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CALCULATING THE UNSATURATED HYDRAULIC CONDUCTIVITY WITH A NEW CLOSED-FORM ANALYTICAL MODEL

by

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Corey (1964) and Jeppson (1974) each used an analytical expression for the conductivity based on the Burdine theory (Burdine, 1953). Brooks and Corey (1964, 1966) obtained fairly accurate predictions with their equations, even though a discontinuity is present in the slope of both the moisture retention curve and the unsaturated hydraulic conductivity curve at some negative value of the pressure head (this point is often referred to as the bubbling pressure). Such a discontinuity sometimes prevents rapid convergence in numerical saturated-unsaturated flow problems. It also appears that predictions based on the Brooks and Corey equations are somewhat less accurate than those obtained with various forms of the (modified) Millington-Quirk method.

Recently Mualem (1976a) derived a new model for predicting the hydraulic conductivity from knowledge of the soil moisture retention curve and the conductivity at saturation. Mualem's derivation leads to a simple integral formula for the unsaturated hydraulic conductivity which enables one to derive closed-form analytical expressions, provided suitable equations for the soil moisture retention curves are available. It is the purpose of this report to derive such closed-form analytical expressions. The theories of both Mualem and Burdine are used for this derivation. The resulting conductivity models generally contain three independent parameters which may be obtained from the soil moisture retention data by means of curve-fitting. Two different methods of curve-fitting are discussed in this paper, a simple graphical method which enables one to obtain the parameters without requiring computer assistance, and a more elaborate non-linear least-squares curve-fitting method requiring the assistance of a digital computer. An existing computer model was modified for this purpose and is included in the appendix. Results

MATHEMATICAL DEVELOPMENT

The following equation was derived by Mualem (1976a) for predicting the relative hydraulic conductivity (K_r) from knowledge of the soil moisture retention curve

$$K_r = \Theta^{\frac{1}{2}} \left[\int_0^{\Theta} \frac{1}{h(x)}\, dx \Bigg/ \int_0^{1} \frac{1}{h(x)}\, dx \right]^2 \tag{1}$$

where $h=h(\Theta)$ is the pressure head, given here as a function of the dimensionless moisture content:

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r}. \tag{2}$$

In this equation, s and r indicate saturated and residual values of the soil moisture content (θ), respectively. To solve (1), an expression relating the dimensionless moisture content to the pressure head is needed. An attractive class of $\Theta(h)$-functions, adopted in this study, is given by the following general equation

$$\Theta = \left[\frac{1}{1+(\alpha h)^n} \right]^m \tag{3}$$

where α, n and m are as yet undetermined parameters. To simplify notation later, h in (3) is assumed to be positive. Equation (3) with $m=1$ has been successfully used in many studies to describe soil moisture retention data (Ahuja and Schwartzendruber, 1972; Endelman *et al.*, 1974;

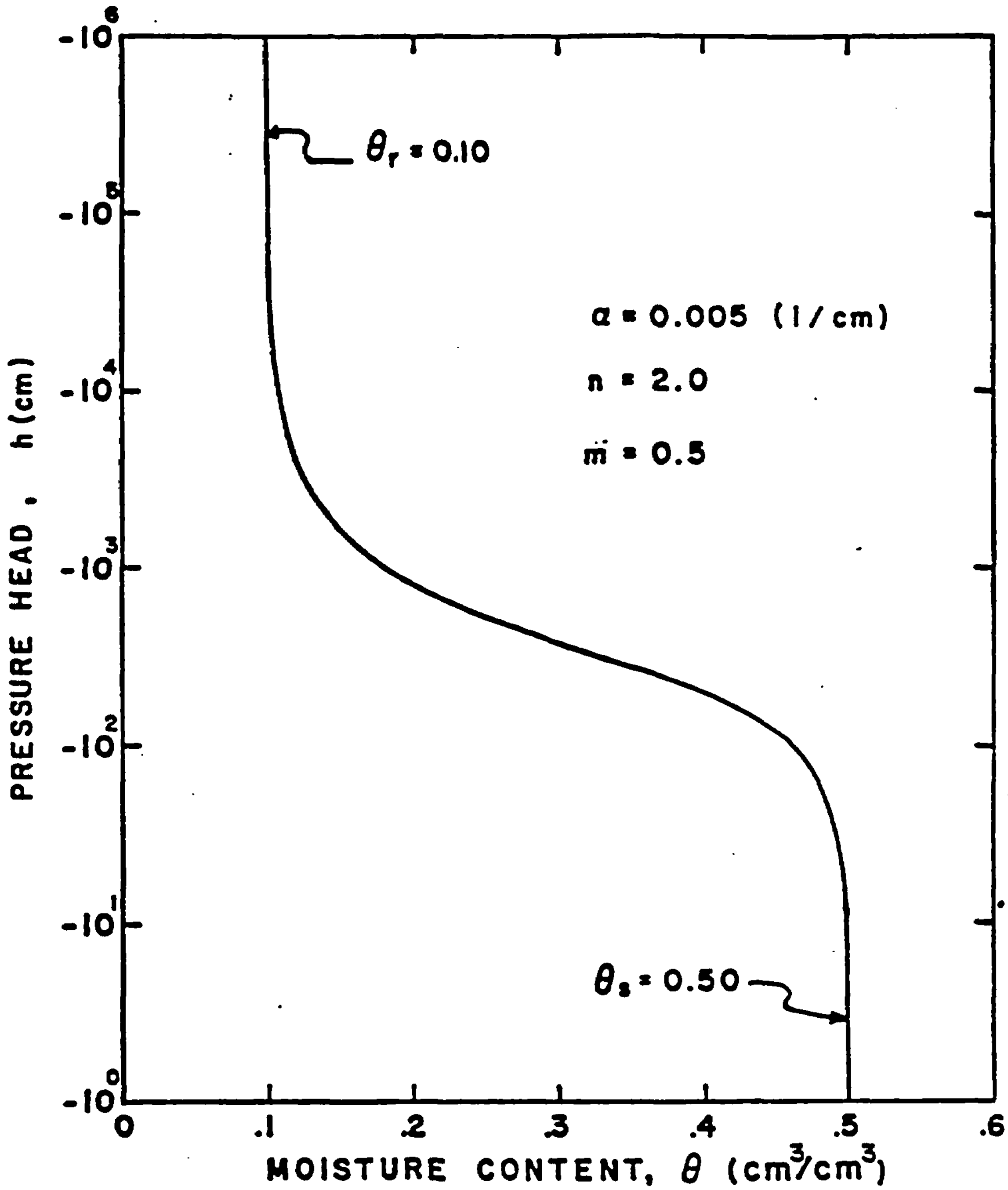

Fig. 1. Typical plot of the soil moisture retention curve based on Eq. (3).

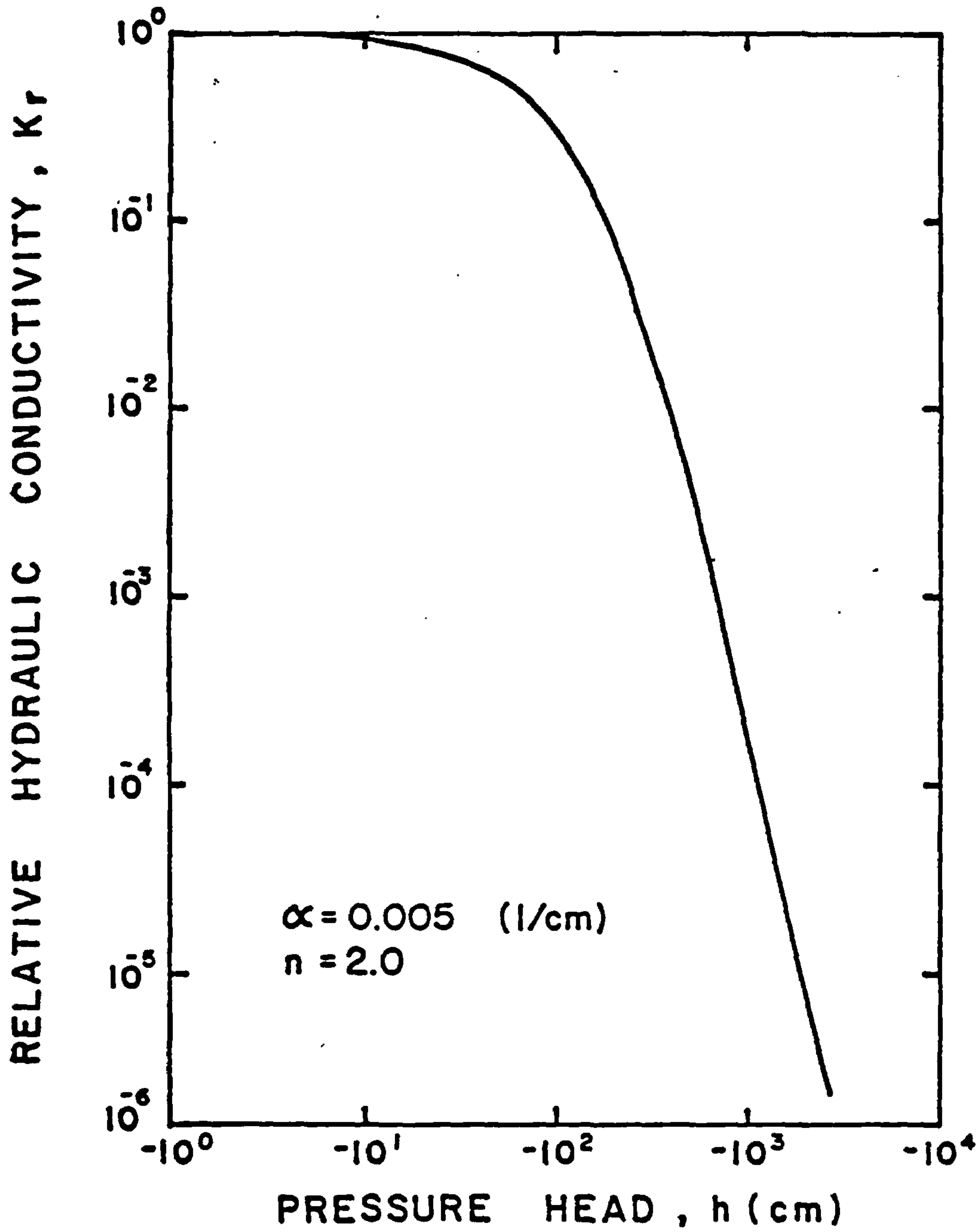

Fig. 2. Plot of the relative hydraulic conductivity versus pressure head as predicted from knowledge of the soil moisture retention curve shown in Fig. 1.

es either θ_r or θ_s. Note that the diffusivity becomes infinite when θ approaches θ_s. Only at intermediate values of the moisture content (approximately between θ=0.25 and θ=0.45 in Fig. 3) does the diffusivity acquire the often assumed exponential dependency on the moisture content. Similar features of the soil moisture diffusivity were obtained and discussed by Ahuja and Schwartzendruber (1972), using the following special form of $D(\theta)$:

$$D(\theta) = \frac{a\,\theta^p}{(\theta_s - \theta)^q} \tag{12}$$

where a, p and q are material characteristic parameters.

The soil hydraulic properties derived above were obtained by assuming that $k=m-1+1/n=0$ in (6). One may also derive closed-form expressions for other integer values of k. For k=1, for example, the conductivity becomes

$$K_r(\Theta) = \Theta^{\frac{1}{2}} \left[1 - m(1-\Theta^{1/m})^{m-1} + (m-1)(1-\Theta^{1/m})^m\right]^2. \qquad (m=2-1/n) \tag{13}$$

While this particular model is not only more complicated than model (8), it also represents only a slight pertubation of the earlier function. Hence, (13) does not present an attractive alternative for (8), and will not be discussed further.

Similar results as above for the Mualem theory may also be obtained when the Burdine theory is taken as a point of departure. The equation given by Burdine (1953) is:

$$K_r(\Theta) = \Theta^2 \int_0^{\Theta} \frac{1}{h^2(x)}\,dx \Bigg/ \int_0^1 \frac{1}{h^2(x)}\,dx\,. \tag{14}$$

The soil moisture diffusivity for this case is given by

$$D(\Theta) = \frac{(1-m)K_s}{2\alpha m(\theta_s-\theta_r)} \Theta^{(3-1/m)/2} [(1-\Theta^{1/m})^{-(m+1)/2} - (1-\Theta^{1/m})^{(m-1)/2}]. \tag{21}$$

Preliminary tests indicated that (8) generated results that were, in most cases, in better agreement with experimental data than (19). Through an extensive series of comparisons, also Mualem (1976a) concluded that predictions based on his theory (i.e., based directly on Eq. (1) by means of numerical approximations) were generally more accurate than those based on various forms of the Burdine theory (including the Millington-Quirk method). It is not the intent of this paper to give accuracy comparisons between various closed-form analytical conductivity expressions. Only a brief discussion of the equations derived by Brooks and Corey (1964) will be given here, since their model of the soil moisture retention curve represents a limiting case of the moisture retention model discussed in this study.

Brooks and Corey (1964; 1966) concluded from comparisons with a large number of experimental data that the soil moisture retention curve $\Theta(h)$ could be described reasonably well with the following general equation

$$\Theta = (h/h_b)^{-\lambda} \qquad (h<h_b) \tag{22}$$

where h_b is the bubbling pressure (approximately equal to the air entry value), and λ a soil characteristic parameter. Comparing Eq. (22) and (3), one sees that (3) reduces to (22) for large values of the pressure head, i.e.

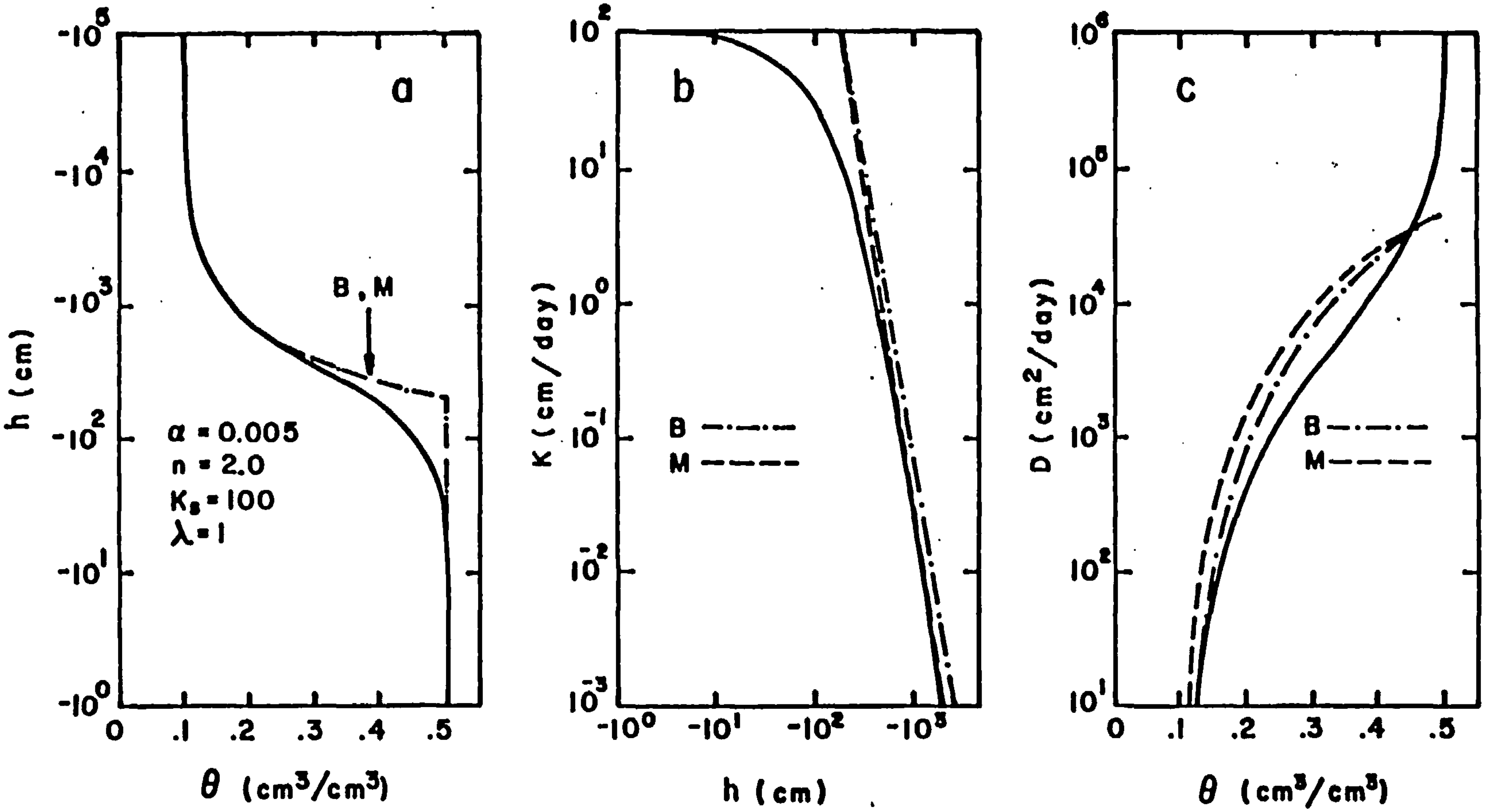

Fig. 4. Comparison of the proposed soil hydraulic functions (solid lines) with curves obtained by applying either the Mualem theory (M; dashed lines) or the Burdine theory (B; dashed-dotted lines) to the Brooks and Corey model of the soil moisture retention curve.

PARAMETER ESTIMATION

The soil moisture content (θ) as a function of the pressure head (h) is given by Eq. (2) and (3), i.e.,

$$\theta = \theta_r + \frac{(\theta_s - \theta_r)}{[1 + (\alpha h)^n]^m} \tag{28}$$

where, as before, it is understood that h is positive, and where for the Mualem model

$$m = 1 - 1/n. \tag{29}$$

Equation (28) contains four independent parameters (θ_r, θ_s, α, and n), which have to be estimated from observed soil moisture retention data. Of these four, the saturated moisture content (θ_s) is probably always available as it is easily obtained experimentally. Also the residual moisture content (θ_r) may be measured experimentally, for example by determining the moisture content on very dry soil. Unfortunately, θ_r-measurements are not always made routinely, and hence have to be estimated by extrapolating existing soil moisture retention data. Assuming for the moment that sufficiently accurate estimates of both θ_r and θ_s are available, the following procedure can then be used to obtain estimates of the remaining parameters α and n.

Differentiation of (28) gives

$$\frac{d\theta}{dh} = \frac{-\alpha m(\theta_s - \theta_r)}{1-m} \Theta^{1/m}(1-\Theta^{1/m})^m \tag{30}$$

Combining (32), (33), and (34b) leads to the following expression for S

$$S = 2.303 \frac{m}{1-m} \Theta (1-\Theta^{1/m}). \tag{35}$$

The best location on the $\theta(h)$ curve for evaluating the slope S is about halfway between θ_r and θ_s. Let P be the point on the soil moisture retention curve for which $\Theta=\frac{1}{2}$ (see Fig. 5). From Eq. (2) and (31) it follows then that the coordinates of P are given by

$$\theta_P = (\theta_s+\theta_r)/2 \tag{36a}$$

$$h_P = \frac{1}{\alpha} (2^{1/m}-1)^{1-m} \tag{36b}$$

while Eq. (35) reduces to

$$S_P(m) = 1.151 \frac{m}{1-m} (1-2^{-1/m}). \tag{37a}$$

The subscript P in these equations is used to indicate evaluation at P. Equation (37a) can also be expressed in terms of n

$$S_P(n) = 1.151 (n-1)(1-2^{n/(1-n)}). \tag{37b}$$

Figure 6 gives a plot of S_P as a function of both n and m. This figure may be used to obtain an estimate of n once the slope S_P is determined graphically from the experimental data. For relatively large values of n, (37b) is closely approximated by

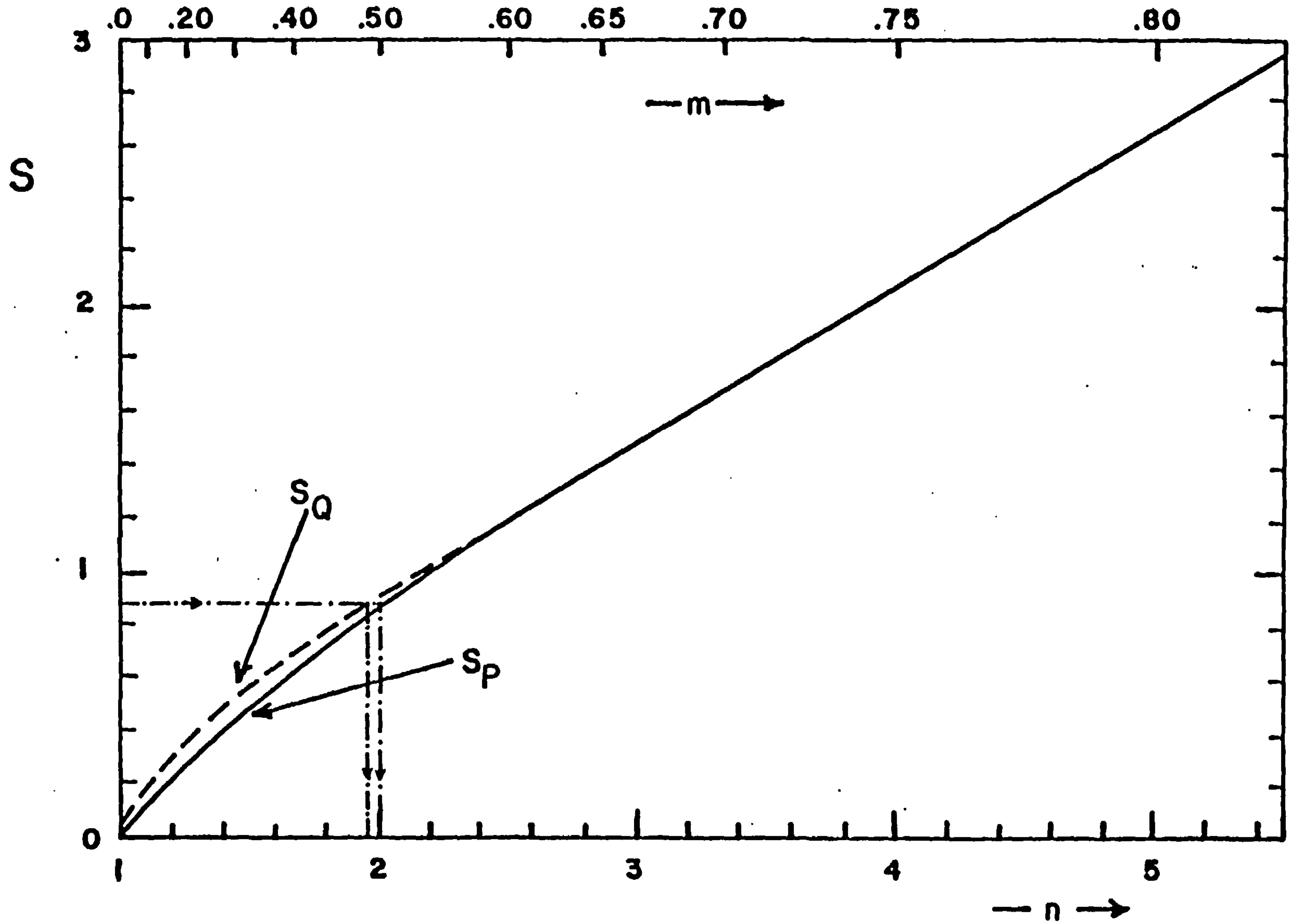

Fig. 6. Plots of the dimensionless slopes S_P and S_Q as functions of the parameters n and m (=1-1/n).

$$\theta_Q = \left[\frac{m}{1+m}\right]^m . \tag{42}$$

Hence, the coordinates of the inflection point are

$$\theta_Q = \theta_r + (\theta_s - \theta_r) \left[\frac{m}{1+m}\right]^m \tag{43a}$$

$$h_Q = \frac{1}{\alpha} \; m^{m-1} . \tag{43b}$$

From (43a) it follows that, at least theoretically, one could estimate the value of m directly by locating the inflection point on the soil moisture retention curve. However, from Fig. 5 it is clear that it is not easy to determine this point accurately (even less so when the curve is based on experimental data). It seems, therefore, better to again estimate m from the slope of the curve. Substitution of (42) into (35) gives

$$S_Q(m) = \frac{2.303}{1-m} \left[\frac{m}{1+m}\right]^{m+1} \tag{44a}$$

or, in terms of n,

$$S_Q(n) = 2.303 \; n \left[\frac{n-1}{2n-1}\right]^{2-1/n} . \tag{44b}$$

Figure 6 shows that $S_P(n)$ and $S_Q(n)$ define approximately the same curve, especially for the larger n-values. This is not surprising since the points P and Q are generally very close together on the soil moisture retention curve. Fig. 5, furthermore, shows that both points define approximately the same gradient. Hence the n-values obtained from the sketched

$$\alpha_Q = \frac{1}{h_Q}\, m^{m-1}$$

$$= 10^{-2.43}(0.49)^{-0.51} = 0.0053.$$

The relative hydraulic conductivities hence are (Eq. 8):

$$K_r(\Theta) = \Theta^{1/2}\left[1-(1-\Theta^{2.00})^{0.50}\right]^2 \quad \text{(based on } S_P\text{)} \qquad (46a)$$

$$K_r(\Theta) = \Theta^{1/2}\left[1-(1-\Theta^{2.04})^{0.49}\right]^2 \quad \text{(based on } S_Q\text{)}. \qquad (46b)$$

Equation (46a) exactly reproduces the conductivity equation one would have obtained if the original data shown in Fig. 5 were used in Eq. (8). Equations (46a) and (46b) generate nearly the same curve when plotted versus or versus h. Minor differences between the curves occur only at the extreme dry side of the curves, and are caused by the fact that the same slope was used to calculate both S_P and S_Q (in reality, S_Q should have been measured somewhat larger than S_P).

The parameters α and n can also be estimated from soil moisture retention data which are plotted on a normal θ versus h scale. The procedure for finding the two parameters is similar to that used before. Equation (37) still holds provided, however, that S is calculated with Eq. (33) and (34). These two equations show that now estimates of both h and the slope, $d\theta/dh$, are necessary for evaluating S. Equations (43) and (44), on the other hand, have to be modified because the inflection point of the $\theta(h)$-curve does not coincide with the inflection point of the $\theta(\log h)$ curve. Contrary to (40), one has now

INFLUENCE OF THE RESIDUAL MOISTURE CONTENT

The foregoing discussion assumes that independent measurements of the saturated and residual moisture contents are available. While θ_s is usually easy to obtain by direct measurement, θ_r is often much more difficult to quantify. In fact, in many cases θ_r may become an ill-defined parameter. The residual moisture content in this report is defined as the moisture content for which the gradient ($d\theta/dh$) becomes zero (excluding the region near θ_s which has also a zero gradient). Also the hydraulic conductivity will approach zero when θ approaches θ_r. From a practical point of view it seems sufficient to define θ_r as the moisture content at some large negative value of the pressure head, e.g., at -10^{-6} cm. Even in that case, however, significant decreases in h are likely to result in further desorption of moisture. It seems that such further changes in θ are fairly unimportant for most practical field problems. In fact, they would be inconsistent with the general shape of the $\theta(h)$-curve defined by (22), and probably invalidate the concept of a residual moisture content itself. A reasonable estimate of θ_r is necessary for an accurate prediction of the hydraulic conductivity, even though its influence on the predictions is generally less than that of α and n. The following example problem demonstrates the effect of θ_r on the conductivity predictions.

Figure 7a shows the soil moisture retention curve of Silt Loam G.E.3, for values of h between zero and 10^{-3} cm. (Reisenauer, 1963). The open circles represent data points of the curve, and were taken from the catalogue of Mualem (1976b). Because only a limited portion of the the curve is defined, an accurate estimate of θ_r is not easy to obtain.

Three different values for θ_r were chosen rather arbitrarily (0.05, 0.10, and 0.15 cm^3/cm^3, respectively), and subsequently used to calculate the hydraulic conductivity. The calculations, based on Eq. (36) and (37), are summarized in Table 1. The slope of the $\theta(\log h)$-curve at $\Theta=\frac{1}{2}$ was assumed to be the same for all three cases (step 6 in Table 1), a sufficiently accurate assumption in this case. Figure 7b compares the calculated retention curves with the experimental curve. Each of the

Table 1. Calculation of the parameters α and n from the observed soil moisture retention curve of Silt Loam G.E.3, using three different values for θ_r (θ_s=0.396)

STEP	θ_r^a	θ_r^b	θ_r^c
1. Estimate θ_r	0.050	0.100	0.150
2. Obtain $(\theta_s-\theta_r)$	0.346	0.296	0.246
3. Calculate $\theta_p=(\theta_s+\theta_r)/2$.	0.223	0.248	0.273
4. Obtain $\log(h_p)$ from data (Fig. 7a)	2.76	2.65	2.55
5. Calculate h_p	575.	447.	355.
6. Estimate $d\theta/d(\log h)$ at θ_p (Fig. 7a) (=0.44/1.8)	0.244	0.244	0.244
7. Calculate S_p [=0.244/$(\theta_s-\theta_r)$] (Eq. 34b)	0.706	0.826	0.994
8. Obtain n from Fig. 6 or Eq. (39)	1.77	1.95	2.21
9. Calculate m (=1-1/n)	0.435	0.487	0.548
10. Calculate α (Eq. 36b)	0.0038	0.0040	0.0043

three curves describes the experimental curve fairly accurately, although curve c (based on θ_r^c) fits the data points somewhat better at the dry end of the curve than the other two. On the other hand, this curve also

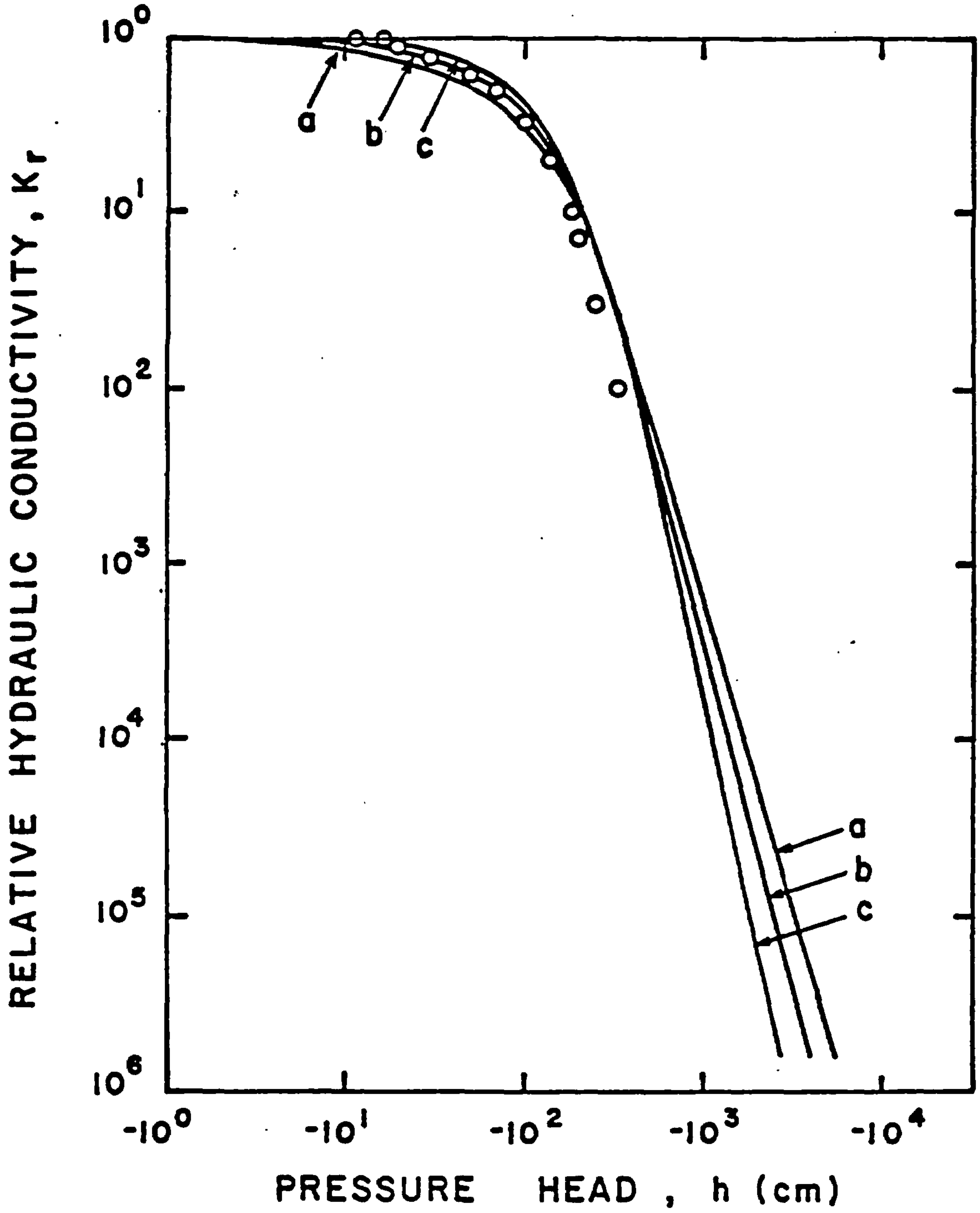

Fig. 8. Comparison of observed (open circles) and calculated curves (solid lines) of the relative hydraulic conductivity of Silt Loam G.E.3. The predicted curves were obtained for three different values of the residual moisture content, θ_r: 0.05 (curve a), 0.10 (curve b), and 0.15 cm^3/cm^3 (curve c).

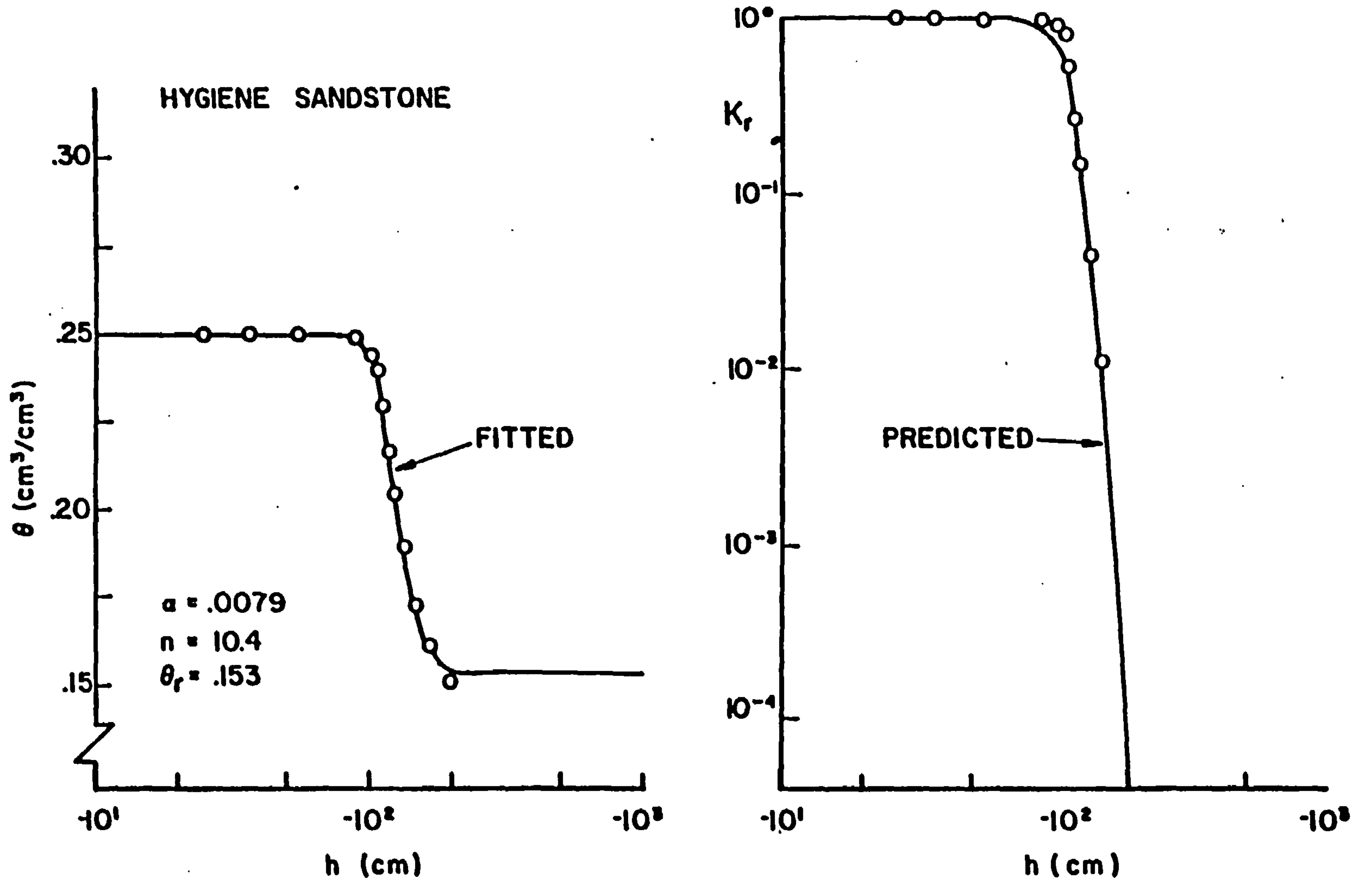

Fig. 9. Observed (open circles) and calculated curves (solid lines) of the soil hydraulic properties of Hygiene Sandstone. The relative hydraulic conductivity was predicted from knowledge of the curve-fitted soil moisture retention curve.

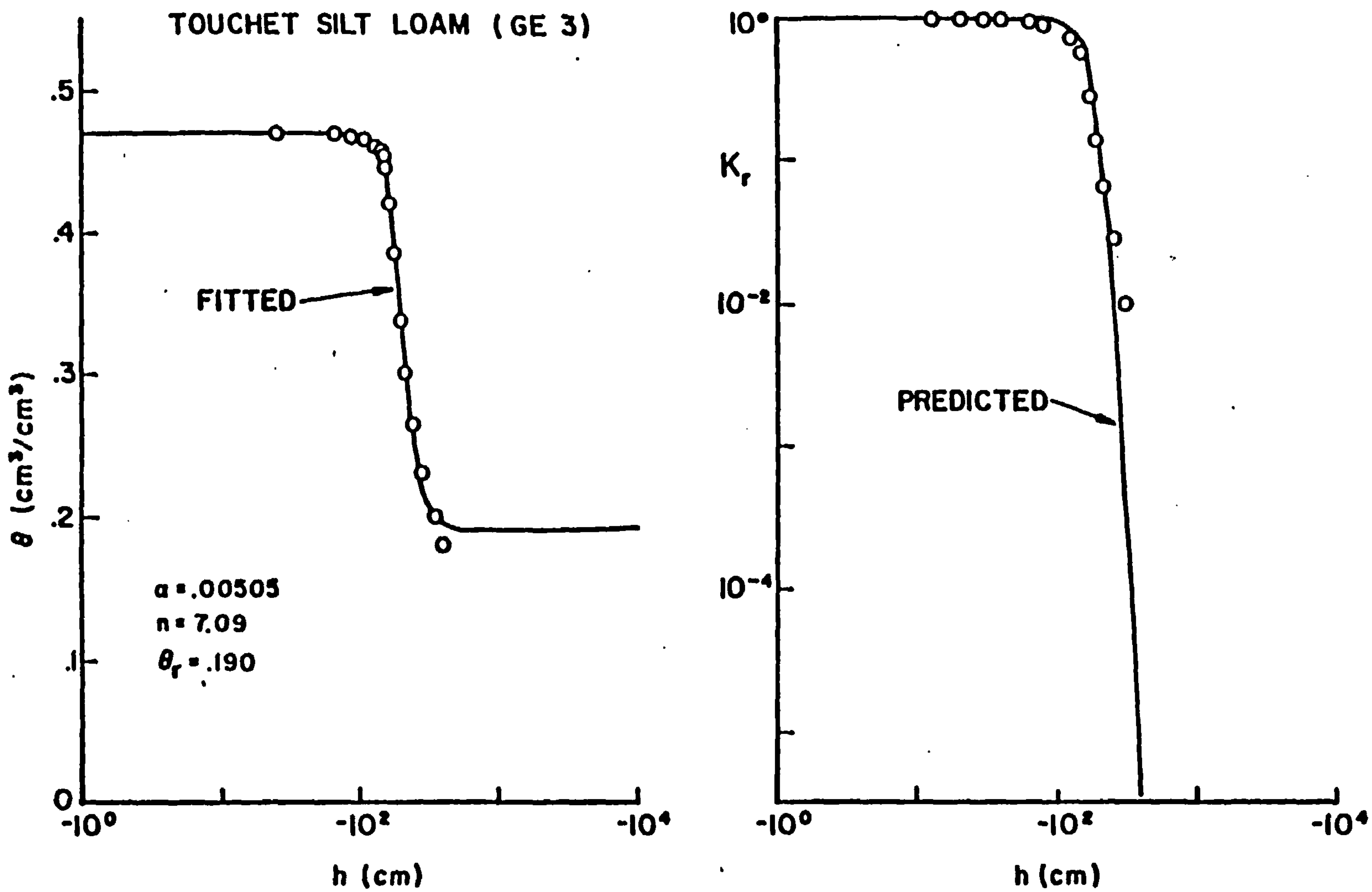

Fig. 10. Observed (open circles) and calculated curves (solid lines) of the soil hydraulic properties of Touchet Silt Loam G.E.3. The relative hydraulic conductivity was predicted from knowledge of the curve-fitted soil moisture retention curve.

program also provides for a correlation matrix between the different parameters. Results, for example, show that θ_r is highly correlated with n but much less than with α, and that α and n are nearly independent of each other. Some of these effects are also noticeable from the calculations in Table 1.

The first three examples each showed excellent agreement between observed and predicted conductivity curves. Predictions obtained for Beit Netofa Clay (Rawitz, 1965), however, were found to be much less accurate (Fig. 12). The higher conductivity values are seriously underpredicted, and also the general shape of the predicted curve is considerably different from the observed one. It seems that much of the poor predictions can be traced back to the inability of equation (28) to match the observed soil moisture retention data. For example, the residual moisture content was estimated to be zero, a rather surprising result since clay soils have generally higher θ_r-values than coarser soils (the saturated hydraulic conductivity of this soil is only 0.082 cm/day). Limited data at the lower moisture contents further increases doubt about the accuracy of the fitted θ_r-value. A careful inspection of the observed curve shows that the gradient of the curve changes fairly suddenly at approximately h=-10,000 cm (the slope suddenly becomes more negative). The location of the last four data points, in particular, appears to be inconsistent with the general shape of curves based on (28). With some imagination one could also identify an inflection point on the observed curve at a pressure head of about -2,000 cm. The observed curve should have become flatter from that point on if equation (28) were to describe the data points. Because of the seemingly unreasonable low value of θ_r, the break in the slope of the curve at h=-10,000 cm, and the presence of an inflection point at h=-2,000 cm, an attempt was made to improve the predictions by deleting rather arbitrarily the last four data points at the dry side of the curve. Fig. 13 shows that the soil moisture retention curve is now much better described (with the obvious exception of the last four data points). Also the description of the conductivity curve is improved somewhat. At least the general shape of the curve is described more accurately, even though the predicted curve is still displaced to the right of the observed one. The example shows that by deleting only four points at the dry end of the curve a completing different value of θ_r is obtained (0.286 versus 0.0 cm^3/cm^3). This case demonstrates again the importance of having some independent procedure for estimating the residual moisture content.

Results for Guelph Loam (Elrick and Bowman, 1964) are given in Fig. 14. This example represents a case in which hysteresis is present in the soil moisture retention curve. The observed data of this example were taken directly from the original study (Figs. 2 and 3 of Elrick and Bowman, 1964). For the wetting branch a maximum ("saturated") value of 0.434 for the moisture content was used, being the highest measured value. Also the wetting branch of the hydraulic conductivity curve was matched to the highest value of K_r measured during wetting (Fig. 14). The value of θ_r, furthermore, was assumed to be the same for drying and wetting, and was obtained from the drying branch of the curve. Both the drying and wetting branches of the soil moisture retention curve are adequately described by (28). Also the conductivity curves are reasonably well described, even though the predicted curves are slightly below the observed ones. Note that some hysteresis is predicted in the relative hydraulic conductivity. Although this is generally to be expected when two different

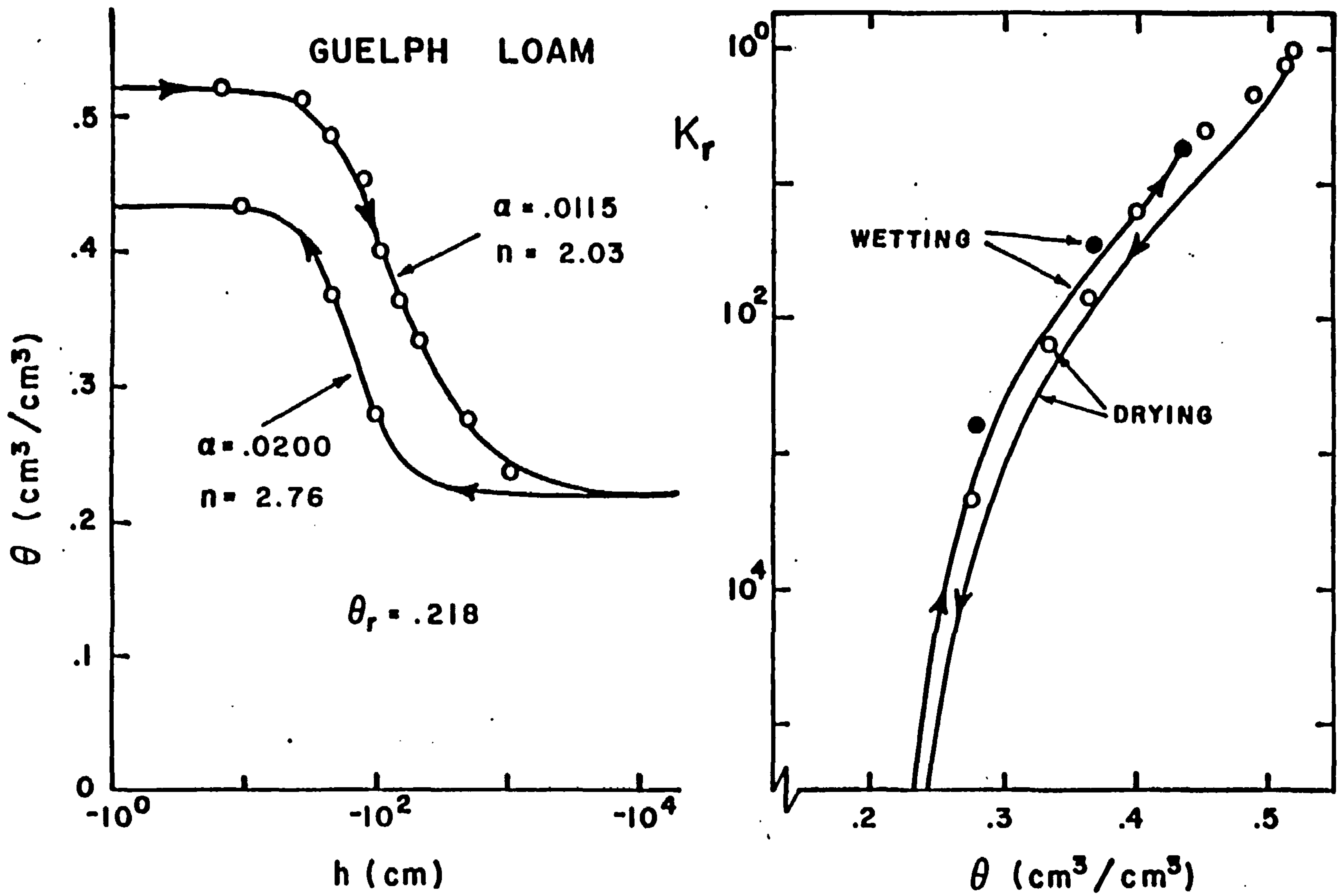

Fig. 14. Observed (circles) and calculated curves (solid lines) of the soil hydraulic properties of Guelph Loam. The drying and wetting branches of the relative hydraulic conductivity curve were predicted from knowledge of the curve-fitted branches of the soil moisture retention curve.

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APPENDIX A

SOHYP:

A COMPUTER MODEL FOR CALCULATING

THE SOIL HYDRAULIC PROPERTIES

FROM SOIL MOISTURE RETENTION DATA.

program only calculates best-fit values of α and n, and assumes that θ_r is known beforehand. The value of θ_r is now given as an input variable (see Table A2). Values of α and n are still calculated by means of Eq. (A1) and (A2) (i.e. the Mualem theory still applies). If MODE equals three, the computer model again calculates best-fit values of the three parameters (θ_r, α, and n), but it is now assumed that the Burdine theory applies. Hence Eq. (A1) and (A3) are now used in the program. In each case the computer program provides for a table of the hydraulic properties of the soil (see Table A4), consistent with the value of MODE selected.

TABLE A1 (CONTINUED):

VARIABLE	*DEFINITION*
STOPCR	Stop Criterion. Iteration process stops when the relative change in each coefficient becomes less than STOPCR.
TITLE(I)	Array containing information of title cards.
WC	Volumetric moisture content (θ).
WCR	Residual moisture content (θ_r).
WCS	Saturated moisture content (θ_s).
X(I)	Array of observed pressure heads (values are assumed to be positive).
Y(I)	Array of observed moisture contents.

Table A3. Input data for example 3 (Silt Loam G.E.3).

```
                     1         2         3         4         5
Column:     12345678901234567890123456789012345678901234567890

Card

  1             1
  2                      SILT LOAM G.E.3
  3             1    3   13    0         0.18     0.396      4.96
  4           0.180    0.002       2.3
  5          WCR      ALPHA       N
  6            10.0   0.396
  7            20.0   0.394
  8            43.0   0.390
  9            60.0   0.3855
 10            80.0   0.379
 11           111.0   0.370
 12           190.0   0.340
 13           285.0   0.300
 14           400.0   0.260
 15           600.0   0.220
 16           800.0   0.200
 17           900.0   0.194
 18          1000.0   0.190
```

NON-LINEAR LEAST SQUARES ANALYSIS

SILT LOAM G.E.3

INPUT PARAMETERS

MODEL NUMBER	1
NUMBER OF COEFFICIENTS	3
NUMBER OF OBSERVATIONS	13
RESIDUAL MOISTURE CONTENT (FOR MODEL 2)	0.1800
SATURATED MOISTURE CONTENT	0.3960
SATURATED HYDRAULIC CONDUCTIVITY	4.9600

OBSERVED DATA

OBS. NO.	PRESSURE HEAD	MOISTURE CONTENT
1	10.00	0.3960
2	20.00	0.3940
3	43.00	0.3900
4	60.00	0.3855
5	80.00	0.3790
6	111.00	0.3700
7	190.00	0.3400
8	285.00	0.3000
9	400.00	0.2600
10	600.00	0.2200
11	800.00	0.2000
12	900.00	0.1940
13	1000.00	0.1900

PRESSURE	LOG P	WC	REL K	LOG RK	ABS K	LOG KA	DIFFUS	LOG D
0.0		0.3960	0.100E 01		0.496E 01			
0.141E 01	0.150	0.3960	0.991E 00	-0.004	0.492E 01	0.692	0.939E 06	5.973
0.168E 01	0.225	0.3960	0.989E 00	-0.005	0.491E 01	0.691	0.781E 06	5.893
0.200E 01	0.300	0.3960	0.987E 00	-0.006	0.490E 01	0.690	0.649E 06	5.812
0.237E 01	0.375	0.3960	0.985E 00	-0.007	0.488E 01	0.689	0.539E 06	5.732
0.282E 01	0.450	0.3960	0.982E 00	-0.008	0.487E 01	0.687	0.448E 06	5.651
0.335E 01	0.525	0.3960	0.978E 00	-0.010	0.485E 01	0.686	0.371E 06	5.570
0.398E 01	0.600	0.3960	0.974E 00	-0.012	0.483E 01	0.684	0.308E 06	5.488
0.473E 01	0.675	0.3960	0.968E 00	-0.014	0.480E 01	0.682	0.255E 06	5.407
0.562E 01	0.750	0.3959	0.962E 00	-0.017	0.477E 01	0.679	0.211E 06	5.325
0.668E 01	0.825	0.3959	0.955E 00	-0.020	0.473E 01	0.675	0.174E 06	5.242
0.794E 01	0.900	0.3959	0.946E 00	-0.024	0.469E 01	0.671	0.144E 06	5.158
0.944E 01	0.975	0.3958	0.935E 00	-0.029	0.464E 01	0.666	0.119E 06	5.074
0.112E 02	1.050	0.3957	0.922E 00	-0.035	0.457E 01	0.660	0.975E 05	4.989
0.133E 02	1.125	0.3956	0.907E 00	-0.043	0.450E 01	0.653	0.800E 05	4.903
0.158E 02	1.200	0.3955	0.888E 00	-0.051	0.441E 01	0.644	0.654E 05	4.815
0.188E 02	1.275	0.3953	0.867E 00	-0.062	0.430E 01	0.633	0.532E 05	4.726
0.224E 02	1.350	0.3949	0.841E 00	-0.075	0.417E 01	0.620	0.432E 05	4.635
0.266E 02	1.425	0.3945	0.811E 00	-0.091	0.402E 01	0.604	0.348E 05	4.542
0.316E 02	1.500	0.3939	0.775E 00	-0.111	0.384E 01	0.585	0.279E 05	4.446
0.376E 02	1.575	0.3930	0.734E 00	-0.134	0.364E 01	0.561	0.222E 05	4.347
0.447E 02	1.650	0.3917	0.686E 00	-0.164	0.340E 01	0.532	0.176E 05	4.245
0.531E 02	1.725	0.3899	0.631E 00	-0.200	0.313E 01	0.496	0.137E 05	4.138
0.631E 02	1.800	0.3874	0.570E 00	-0.244	0.283E 01	0.451	0.106E 05	4.026
0.750E 02	1.875	0.3640	0.502E 00	-0.299	0.249E 01	0.396	0.811E 04	3.909
0.891E 02	1.950	0.3794	0.430E 00	-0.367	0.213E 01	0.329	0.611E 04	3.786
0.106E 03	2.025	0.3732	0.355E 00	-0.450	0.176E 01	0.246	0.453E 04	3.656
0.126E 03	2.100	0.3650	0.281E 00	-0.551	0.139E 01	0.144	0.330E 04	3.518
0.150E 03	2.175	0.3547	0.212E 00	-0.675	0.105E 01	0.021	0.236E 04	3.373
0.178E 03	2.250	0.3421	0.151E 00	-0.822	0.747E 00	-0.127	0.166E 04	3.221
0.211E 03	2.325	0.3272	0.101E 00	-0.996	0.500E 00	-0.301	0.115E 04	3.061
0.251E 03	2.400	0.3105	0.634E-01	-1.198	0.315E 00	-0.502	0.783E 03	2.894
0.299E 03	2.475	0.2926	0.375E-01	-1.426	0.186E 00	-0.730	0.526E 03	2.721
0.355E 03	2.550	0.2743	0.210E-01	-1.678	0.104E 00	-0.983	0.349E 03	2.543
0.422E 03	2.625	0.2563	0.112E-01	-1.953	0.553E-01	-1.257	0.230E 03	2.361
0.501E 03	2.700	0.2394	0.569E-02	-2.245	0.282E-01	-1.549	0.150E 03	2.176
0.596E 03	2.775	0.2238	0.281E-02	-2.551	0.139E-01	-1.856	0.973E 02	1.988
0.708E 03	2.850	0.2100	0.135E-02	-2.869	0.670E-02	-2.174	0.629E 02	1.798
0.841E 03	2.925	0.1978	0.637E-03	-3.196	0.316E-02	-2.500	0.405E 02	1.608
0.100E 04	3.000	0.1873	0.296E-03	-3.529	0.147E-02	-2.833	0.260E 02	1.416

Table A5. Fortran listing of SOHYP.

```
      IF(MODE.NE.2) WRITE(6,1026) NIT,B(1),B(2),B(3),SSQ,MODE
C
C     ----- BEGIN OF ITERATION -----
   34 NIT=NIT+1
      GA=0.1*GA
      DO 38 J=1,NP
      TEMP=TH(J)
      TH(J)=1.01*TH(J)
      Q(J)=0
      CALL MODEL(TH,DELZ(1,J),NOB,X,WCS,MODE,NP,WCR)
      DO 36 I=1,NOB
      DELZ(I,J)=DELZ(I,J)-F(I)
   36 Q(J)=Q(J)+DELZ(I,J)*R(I)
      Q(J)=100.*C(J)/TH(J)
C
C     ----- STEEPEST DESCENT -----
   38 TH(J)=TEMP
      DO 44 I=1,NP
      DO 42 J=1,I
      SUM=0
      DO 40 K=1,NOB
   40 SUM=SUM+DELZ(K,I)*DELZ(K,J)
      D(I,J)=10000.*SUM/(TH(I)*TH(J))
   42 D(J,I)=D(I,J)
C
C     ----- D = MOMENT MATRIX -----
   44 E(I)=SQRT(D(I,I))
   50 DO 52 I=1,NP
      DO 52 J=1,NP
   52 A(I,J)=D(I,J)/(E(I)*E(J))
C
C     ----- A IS THE SCALED MOMENT MATRIX -----
      DO 54 I=1,NP
      P(I)=Q(I)/E(I)
      PHI(I)=P(I)
   54 A(I,I)=A(I,I)+GA
      CALL MATINV(A,NP,P)
C
C     ----- P/E IS THE CORRECTION VECTOR -----
      STEP=1.0
   56 DO 58 I=1,NP
   58 TB(I)=P(I)*STEP/E(I)+TH(I)
      DO 62 I=1,NP
      IF(TH(I)*TB(I))66,66,62
   62 CONTINUE
      SUMB=0.0
      CALL MODEL(TB,F,NOB,X,WCS,MODE,NP,WCR)
      DO 64 I=1,NCB
      R(I)=Y(I)-F(I)
   64 SUMB=SUMB+R(I)*R(I)
   66 SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      DO 68 I=1,NP
```

```
      TSEC=TVAR*SECOEF
      TMCOE=TH(I)-TSEC
      TPCOE=TH(I)+TSEC
      K=2*I
      J=K-1
  108 WRITE(6,1058) BI(J),BI(K),TH(I),SECOEF,TVALUE,TMCOE,TPCOE
C
C     ----- PREPARE FINAL OUTPUT -----
      LSORT(1)=1
      DO 116 J=2,NOB
      TEMP=R(J)
      K=J-1
      DO 111 L=1,K
      LL=LSORT(L)
      IF(TEMP-R(LL)) 112,112,111
  111 CONTINUE
      LSORT(J)=J
      GO TO 116
  112 KK=J
  113 KK=KK-1
      LSORT(KK+1)=LSORT(KK)
      IF(KK-L) 115,115,113
  115 LSORT(L)=J
  116 CONTINUE
      WRITE(6,1066)
      DO 118 I=1,NOB
      J=LSORT(NOB+1-I)
  118 WRITE(6,1068) I,X(I),Y(I),F(I),R(I),J,X(J),Y(J),F(J),R(J)
C
C     ----- WRITE SOIL HYDRAULIC PROPERTIES -----
      WRITE(6,1069)
      PRESS=1.18850
      RN1=0.0
      RKLN=1.0
      WRITE(6,1072) RN1,WCS,RKLN,SATK
      DO 140 I=1,75
      IF(RKLN.LT.(-16.)) GO TO 142
      PRESS=1.18850*PRESS
      IF(MODE-2) 120,122,120
  120 WCR=TH(1)
      ALPHA=TH(2)
      RN=TH(3)
      GO TO 124
  122 ALPHA=TH(1)
      RN=TH(2)
  124 RM=1.-1./RN
      IF(MODE.EQ.3) RM=1.-2./RN
      RN1=RM*RN
      RWC=1./(1.+(ALPHA*PRESS)**RN)**RM
      WC=WCR+(WCS-WCR)*RWC
      TERM=1.-RWC*(ALPHA*PRESS)**RN1
      IF((TERM.LT.5.E-05).OR.(RWC.LT.0.06)) TERM = RM*RWC**(1./RM)
      IF(MODE.EQ.3) RK=RWC*RWC*TERM
      IF(MODE.NE.3) RK=SQRT(RWC)*TERM*TERM
```

MATINV

```
      SUBROUTINE MATINV(A,NP,B)
      DIMENSION A(3,3),B(3),INDEX(3,2)
      DO 2 J=1,4
    2 INDEX(J,1)=0
      I=0
    4 AMAX=-1.0
      DO 10 J=1,NP
      IF(INDEX(J,1)) 10,6,10
    6 DO 10 K=1,NP
      IF(INDEX(K,1)) 10,8,10
    8 P=ABS(A(J,K))
      IF(P.LE.AMAX) GO TO 10
      IR=J
      IC=K
      AMAX=P
   10 CONTINUE
      IF(AMAX) 30,30,14
   14 INDEX(IC,1)=IR
      IF(IR.EQ.IC) GO TO 18
      DO 16 L=1,NP
      P=A(IR,L)
      A(IR,L)=A(IC,L)
   16 A(IC,L)=P
      P=B(IR)
      B(IR)=B(IC)
      B(IC)=P
      I=I+1
      INDEX(I,2)=IC
   18 P=1./A(IC,IC)
      A(IC,IC)=1.0
      DO 20 L=1,NP
   20 A(IC,L)=A(IC,L)*P
      B(IC)=B(IC)*P
      DO 24 K=1,NP
      IF(K.EQ.IC) GO TO 24
      P=A(K,IC)
      A(K,IC)=0.0
      DO 22 L=1,NP
   22 A(K,L)=A(K,L)-A(IC,L)*P
      B(K)=B(K)-B(IC)*P
   24 CONTINUE
      GO TO 4
   26 IC=INDEX(I,2)
      IR=INDEX(IC,1)
      DO 28 K=1,NP
      P=A(K,IR)
      A(K,IR)=A(K,IC)
   28 A(K,IC)=P
      I=I-1
   30 IF(I) 26,32,26
   32 RETURN
      END
```