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To: Jeff Pohle
Fm: Mark J. Logsdon

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U.S NUCLEAR REGULATORY COMMISSION
DIVISION OF WASTE MANAGEMENT

Mini-Report #6

ANALYSES OF THERMAL CONVECTION OF GROUND WATER INDUCED BY RADIOGENIC HEATING WITHIN THE WASTE REPOSITORY

Salt Repository Project
Subtask 3.5

Prepared by

Daniel B. Stephens & Associates, Inc.

for

Nuclear Waste Consultants

TECHNICAL ASSISTANCE IN HYDROGEOLOGY
PROJECT B - ANALYSIS
RS-NMS-85-009

JANUARY, 1987

D-1021

PDR
LPDR-WM-10(2)
WM-11(2)
WM-16(2)

NUCLEAR WASTE CONSULTANTS INC.
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'87 FEB 12 A10:59

February 5, 1987

009/3.5/DBS.002
RS-NMS-85-009
Communication No. 135

U.S. Nuclear Regulatory Commission
Division of Waste Management
Geotechnical Branch
MS-623-SS
Washington, DC 20555

Attention: **Mr. Jeff Pohle, Project Officer**
Technical Assistance in Hydrogeology - Project B (RS-NMS-85-009)

Re: **Semi-Annual Update of SALT Numerical Evaluations Report**

Dear Mr. Pohle:

This cover letter transmits to the NRC staff Daniel B. Stephens and Associates (DBS) semi-annual update of the Numerical Evaluation of Conceptual Models Report for Salt (Subtask 3.5). The report has received a management and technical review by M. Logsdon of Nuclear Waste Consultants. The reports have been completed under DBS's quality assurance procedures in compliance with NWC's project-specific QA Plan.

The update report includes two new "mini-reports":

- o DBS Mini-Report #6: Analysis of Thermal Convection of Ground Water Induced by Radiogenic Heating with the Waste Repository.
- o DBS Mini-Report #8: Analysis of the Importance of Diffusion of Radionuclides Within the Proposed Repository Host Horizon.

WM Record File: WM-RES D1021 NWCI
WM Project: 10, 11, 16
Docket No.
XPDR ✓
X LPDR ✓ (B, N, S)

Distribution:
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In addition to these completed reports, DBS has two additional reports in progress, dealing with the following topics:

- o DBS Report #7, dealing with the sensitivity of hydraulic conditions at the Deaf Smith site to recharge in New Mexico;
- o DBS Report #9, dealing with the importance of aquitard diffusion of radionuclides during horizontal groundwater flow.

DBS anticipates completing these reports and submitting them to the NRC Staff within the next few months, without waiting for another formal update document.

Submission of this update report completes the contract deliverable for Subtask 3.5 at this time. DBS will update the Numerical Evaluation of Conceptual Models Report on a semi-annual basis, as directed in the current contract.

If you have any questions concerning this letter or the attached reports, please contact me immediately.

Respectfully submitted,
NUCLEAR WASTE CONSULTANTS, INC.

Mark J. Logsdon

Mark J. Logsdon, Project Manager

Att: Salt Numerical Evaluations of Conceptual Models Report Update

cc: US NRC - Director, NMSS (ATTN: PSB)
DWM (ATTN: Division Director) - 2
Mary Little, Contract Administrator
WMGT (ATTN: Branch Chief)

L. Davis, WWL
M. Galloway, TTI

bc: J. Minier, DBS

DANIEL B. STEPHENS & ASSOCIATES, INC.
CONSULTANTS IN GROUND-WATER HYDROLOGY

• GROUND-WATER CONTAMINATION • UNSATURATED ZONE INVESTIGATIONS • WATER SUPPLY DEVELOPMENT •

FEB 03 1987

January 31, 1987

Mr. Mark Logsdon
Nuclear Waste Consultants
8341 S. Sangre de Cristo Rd.
Littleton, CO 80127

Dear Mark:

Daniel B. Stephens & Assoc. submit the attached as the update report for Subtask 3.5, Numerical Evaluation of Conceptual Models.

This report contains the following "mini-reports":

6. Analysis of thermal convection of ground water induced by radiogenic heating within the waste repository

8. Analysis of the importance of diffusion of radionuclides within the proposed repository host horizon.

At present, two additional mini-reports, originally scheduled to be included in this Subtask 3.5 update report, are being revised as a result of report review. The additional mini-reports, which will be submitted after the revisions are completed, are:

7. The influence of increased recharge in New Mexico on ground-water travel time in the Wolfcamp

9. Analysis of the importance of aquitard diffusion of radionuclides during transport by horizontal ground-water flow.

Please contact me if you have any questions concerning this Subtask 3.5 update report.

Yours truly,
Daniel B. Stephens & Assoc.

Jeffrie D. Minier

Jeffrie D. Minier
Project Manager

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ANALYSIS OF THERMAL CONVECTION OF GROUND WATER INDUCED BY RADIOGENIC HEATING WITHIN THE WASTE REPOSITORY

Numerical Evaluation of Conceptual Models
Subtask 3.5
Mini-Performance Assessment #6

January 1987

DANIEL B. STEPHENS & ASSOCIATES, INC.

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1.0 INTRODUCTION

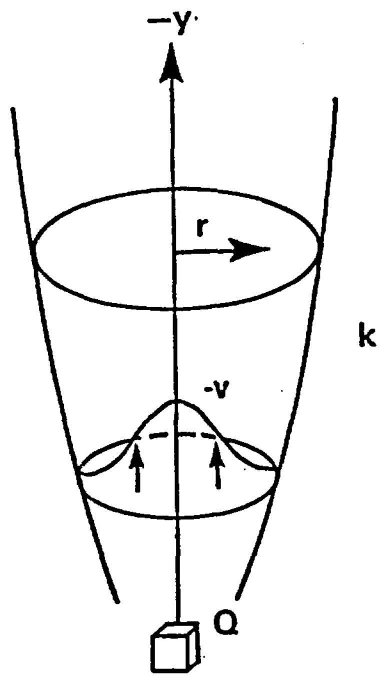
1.1 General Statement of the Problem

Radioactive decay within the waste repository will heat ground water in the vicinity of the repository, thus locally decreasing ground-water density. The heated buoyant ground water will rise from the repository location in the form of a plume (Figure 1). Such a plume may be a significant pathway for the transport of radionuclides to the accessible environment.

1.2 Statement of Relevance to NRC

The location of the proposed repository is within the Permian evaporite aquitard in the Palo Duro Basin, Texas. Ground-water flow within the Permian aquitard is generally considered to be downward, a favorable condition with regard to the ability of the repository to isolate waste (DOE, 1986; 10CFR60, section 60.122). Buoyancy forces resulting from heating by radioactive decay within the repository may induce upward ground-water flow (Figure 1). Potential for changes in hydrologic conditions that would affect the migration of radionuclides to the accessible environment is a potentially adverse condition and therefore must be investigated (10CFR60, section 60.122).

1.3 Relationship to other analyses, documents, tasks and subtasks



Project Number: 85-130
Date: 7-18-86

Figure 1. Axisymmetric plume of buoyant ground water rising above the proposed repository which is emitting heat at the rate Q due to radioactive decay within the repository. The repository is located in a homogeneous, isotropic, unfractured saturated medium of permeability k. (After Turcotte and Schubert, 1982).

Most analyses are performed under the assumption that flow within the Permian evaporite aquitard is generally downward under saturated conditions (DOE, 1986; Stephens & Assoc., 1986a). In addition, most analyses do not incorporate buoyancy forces due to salinity or thermal effects (Stephens & Assoc., 1986a). Favorable and potentially adverse conditions for site criteria relate to downward flow and potential for changes in the hydrologic conditions (10CFR60, section 60.122). Thermally induced buoyancy forces may provide a mechanism for the transport of radionuclides to the accessible environment through a previously unconsidered pathway, i.e. upward ground-water flow in a buoyancy driven convection system.

2.0 OBJECTIVE

The objective of this analysis is to estimate the velocity profile of the thermal ground-water plume generated by heating due to radioactive decay within the repository. The results of this analysis will relate to environmental standards and the transport of radionuclides to the accessible environment (e.g. 40CFR191, 10CFR60), because the transport of radionuclides is dependent upon the volumetric flow rate of ground-water in the host rock.

3.0 OPERATIONAL APPROACH - CONCEPTS AND GENERAL ASSUMPTIONS

3.1 Geology

In this analysis the Permian shale-evaporite aquitard (HSU B; DOE, 1986) is modeled as a three-dimensional, homogeneous, isotropic, unfractured, infinite, saturated porous medium (Figure 1). Although this is a highly simplified model of a complicated system, the model may provide a reasonable and useful estimate of the velocity profile of the thermal ground-water plume which may be generated by the repository.

The Permian aquitard is a sedimentary sequence of shales, evaporites, carbonates and siltstones, and as such should exhibit some heterogeneity and anisotropy in permeability (Palo Duro Basin Field Trip, 1986; DOE, 1986; Stephens & Assoc., 1986a).

3.2 Flow/Transport System

Ground-water flow is assumed to occur in response to buoyancy forces generated by thermal expansion of water due to heating by radioactive decay within the repository. In this analysis, it is assumed that the physical properties of the ground water (including viscosity but with the exception of density) are unaffected by temperature and pressure variations (see Appendix). In order to obtain a relatively simple solution it is necessary to assume that flow occurs in a homogeneous, isotropic, unfractured medium. These assumptions may lead to inaccurate estimates of flow velocities and convection plume dimensions. Should this preliminary 'back of the envelope' analysis indicate that a significant thermal convection system is likely to be induced, then more complicated models incorporating anisotropy, heterogeneity and fracture flow may be pursued to more accurately estimate the thermal plume convection system.

3.3 Repository and Source Term

In this analysis the repository is represented by a point source emitting a constant amount of heat per unit time; the standard solution is steady state. The rate that heat will actually be emitted from the repository will not be constant but rather will increase initially to some maximum value and then decrease as the radioactive decay rate decreases (the decay rate is proportional to the amount of parent material). Therefore, if integrated over time, the standard solution will overestimate the vertical flux of ground water.

4. TECHNICAL APPROACH

4.1 Formal Statement of Problem

Within the limits of the assumptions given within, an analysis (using a boundary layer approximation) will be performed to determine the velocity profile within a rising plume of buoyant ground water (generated by heating due to radioactive decay within the repository) as a function of permeability. The vertical flux of ground water may be determined by integrating the velocity profile.

4.2 Identification of Solution Techniques

This analysis utilizes the "Boussinesq approximation" where the fluid is considered to be incompressible except for a buoyancy term (density variation due to thermal expansion) in the vertical Darcy velocity component. The equations for conservation of energy and continuity are solved subject to the appropriate boundary conditions. Stream functions and similarity variables are introduced in order to obtain a solution. The governing equations, boundary conditions and assumptions used to obtain a solution are presented in the Appendix with the mathematical statement and analysis of the problem.

4.3 Definitions and Assumptions

The velocity profile of the thermal plume is described by several parameters which are defined in the following list (see standard solution, section 4.5 of this report):

v - vertical darcy velocity

k - hydraulic permeability

α_f - fluid coefficient of volumetric thermal expansion g - gravitational acceleration

ρ_f - fluid density

Q - rate heat is emitted from the repository, constant

λ_m - thermal conductivity of the saturated medium

μ - fluid dynamic viscosity

C_f - fluid specific heat

In general, the simplifying assumptions made in this analysis are required in order to obtain an analytical solution. Assumptions which may be somewhat more realistic in terms of the observed characteristics and conceptual model will require more complicated analyses, most likely numerical simulations. Some of the assumptions used in this analysis are listed below and discussed in the Appendix with the additional assumptions which are required for this analysis.

A. Flow occurs in a homogeneous, isotropic, unfractured, saturated, permeable medium. The stratified nature of the sedimentary Permian aquitard suggests that the aquitard will exhibit considerable heterogeneity and anisotropy (DOE, 1986; Stephens & Assoc., 1986a). Salt dissolution and movement along basement faults may produce fractures within the aquitard (Gustavson and Budnik, 1984; DOE, 1986). Stephens & Assoc. (1986a, 1986b) discuss and demonstrate the possibility of unsaturated conditions within the Permian aquitard. The assump-

assumption of a homogeneous, isotropic, unfractured, saturated medium is made in order to obtain a relatively simple, analytical solution. The effect of characteristics different from those assumed here may be analyzed in a subsequent mini-performance assessment report, perhaps by using a numerical model.

B. The fluid is incompressible, except for a buoyancy term (due to the density variation caused by thermal expansion), also known as the "Boussinesq approximation"; the fluid is fresh water. The effects of using a fluid density appropriate for saline water is apparent in the standard solution. Spatial variability of fluid density cannot be incorporated in this simple analysis and may be pursued in a subsequent mini-performance assessment report, perhaps using a numerical model.

C. Heat is emitted from the repository at a constant rate. This assumption clearly differs from the expected rate of radiogenic heat production since the decay rates of the radionuclides are proportional to the amount of remaining parent materials. Thus the rate of radiogenic heat production is expected to decrease with time. Given a constant source strength, the steady state would be arrived at only after some time after emplacement of the source. Norton and Knight (1977) use numerical modelling techniques to evaluate the unsteady convection resulting from a transient heat source.

D. The plume of buoyant ground water remains thin as it moves upward. This allows the boundary layer approximation to be used (see Appendix).

4.4 Identification of Output and Criteria

Permeability is the only parameter in the standard solution which is expected to vary by much more than an order of magnitude and thus is expected to be a critical parameter in this analysis. Graphical representations of the standard solution as a function of permeability will be presented.

4.5 Standard Solution

The vertical velocity distribution in the thermal plume is given by Turcotte and Schubert (1982):

$$v = \frac{3}{8\pi y} \frac{k\alpha_f g \rho_f Q}{\mu \lambda_m} \left\{ 1 + \frac{3}{64\pi} \frac{r^2}{y^2} \frac{k\alpha_f g \rho_f^2 c_{p_f} Q}{\mu \lambda_m^2} \right\}^{-2}$$

The dimensionless velocity profile for the axisymmetric plume is shown in Figure 2.

5. ANALYSIS

The following parameter values are considered representative of the repository/aquitard system:

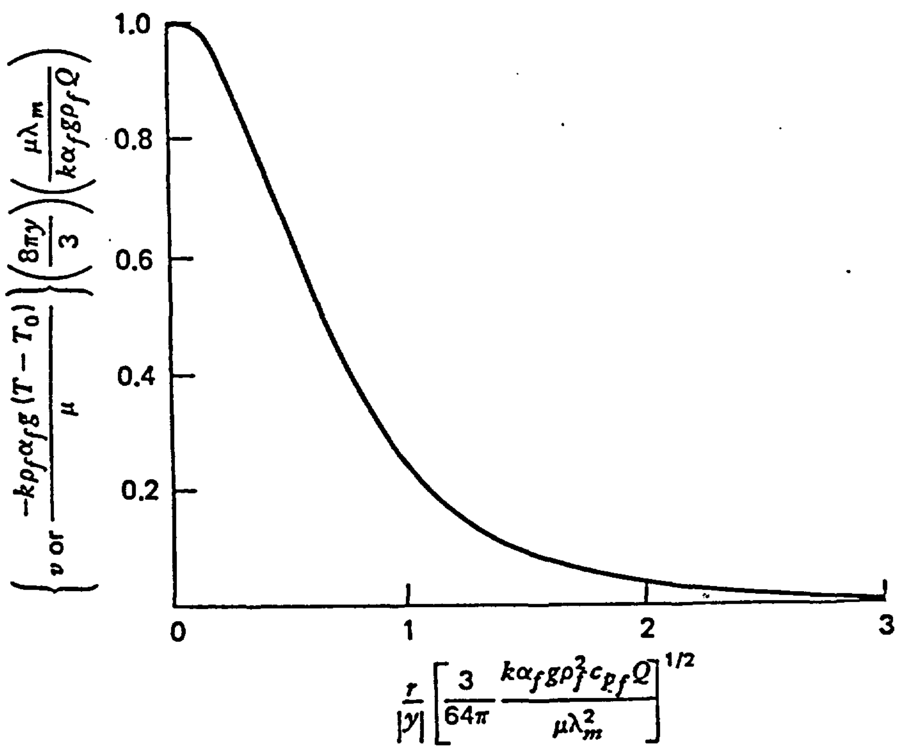


Figure 2. Profile of the Dimensionless Velocity and Temperature in an Axisymmetric Plume (Turcotte and Schubert, 1982).

k = variable, units: m^2 (see Freeze and Cherry, 1979)

$\alpha_f = 5 \times 10^{-4}\ K^{-1}$ (T=56°C, CRC, 1978, p. F-5)

$g = 9.78\ m\ s^{-2}$ (Turcotte and Schubert, 1982, p.430)

$\rho_f = 10^6\ g\ m^{-3}$ (Hillel, 1980, p. 294)

$Q = 2.1 \times 10^3\ W$ (estimated from Wagner and Others, 1986)

$\lambda_m = 1.5\ W\ m^{-1}\ K^{-1}$ (Reiter and Tovar, 1982; DOE, 1986)

$\mu = 0.49\ g\ m^{-1}\ s^{-1}$ (T=56°C, CRC, 1978, p. F-51)

$C_f = 4.2\ W\ s\ g^{-1}\ K^{-1}$ (Hillel, 1980, p. 294)

Substituting the parameter values cited above into the standard solution will illustrate the effect of different permeabilities on the velocity profile of the plume (Figure 3).

6. RESULTS

Figure 3 illustrates the effect of permeability on the velocity profile of the plume. Increasing the permeability by an order of magnitude approximately increases the velocity by an order of magnitude for these two values of permeability. The characteristic vertical permeability of the Permian aquitard is much lower than the values used to generate Figure 3 (DOE, 1986). The plume diameter 1 km above the repository is approximately 100-400 m for permeability values of $10^{-9} - 10^{-10}\ m^2$, respectively (based on temperature considerations, see Turcotte and Schubert, 1982). For values of permeability less than $10^{-10}\ m^2$, the aspect ratio of the plume is less than 2-3 (h/d = 1000m/400m) thus the plume does not remain thin as it moves upward.

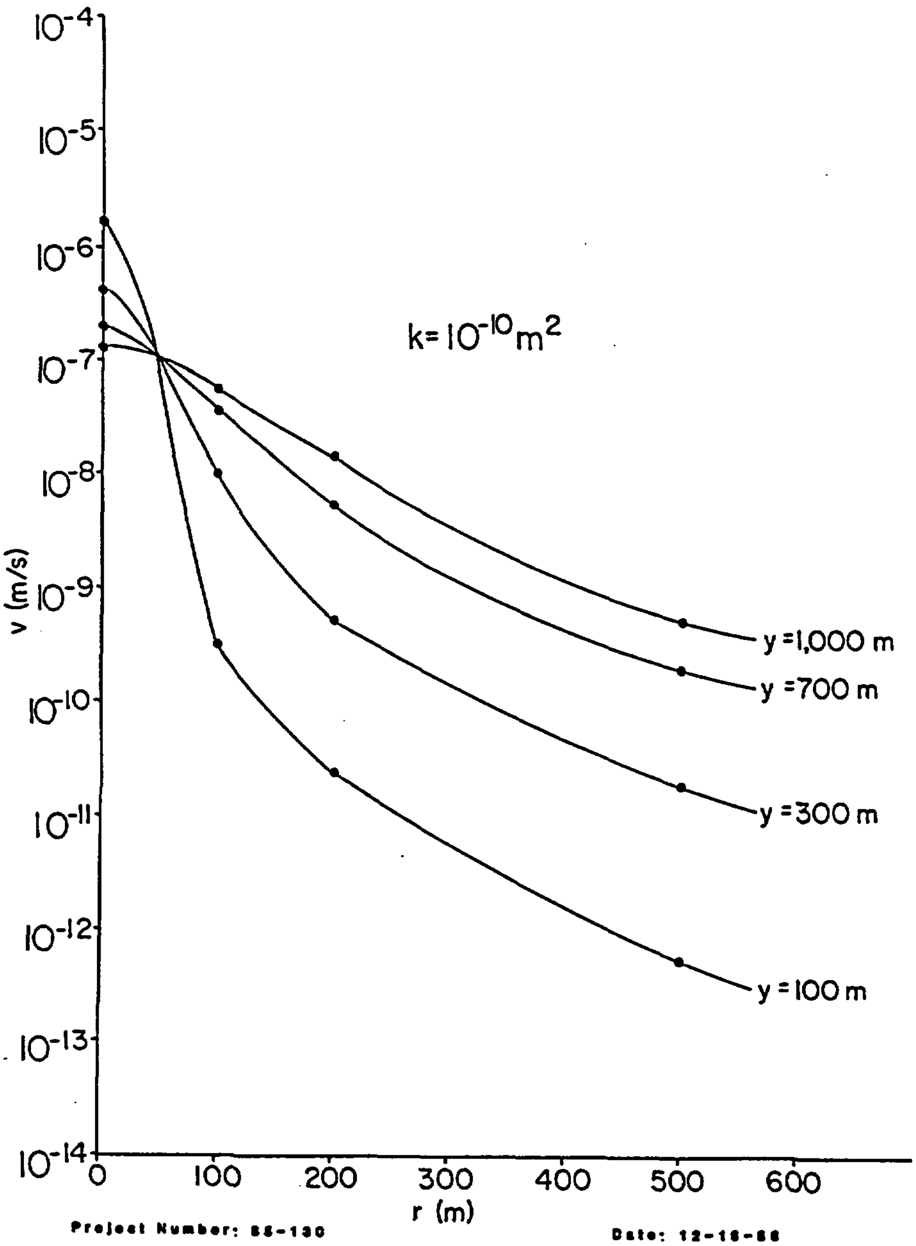


Figure 3a. Velocity Profile in Axisymmetric Plume as a Function of Permeability and Distance y above the Repository. Permeability $k = 10^{-10} m^2$.

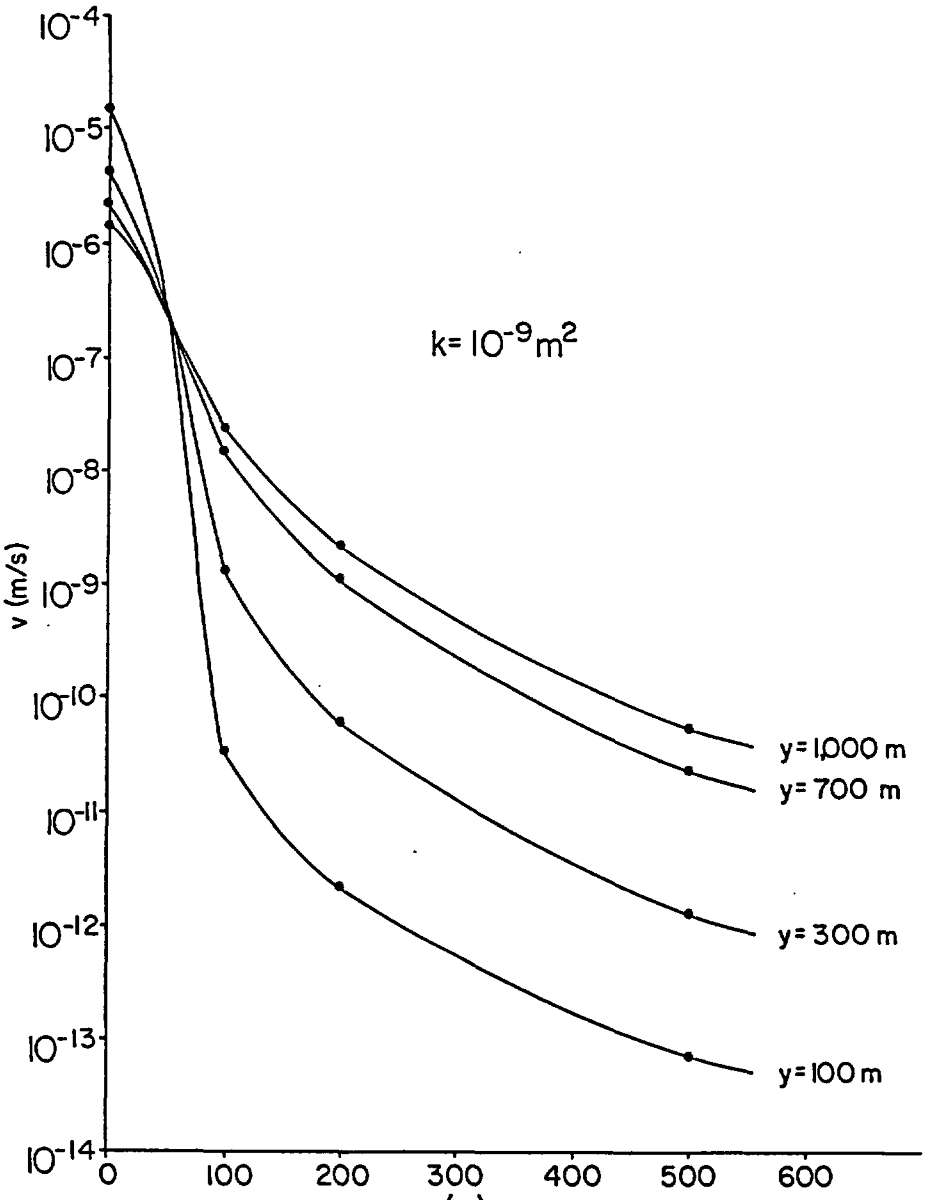


Figure 3b. Velocity Profile in Axisymmetric Plume as a Function of Permeability and Distance y above the Repository. Permeability $k = 10^{-9} m^2$.

The boundary layer approximations used to obtain the analytical solution may not be valid for lower values of permeability (Appendix).

7. CONCLUSIONS

Assumptions made in this analysis limit the usefulness of the results to systems with permeability greater than $10^{-10} m^2$. The characteristic permeability for the evaporite equitard is expected to be less than $10^{-10} m^2$. Therefore conclusions related to the significance of ground-water velocities (generated by buoyancy forces resulting from radioactive heating of the repository) may not be drawn from the analyses in this report. It is therefore recommended that a more complex solution, for example numerical simulation, be pursued to investigate the significance of a thermal plume (of ground-water rising from the repository) on the ability of the repository to isolate waste.

8. DISCUSSION

This analysis assumed that the buoyant ground-water plume remains thin as it rises from the vicinity of the repository. This assumption allows for boundary layer approximations which simplify the governing equations resulting in a relatively simple analytical solution (see Appendix). The analytical solution is valid for values of permeability greater than about $10^{-10} m^2$ which is higher than what would be expected for an aquitard (see Freeze and Cherry, 1979). The extrapolation of the results by using much lower values of permeability may violate assumptions used to obtain the analytical solution and yield results which may not be valid.

While this simple analysis is not directly applicable to the proposed repository host horizon it does indicate that permeability may be an important parameter in the thermal plume ground-water transport mechanism. In addition, this analysis provides some justification for the use of more complex solutions, for example numerical simulation. An analysis to determine the velocity profile of a thermal buoyancy plume (as is described in this report) using a numerical model, perhaps SUTRA, is currently being investigated by the DBS team.

9. REFERENCES

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APPENDIX

Mathematical Analysis

The following mathematical analysis (taken from section 9-10 of Turcotte and Schubert, 1982) is not meant to be a rigorous derivation, but rather is intended to be a guide illustrating the method of solution. The presentation of the governing equations is followed by a list of the assumptions, boundary conditions and initial conditions. The mathematical approach to the solution is then discussed.

A1. GOVERNING EQUATIONS

The continuity equation for axisymmetric, incompressible flow is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial v}{\partial y}=0 \qquad (1)$$

where r and y are the radial and vertical coordinates, and u_r and v are the radial and vertical darcy velocities (Figures 1, 4).

Stream functions which satisfy the continuity equation are

$$u_r=-\frac{1}{r}\frac{\partial\psi}{\partial y} \qquad (2)$$

$$v=\frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (3).$$

The energy conservation equation in cylindrical coordinates for

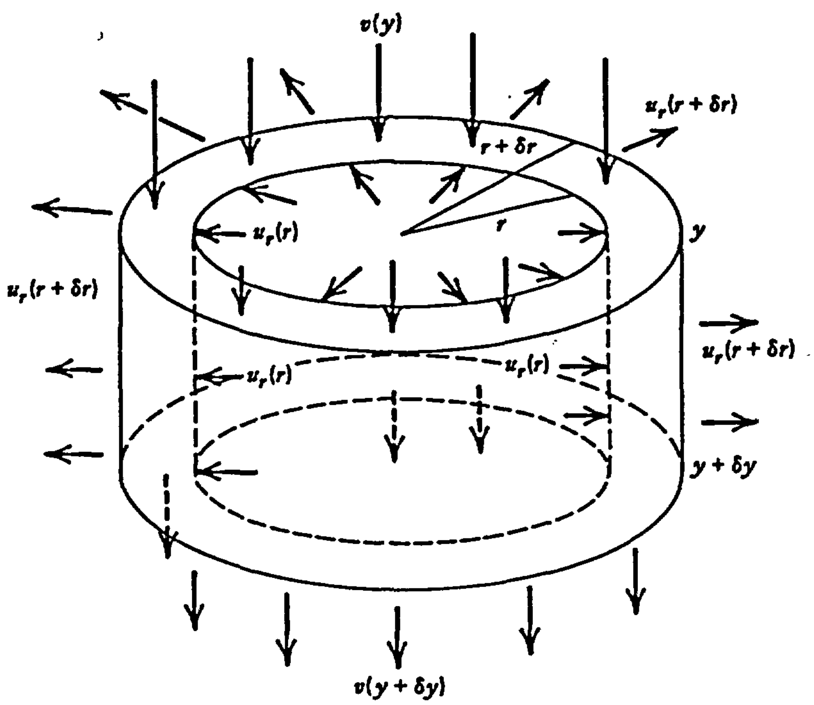

Figure 4. Flows Into and Out of an Infinitesimal Tubular Cylindrical Volume in Homogeneous, Isotropic, Permeable Medium (Turcotte and Schubert, 1982)

this problem is

$$u_r\frac{\partial T}{\partial r}+v\frac{\partial T}{\partial y}=\frac{\lambda_m}{\rho_f c_{p_f}}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right) \quad (4)$$

where T is temperature, λ_m is the thermal conductivity of the fluid-saturated porous medium, and ρ_f and c_f are, respectively, the density and specific heat of the fluid.

The boundary layer approximation for the vertical darcy velocity is

$$v=-\frac{k\rho_f\alpha_f g}{\mu}(T-T_0) \quad (5)$$

where g is the gravitational acceleration, k is the permeability of the porous medium, μ and α_f are, respectively, the dynamic viscosity and coefficient of thermal expansion of the fluid and T is the ambient temperature.

The vertical heat flux for any y is

$$Q=-2\pi\int_0^\infty \rho_f c_{p_f} r v(T-T_0)\,dr \quad (6).$$

2.0 ASSUMPTIONS AND BOUNDARY AND INITIAL CONDITIONS

2.1 The medium is homogeneous, isotropic, permeable, rigid, unfractured and saturated.

2.2 The fluid is incompressible, except for the buoyancy term in the vertical darcy velocity.

2.3 The physical properties of the fluid, including viscosity but with the exception of density, are constant, i.e. the physical properties are unaffected by temperature and pressure.

2.4 The velocity profile of the plume and the amount of heat emitted from the repository due to radioactive decay are constant in time (steady state).

2.5 The velocity profile of the thermal plume is radially symmetric about its center (see Figures 1, 4), i.e.

$$u_r = \frac{\partial v}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{7}$$

2.6 At large distances from the plume the ambient temperature is T and the fluid is motionless, i.e.

$$T \rightarrow T_0, v \rightarrow 0, \quad \text{as} \quad r \rightarrow \infty \tag{8}$$

2.7 The boundary layer approximation assumes that the plume remains thin as it moves upward and allows the general darcy velocity and energy conservation equations to be simplified to the forms given above in equations (4) and (5). The requirement that the plume be thin is equivalent to

$$d \ll h$$

where d is the diameter of the plume at a distance h above the repository (Figure 5). The plume width is a function of the distance above the repository and the rate that heat is emitted from the repository. For a large heat emission rate Q, the ground water in the plume will be very buoyant and will rise at a relatively rapid rate. The narrowness of the plume results since the plume material will move upward quite far before it has an opportunity to spread out in the radial direction. In a narrow plume, velocity and temperature vary much more rapidly with distance across the plume than they do with distance along the plume (Figure 5).

The boundary layer approximation estimates the vertical and horizontal darcy velocities where the plume thickness is d at a distance h above the repository (Figure 5). These estimates combined with equation (1), the continuity equation, require the magnitude of the radial darcy velocity to be small compared to the magnitude of the vertical velocity in a narrow plume. One result of the boundary layer approximation is that heat conduction along the plume is small compared to heat conduction across the plume. In the right hand side of equation (4) the vertical heat conduction term has been neglected. Also, since the upward movement of the plume is driven by the buoyancy force the vertical darcy velocity can be approximated by equation (5).

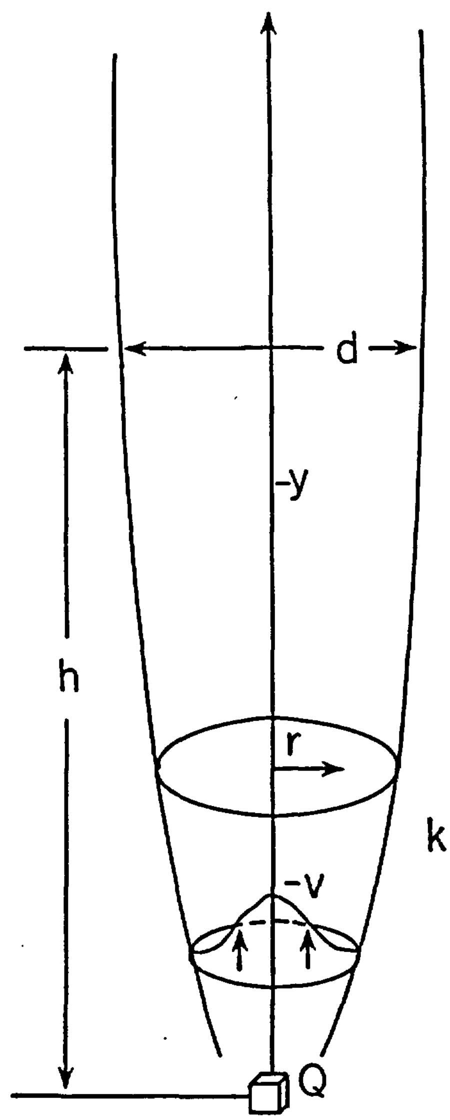

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Figure 5. Notation for Boundary Layer Approximation (After Turcotte and Schubert, 1982).

Turcotte and Schubert (1982, section 9-10) present a detailed discussion and analysis of the boundary layer approximation.

3.0 APPROACH

Combining equations (3) and (5) one gets

$$T-T_0=\frac{-\mu}{k\rho_f\alpha_f g r}\frac{\partial\psi}{\partial r} \quad (9).$$

The energy conservation equation in terms of ψ may be obtained by substituting equations (2), (3) and (9) into equation (4) resulting in

$$\frac{1}{r^2}\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r\partial y} =\frac{\lambda_m}{\rho_f c_{p_f}}\left\{\frac{1}{r^2}\frac{\partial\psi}{\partial r}-\frac{1}{r}\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^3\psi}{\partial r^3}\right\} \quad (10).$$

The vertical heat flux in terms of ψ (equation (9) into equation (6)) is

$$Q=\frac{2\pi c_{p_f}\mu}{k\alpha_f g}\int_0^\infty\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)^2 dr \quad (11)$$

and the boundary conditions (equations (7) and (8)) are

$$-\frac{1}{r}\frac{\partial\psi}{\partial y}\to 0,\quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\to 0,\quad \text{as}\quad r\to 0 \quad (12)$$

$$\frac{1}{r}\frac{\partial\psi}{\partial r}\to 0,\quad \text{as}\quad r\to\infty \quad (13).$$

To solve equation (10) given equations (11) to (13), introduce stitute equation (3) into (14), differentiate equation (15) with respect to r, and compare the results (use the chain rule). One finds that

$$df/d\eta \rightarrow 0 \text{ as } \eta \rightarrow 0.$$

therefore c_1 =0 and equation (20) reduces to

$$\frac{f}{\eta}\frac{df}{d\eta} = \frac{1}{\eta}\frac{df}{d\eta} - \frac{d^2f}{d\eta^2} \qquad (22).$$

Equation (22) is a solution to equation (22) and satisfies the boundary conditions, equations (18) and (19):

$$f = \frac{4c_2\eta^2}{1+c_2\eta^2} \qquad (23).$$

Use of the magnitude of vertical heat flux, equation (17), with equation (23) allows the constant of integration c_2 to be determined:

$$c_2 = \frac{3}{64\pi} \qquad (24).$$

Thus the similarity solution is

$$f = \frac{3}{16\pi}\frac{\eta^2}{\left(1+\frac{3\eta^2}{64\pi}\right)} \qquad (25).$$

By combining equations (3), (9), (14), (15), and (25), the vertical velocity distribution in the plume is found to be

$$v=\frac{-k\rho_f\alpha_f g}{\mu}(T-T_0)$$
$$=\frac{3}{8\pi y}\frac{k\alpha_f g\rho_f Q}{\mu\lambda_m}\left\{1+\frac{3}{64\pi}\frac{r^2}{y^2}\frac{k\alpha_f g\rho_f^2 c_{p_f} Q}{\mu\lambda_m^2}\right\}^{-2} \qquad (26).$$