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R.W. Zimmerman and G.S. Bodvarsson

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R. W. Zimmerman and G. S. Bodvarsson

Earth Sciences Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
Berkeley, California 94720

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R.W. Zimmerman and G.S. Bodvarsson

**Earth Sciences Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
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ABSTRACT

A simple approximate solution is derived for the problem of one-dimensional absorption in a porous medium characterized by the Brooks-Corey equations. A piecewise-linear saturation profile, which satisfies flux continuity at the edge of the tension-saturated zone as well as an integrated form of the Richards' equation, is assumed. The predicted sorptivity agrees very well with the results of numerical simulations.

Key words: Infiltration, sorptivity, Brooks-Corey

Running title: Absorption in a Brooks-Corey Medium

1. Notation

D	diffusivity function [m^2/s]
k	absolute permeability [m^2]
k_r	relative permeability
n	Brooks-Corey parameter for characteristic curves
q	instantaneous liquid flux [$\text{m}^3/\text{m}^2\text{s}$]
Q	cumulative liquid flux [m^3/m^2]
S	liquid saturation
S_r	residual liquid saturation
S_s	liquid saturation at zero potential
\hat{S}	normalized liquid saturation, = $(S - S_r)/(S_s - S_r)$
S	sorptivity, = Q/\sqrt{t} [$\text{m}/\text{s}^{-1/2}$]
t	time since start of absorption [s]
x	distance into half-space [m]

Greek letters

α	Brooks-Corey pressure-scaling parameter [ms^2/kg]
δ	width of saturation front, see Fig. 3
η	similarity variable, = $[\alpha\mu\phi(S_s - S_r)x^2/kt]^{1/2}$
λ	width of fully-saturated zone, see Fig. 3
μ	viscosity [kg/ms]
ϕ	porosity
ψ	potential [kg/ms ²]
$\hat{\psi}$	normalized potential, = $\alpha\psi$

Subscripts, superscripts, etc.

i	initial value at $t = 0$
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2. Introduction

Horizontal flow of water in an unsaturated medium is generally described by Richard's partial differential equation (Hillel, 1980):

$$\frac{\partial}{\partial x} \left[\frac{kk_r}{\mu\phi} \frac{\partial \psi}{\partial x} \right] = \frac{\partial S}{\partial t} \quad (1)$$

In this equation, S is the liquid saturation, which is equal to the fraction of the pore space that is filled with water. ψ is the potential, or capillary pressure, and is related to the saturation through a capillary pressure function $\psi = \psi(S)$. k is the absolute (i.e., fully-saturated) permeability of the medium, ϕ is its porosity (assumed constant), and μ is the viscosity of water. k_r is the dimensionless relative permeability function, which measures the decrease in the permeability of the medium to water due to incomplete saturation; if hysteretic effects are neglected, k_r is a single-valued function of S (or ψ).

Equation (1) embodies the principal of conservation of mass for the water, along with Darcy's law to relate the volumetric flux to the potential gradient. Although the physical problem of flow in an unsaturated medium actually involves both the water and the air phase, it is conventional to ignore the air by implicitly assuming it to be infinitely mobile, and at one atmosphere pressure. Note that Equation (1) can be put into the form of a diffusion equation with a saturation-dependent diffusivity as follows:

$$\frac{\partial}{\partial x} \left[\frac{kk_r(S)}{\mu\phi} \frac{d\psi}{dS} \frac{\partial S}{\partial x} \right] = \frac{\partial}{\partial x} \left[D(S) \frac{\partial S}{\partial x} \right] = \frac{\partial S}{\partial t} \quad (2)$$

where $D(S) = kk_r(S)\psi'(S)/\mu\phi$. Although this form of the governing equation is fre-

quently used in soil physics (e.g., Kutílek and Valentová, 1986), we will find it convenient to use the form given in Equation (1).

The capillary pressure and relative permeability functions depend on the pore geometry of the medium (Bear, 1988), and have different forms for different media. One form of the "characteristic equations" that has been widely used is the Brooks-Corey (Brooks and Corey, 1966) model:

$$\hat{S}(\psi) = \frac{S(\psi) - S_r}{S_s - S_r} = 1 \quad \text{if } |\alpha\psi| \leq 1, \\ = |\alpha\psi|^{-n} \quad \text{if } |\alpha\psi| > 1; \quad (3)$$

$$k_r(\psi) = 1 \quad \text{if } |\alpha\psi| \leq 1, \\ = |\alpha\psi|^{-(3n+2)} \quad \text{if } |\alpha\psi| > 1, \quad (4)$$

where S_r is the residual saturation, S_s is the saturation at zero potential, \hat{S} is the normalized saturation, and α is a scaling parameter that is inversely proportional to the mean pore diameter. These functions are plotted in Figs. 1 and 2, for a few different values of n . The normalized saturation equals 1 for all $|\psi| < 1/\alpha$, after which it drops off to zero as $|\psi|$ increases, according to a power law. The relative permeability monotonically decreases from 1 to 0 as the normalized saturation decreases from 1 to 0.

2. Formulation of Problem

A basic problem in hydrology and soil physics is that of absorption of water into a semi-infinite medium from a planar boundary which is saturated at zero potential. If the medium is initially at some uniform saturation, the problem can be specified by

$$\frac{\partial}{\partial x} \left[\frac{kk_r}{\mu\phi} \frac{\partial \psi}{\partial x} \right] = \frac{\partial S}{\partial t} \quad (5)$$

$$\psi(0,t) = 0 \quad (6)$$

$$\psi(x,0) = \psi_i < 0 \quad (7)$$

$$\lim_{x \rightarrow \infty} \psi(x,t) = \psi_i \quad (8)$$

The initial potential ψ_i corresponds to the initial saturation S_i through the capillary pressure Equation (3).

Before attempting to solve this problem, it is convenient to normalize it, and transform it into an ordinary differential equation using a Boltzmann-type transformation (Bruce and Klute, 1956; Zimmerman and Bodvarsson, 1989). If we define a normalized potential as $\hat{\psi} = \alpha\psi$, and a similarity variable η as

$$\eta = \left[\frac{\alpha\mu\phi(S_s - S_r)x^2}{kt} \right]^{1/2} \quad (9)$$

then Equations (5-8) are transformed into

$$\frac{d}{d\eta} \left[k_r(\hat{S}) \frac{d\hat{\psi}}{d\eta} \right] + \frac{\eta}{2} \frac{d\hat{S}}{d\eta} = 0, \quad (10)$$

$$\hat{\psi}(0) = 0, \quad (11)$$

$$\lim_{\eta \rightarrow \infty} \hat{\psi}(\eta) = \hat{\psi}_i, \quad (12)$$

where $\hat{S} = (S - S_r)/(S_s - S_r)$ is the normalized saturation.

4. Boundary-Layer Solution

Many approximate solutions to the above problem have been found in terms of an effective diffusivity that is some weighted integral of the diffusivity over the range $S_i < S < S_s$ (see Brutsaert, 1976; Kutilék and Valentová, 1986). Recently, Zimmerman and Bodvarsson (1989) used a boundary-layer method to derive a completely closed-form solution for one-dimensional absorption in a van Genuchten (van Genuchten, 1980) medium. A similar approach will now be used for a Brooks-Corey medium, although the algebra will be simplified by working with the transformed Equation (10) instead of with the original partial differential equation.

The basic idea behind the simplest form of the boundary-layer method is to approximate the solution with some simple function that contains an adjustable parameter, and then fix the value of this parameter by requiring the solution to satisfy the differential equation in an integrated sense. An appropriate algebraic form for the saturation and capillary pressure profiles can best be ascertained by examining a few numerical solutions to the problem (see Figs. 4 and 5, which are discussed in more detail below). These simulations suggest the following profile (Fig. 3):

$$0 < \eta < \lambda: \quad \hat{\psi} = -\eta/\lambda, \quad \hat{S} = 1;$$

$$\lambda < \eta < \lambda + \delta: \quad \hat{S} = 1 - (1 - \hat{S}_i) \frac{\eta - \lambda}{\delta}, \quad \hat{\psi} = -\hat{S}^{-1/n};$$

$$\lambda + \delta < \eta < \infty: \quad \hat{S} = \hat{S}_i, \quad \hat{\psi} = -\hat{S}_i^{-1/n}. \quad (13)$$

In the region $0 < \eta < \lambda$, $\hat{S} = 1$ and $k_r = 1$, so Equation (10) implies that $\hat{\psi}$ will be a linear function of η . For all η greater than some value (which we will denote by $\lambda + \delta$), the saturation will equal its undisturbed value \hat{S}_i . In the intermediate region, a linear saturation profile is used because of its simplicity.

A relationship between the parameters λ and δ can be found by requiring continuity of the capillary pressure gradient at $\eta = \lambda$:

$$\left. \frac{\partial \hat{\psi}}{\partial \eta} \right|_{\lambda^-} = \frac{1}{\lambda},$$

$$\left. \frac{\partial \hat{\psi}}{\partial \eta} \right|_{\lambda^+} = \left. \frac{d\hat{\psi}}{d\hat{S}} \right|_{\hat{S}=1} \left. \frac{\partial \hat{S}}{\partial \eta} \right|_{\eta=\lambda}$$

$$= \left[\frac{-1}{n} \hat{S}^{-(1+1/n)} \right]_{\hat{S}=1} \left[\frac{-(1 - \hat{S}_i)}{\delta} \right]_{\eta=\lambda} = \frac{(1 - \hat{S}_i)}{n\delta},$$

$$\text{so } \lambda = \frac{n\delta}{(1 - \hat{S}_i)}. \quad (14)$$

Note that we do not require the capillary pressure gradient to be continuous at $\eta = \lambda + \delta$; as explained by Zimmerman and Bodvarsson (1989), imposition of this condition leads to only slight increases in accuracy, at the expense of much additional algebraic complexity.

To find an expression for δ , we integrate Equation (10) from $\eta = 0$ to $\eta = \infty$. The first term in Equation (10) integrates out to

$$k_r(\hat{S}) \left. \frac{d\hat{\psi}}{d\eta} \right|_0^{\infty} = \frac{-1}{\lambda} = \frac{-(1-\hat{S}_i)}{n\delta}. \quad (15)$$

Since the term $d\hat{S}/d\eta$ is non-zero only in the range $\lambda < \eta < \lambda + \delta$, the second term integrates out to

$$\begin{aligned} \int_0^{\infty} \frac{\eta}{2} \frac{d\hat{S}}{d\eta} d\eta &= - \int_{\lambda}^{\lambda+\delta} \frac{\eta}{2} \frac{(1-\hat{S}_i)}{\delta} d\eta \\ &= \frac{-(1-\hat{S}_i)}{2\delta} \left. \frac{\eta^2}{2} \right|_{\lambda}^{\lambda+\delta} = \frac{-(1-\hat{S}_i)}{4\delta} [(\lambda+\delta)^2 - \lambda^2]. \end{aligned} \quad (16)$$

Combining Equations (15) and (16) leads to

$$\frac{4}{n} = \delta^2 + 2\lambda\delta. \quad (17)$$

Using Equation (14) to eliminate λ from Equation (17) leads to

$$\frac{4}{n} = \delta^2 \left[1 + \frac{2n}{(1-\hat{S}_i)} \right], \quad (18)$$

which can be solved for

$$\delta = 2 \left[n \left[1 + \frac{2n}{(1-\hat{S}_i)} \right] \right]^{-1/2}. \quad (19)$$

Equations (9,13,14,19) specify the approximate solution to the problem.

The instantaneous liquid flux into the medium, per unit surface area, can be found from Darcy's law as follows:

$$\begin{aligned} q &= \frac{-kk_r}{\mu} \frac{\partial \psi}{\partial x} \Big|_{x=0} = \frac{-k}{\mu} \frac{d\psi}{d\hat{\psi}} \frac{\partial \hat{\psi}}{\partial \eta} \Big|_{\eta=0} \frac{\partial \eta}{\partial x} \Big|_{x=0} \\ &= \frac{-k}{\mu} \frac{1}{\alpha} \frac{-1}{\lambda} \left[\frac{\alpha \mu \phi (S_s - S_r)}{kt} \right]^{1/2} \\ &= \frac{k}{\mu} \frac{(1-\hat{S}_i)}{\alpha n \delta} \left[\frac{\alpha \mu \phi (S_s - S_r)}{kt} \right]^{1/2} \\ &= \left[\frac{k \phi (S_s - S_i)}{2\alpha \mu t} \left[1 + \frac{(S_s - S_i)}{2n(S_s - S_r)} \right] \right]^{1/2} \end{aligned} \quad (20)$$

If we write the instantaneous flux $q(t)$ as $S/2\sqrt{t}$, then the cumulative flux up to some time t is equal to

$$Q(t) = \int_0^t q(\tau) d\tau = \int_0^t \frac{S}{2} \tau^{-1/2} d\tau = S t^{1/2} . \quad (21)$$

The constant S is often referred to as the sorptivity (Brutsaert, 1976; Kutilék and Valentová, 1986). From Equation (20) and (21), S can be expressed as

$$S = \left[\frac{2k \phi(S_s - S_i)}{\alpha \mu} \left[1 + \frac{(S_s - S_i)}{2n(S_s - S_r)} \right] \right]^{1/2} . \quad (22)$$

5. Comparison with Numerical Solutions

In order to judge the accuracy of the approximate solution, we can compare it to the results of numerical solutions to Equations (10-12). These equations represent a two-point (ordinary differential equation) boundary-value problem, which can be solved using the shooting method described by Press et al. (1986) and Zimmerman and Bodvarsson (1989). Figs. 4 and 5 compare the capillary pressure and saturation profiles of the approximate solution and the numerical solution for a Brooks-Corey medium with $n = 2$, for two different initial saturations. This value of n is close to the values that have been estimated (Rulon, Bodvarsson and Montazer, 1986) for the welded tuffs at Yucca Mountain, Nevada, the proposed site of an underground nuclear waste repository. The approximate solution very accurately predicts λ , the length of the fully saturated zone, but slightly overpredicts δ , the width of the wetting front. Since the sorptivity is proportional to the slope of the capillary pressure profile at $\eta = 0$, and to the area bounded by the saturation profile and the line $S = S_i$, it is clear that the approximate solution estimates S very accurately. Fig. 6 shows the normalized sorptivity $S/[k \phi / \alpha \mu]^{1/2}$ plotted against the initial saturation, for a few different values of n . For simplicity, S_r is taken to be 0, and S_s is taken to be 1. The approximate

solution is seen to estimate the sorptivity very accurately, over the entire range of initial saturations.

Conclusions

A simple approximate solution has been derived for absorption of water into an unsaturated porous medium whose characteristic curves are of the Brooks-Corey type. A piecewise-linear saturation profile has been assumed, which is then forced to satisfy an integrated form of the Richards' equation. The length (λ) of the tension-saturated zone was predicted very accurately, while the length (δ) of the wetting front was somewhat overpredicted. For values of the Brooks-Corey parameter representative of the welded tuffs found at Yucca Mountain, Nevada, *i.e.*, $n=2$, the error in the predicted sorptivity was less than 7%, over the entire range of initial saturations.

Acknowledgements

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Figure Captions

Fig. 1. Normalized capillary pressure curves of Brooks-Corey media, for different values of n .

Fig. 2. Relative permeability curves of Brooks-Corey media, for different values of n .

Fig. 3. Assumed saturation profile for one-dimensional absorption into a Brooks-Corey medium.

Fig. 4. Capillary pressure profiles for one-dimensional absorption in Brooks-Corey medium with $n = 2$, for various values of the initial saturation.

Fig. 5. Saturation profiles for one-dimensional absorption in a Brooks-Corey medium with $n = 2$, for various values of the initial saturation.

Fig. 6. Sorptivity of a Brooks-Corey medium as a function of the initial saturation, for various values of n . To simplify the figures, we take $S_e = 1$ and $S_r = 0$.

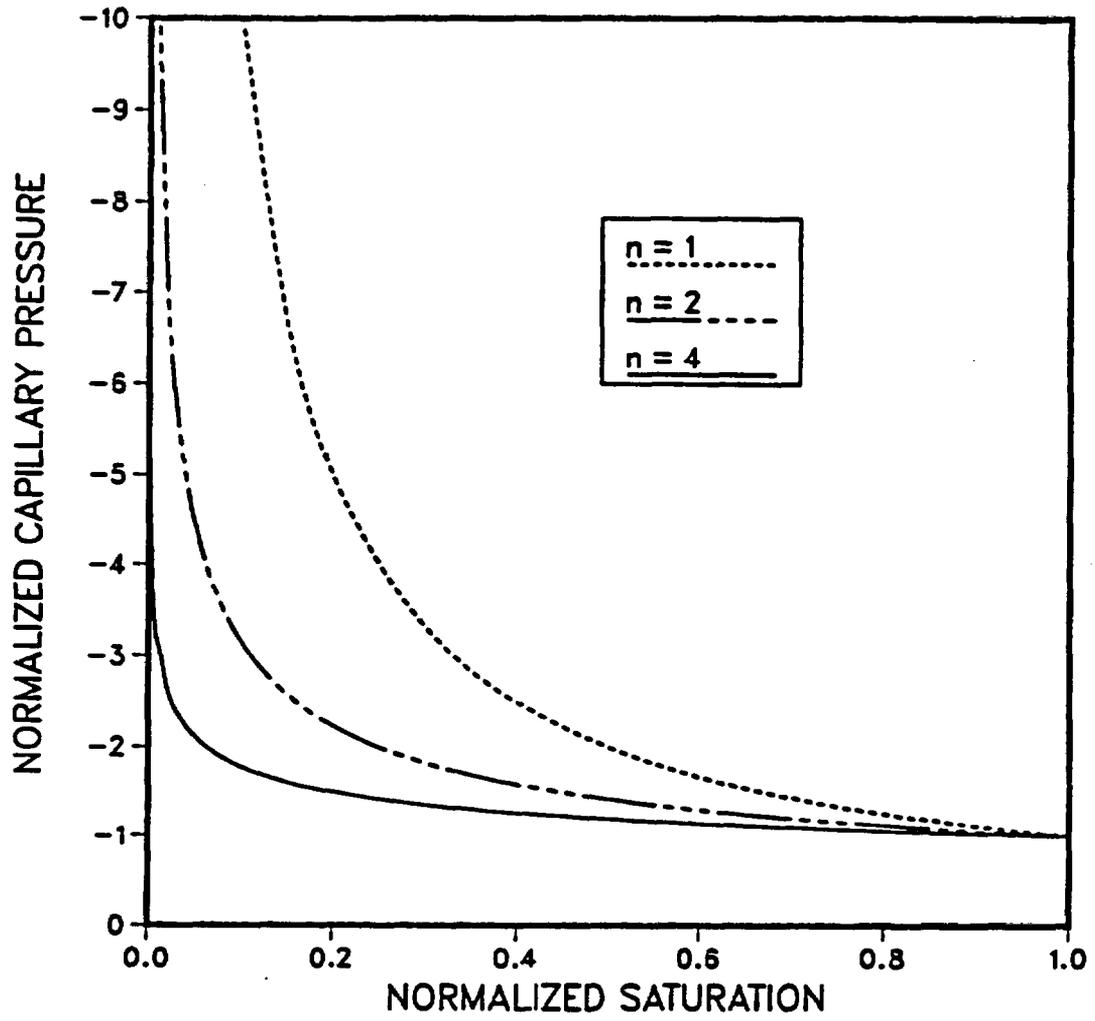


Fig. 1. Normalized capillary pressure curves of Brooks-Corey media, for different values of n .

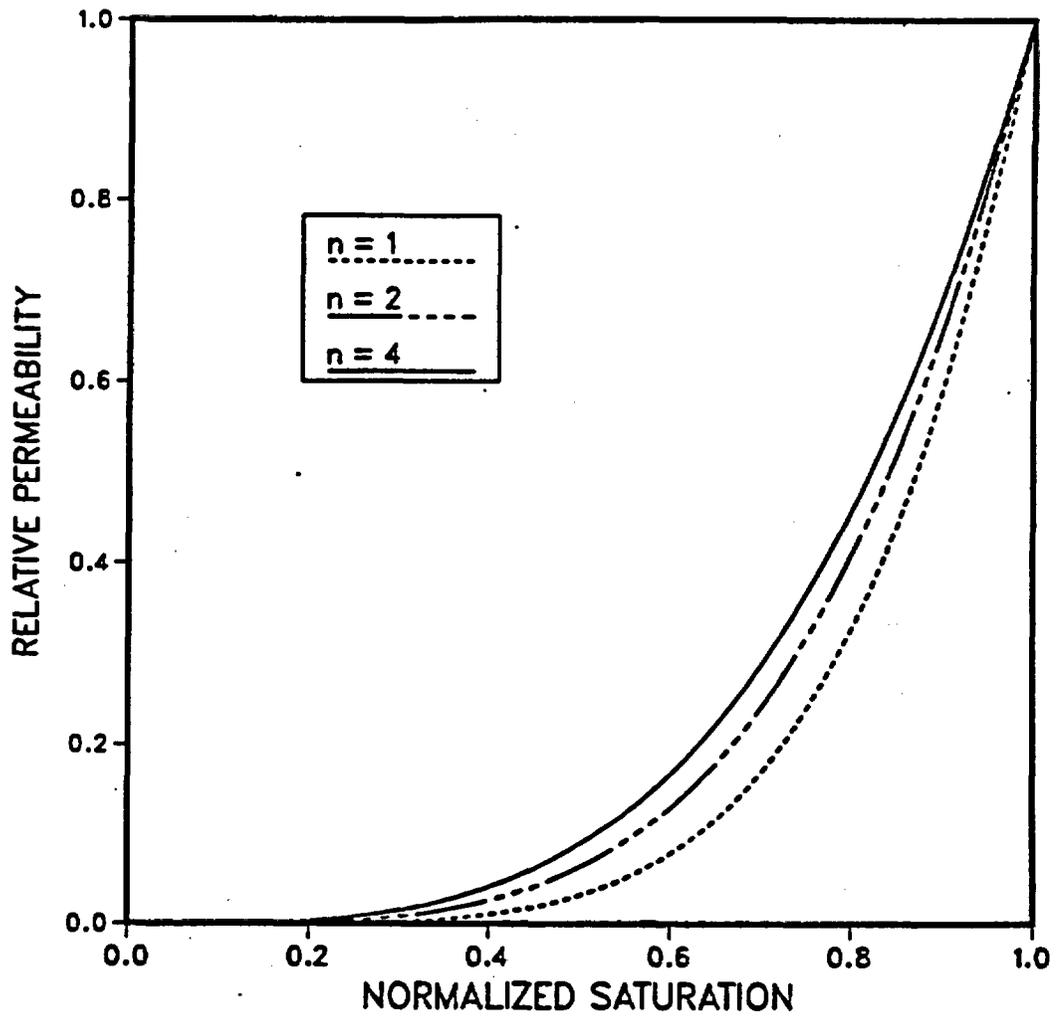


Fig. 2. Relative permeability curves of Brooks-Corey media, for different values of n .

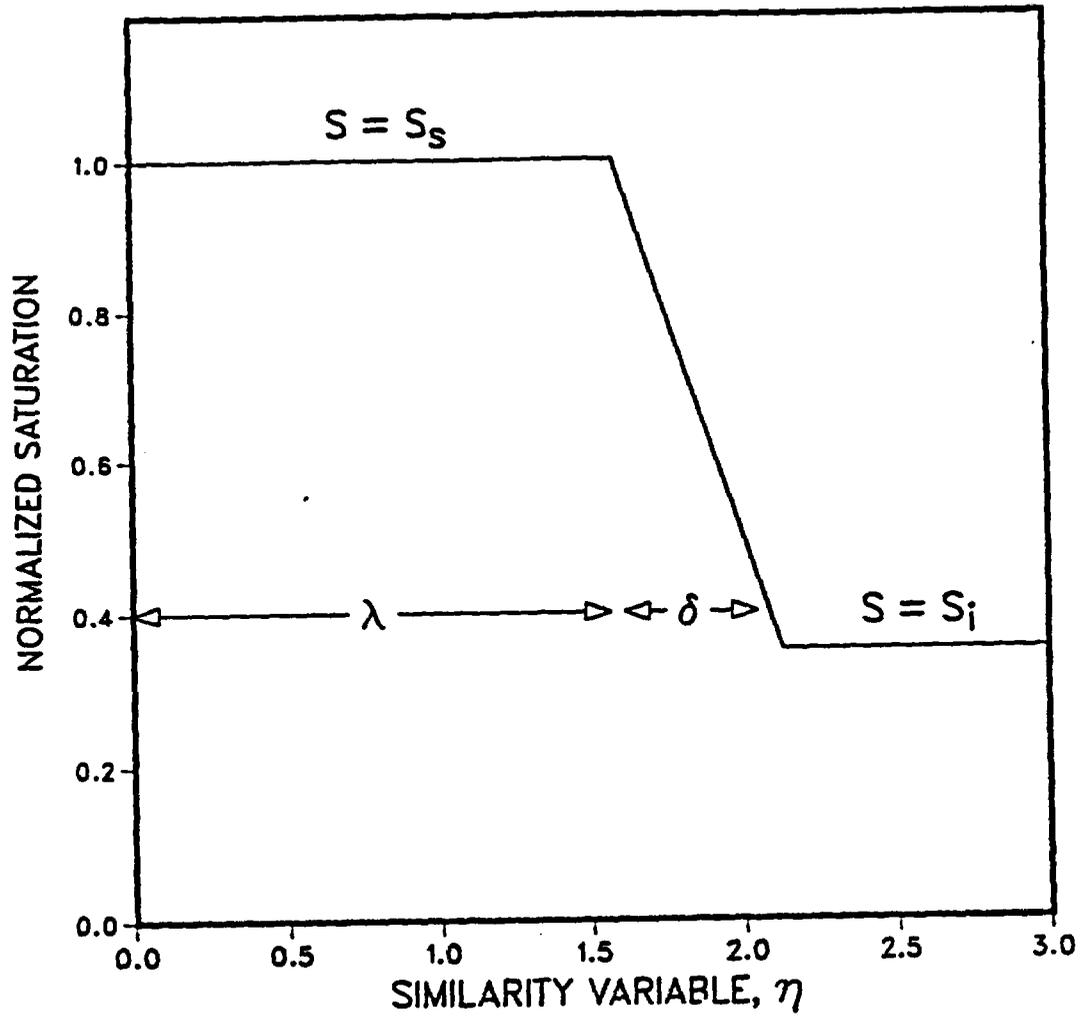


Fig. 3. Assumed saturation profile for one-dimensional absorption into a Brooks-Corey medium.

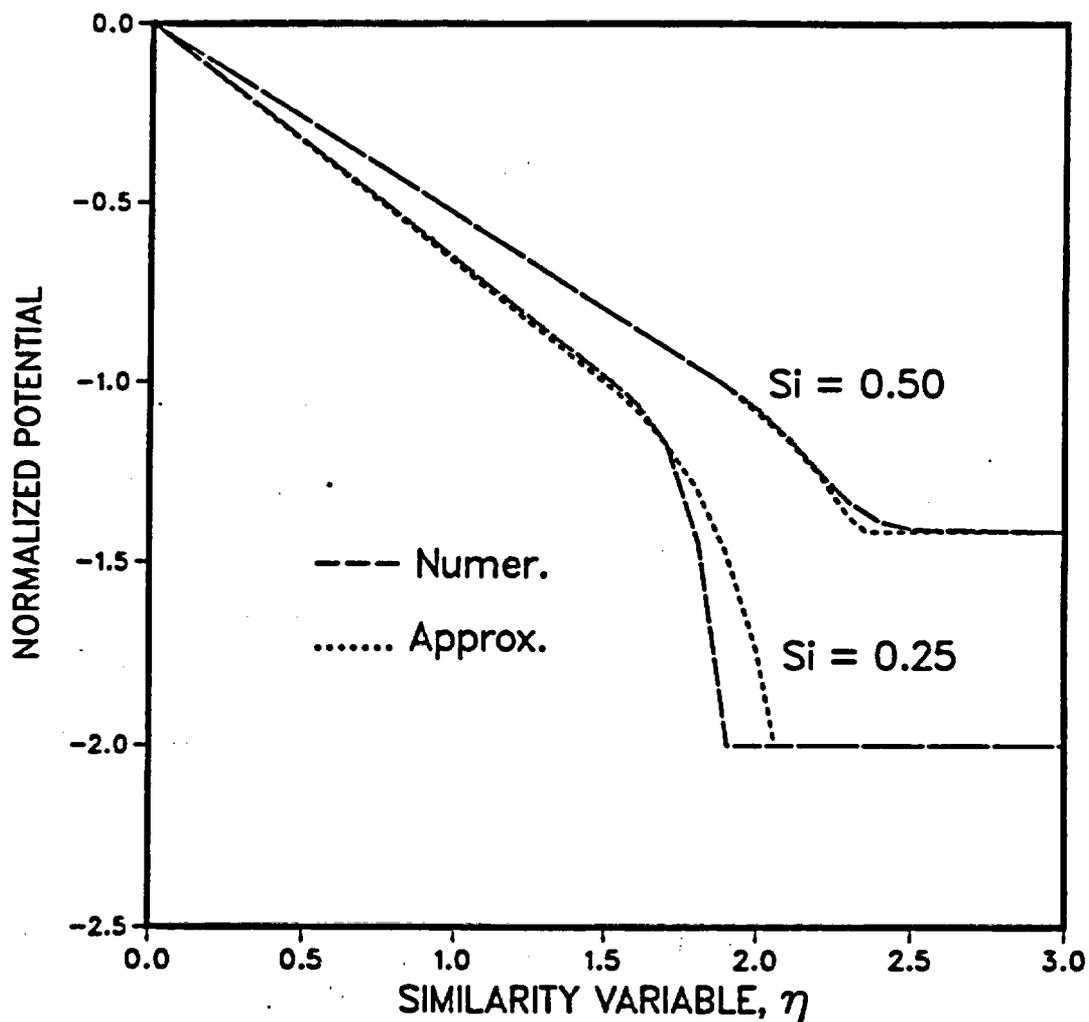


Fig. 4. Capillary pressure profiles for one-dimensional absorption in Brooks-Corey medium with $n = 2$, for various values of the initial saturation.

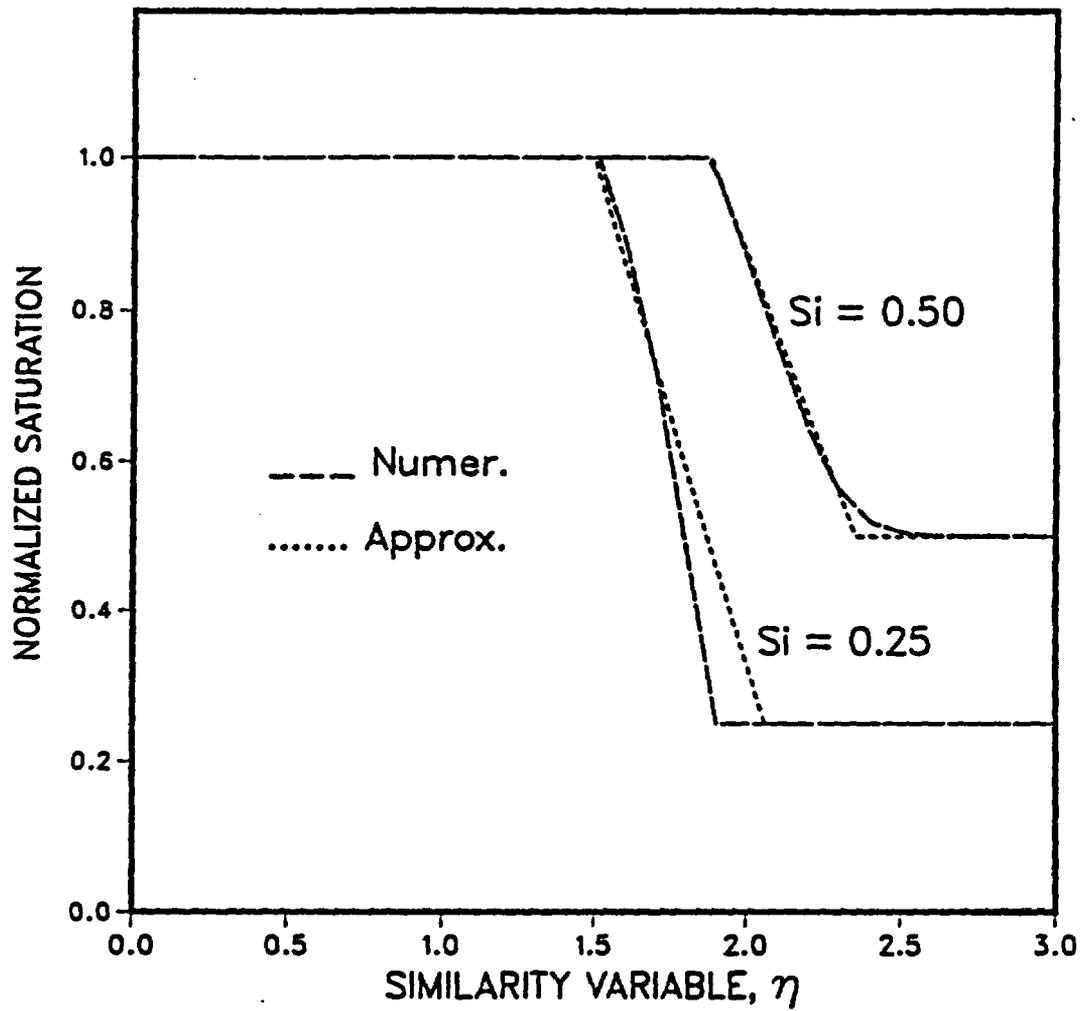


Fig. 5. Saturation profiles for one-dimensional absorption in a Brooks-Corey medium with $n = 2$, for various values of the initial saturation.

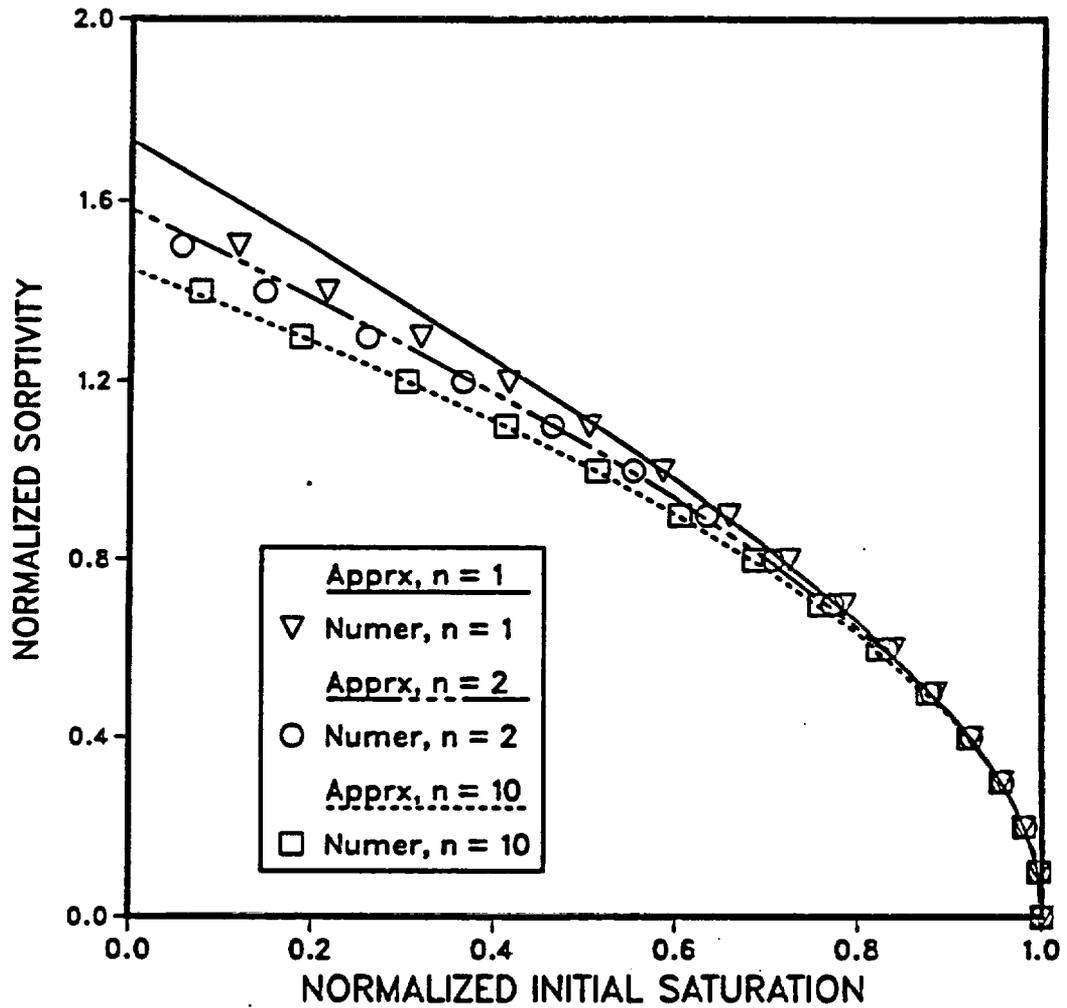


Fig. 6. Sorptivity of a Brooks-Corey medium as a function of the initial saturation, for various values of n . To simplify the figures, we take $S_r = 1$ and $S_r = 0$.

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TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720*