqrt{\frac{\lambda}{\varkappa+\lambda\eta}}\,r\right) \tag{5.19}$$

(K_0 is the symbol of the Macdonald function), and by the general rule of inversion [14] we obtain

$$u(r,t) = \frac{\varkappa}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda(\varkappa+\lambda\eta)}K_0\left(\sqrt{\frac{\lambda}{\varkappa+\lambda\eta}}\,r\right)d\lambda \tag{5.20}$$

In an entirely analogous manner, after calculating the integral and reverting to the variable p_1, we obtain the pressure distribution in the form

$$p_1(r,t) = P + \frac{Q\mu}{2\pi k_1 h}\int_0^\infty \frac{J_0(\nu r)}{\nu}\left[1-\exp\left(-\frac{\nu^2\varkappa t}{1+\nu^2\eta}\right)\right]d\nu \tag{5.21}$$

The known self-similar solution of the corresponding problem of the theory of seepage in a porous medium is obtained from Equation (5.21) for $\eta = 0$:

$$p_1(r,t) = P + \frac{Q\mu}{2\pi k_1 h}\int_0^\infty \frac{J_0(\nu r)}{\nu}\left(1 - e^{-\nu^2 \varkappa t}\right) d\nu = P - \frac{Q\mu}{4\pi k_1 h}\,\mathrm{Ei}\left(-\frac{r^2}{4\varkappa t}\right) \qquad (5.22)$$

It can be seen, in the same way as in the previous problem, that for increasing values of $\kappa t/\eta$, the solution (5.21) of the problem of seepage in a fissured rock tends asymptotically to the solution (5.22) of the problem of seepage in a porous medium.

It can also be seen from the examples investigated that the most characteristic property of the non-steady-state flow of liquids in fissured rocks is the occurrence of some delay in the transient processes; the characteristic time of this delay is

$$\tau = \eta/\varkappa \qquad (5.23)$$

Thus, the following general conclusion can be advanced: in considering processes of non-steady-state infiltration in fissured rocks, the ordinary equations of non-steady-state flow in a porous medium can be applied only if the characteristic times of the process under consideration are long compared to the delay times τ. However, if the characteristic times of the process are comparable to τ, it is necessary to apply the model and the basic equations presented in this paper. The estimates which have been carried out have shown that fissuring must be taken into consideration in many cases when investigating such processes as the restoration of pressure in shut-down wells and, generally, transient processes during changes of the operating conditions of the well.

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