Department of Energy
Washington, DC 20585

SEP 14 1992

Mr. Joseph J. Holonich, Director
Repository Licensing & Quality Assurance
Project Directorate
Division of High-Level Waste Management
Office of Nuclear Material Safety
and Safeguards
U.S. Nuclear Regulatory Commission
Washington, D.C. 20555

Dear Mr. Holonich:

The U.S. Nuclear Regulatory Commission (NRC) offered Phase I review comments on Study Plan 8.3.1.2.3.1, "Characterization of the Site Saturated Zone Ground Water Flow System," on December 6, 1991 (enclosure 1). Activities 1-6 are the responsibility of the U.S. Geological Survey (USGS), and Los Alamos National Laboratory is responsible for Activity 7. Several concerns were raised in NRC's letter. In responding to these concerns, the U.S. Department of Energy's (DOE) response to NRC's Phase I review comments on Study Plan 8.3.1.14.2, "Studies to Provide Soil and Rock Properties for Potential Locations of Surface Facilities," should also be consulted.

The first concern pertained to technical procedures in general. The NRC has asked for a schedule as to availability of technical procedures needed for conduct of the work and that the procedures themselves be supplied upon availability. The DOE does not maintain a schedule as to availability of these procedures and only requires that technical procedures be in place prior to beginning work. Verification that technical procedures are available takes place during the preparation of job packages under Yucca Mountain Site Characterization Project Office (YMPO) Administrative Procedure (AP) 5.21Q, Field Work Activation. If it was NRC's intent to request specific technical procedures, please make a specific request through the on-site representative (OR) by having the OR request them from YMPO.

Another concern was the potential test-to-test interference in the C-holes cluster and the availability of a graphic showing how this study intends to pass information to other studies requiring it, or receive information from other studies supplying it.

With respect to test-to-test interferences, the study plan's discussion of test-to-test interferences is not exhaustive, nor is it meant to be. Detailed consideration of test-to-test interferences, for specific tests at specific locations, are

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dealt with under AP 5.32, Test Planning and Implementation Requirements. The planning package resulting from this process includes details on interferences that is explicit to the operational test program to be performed. Interference among tests is not anticipated to be a major problem that would impact schedules. Past experience at the C-wells complex indicates that after a typical hydraulic test, the pressures return to pretest conditions within hours to days. The large scale hydraulic tests certainly take longer, but the time frame would be days to weeks rather than months to years. To minimize interference among tracer tests, a suite of tracers rather than a single tracer would be used. Ideally, each test would use a separate tracer. Even if some tracer remains from an early test, the breakthrough curve for a later test using the same tracer would still be distinguishable from the background concentration caused by the early test.

With respect to a schedule with test start/duration information, in general the exact type, start date, or duration of tests at multiple-well sites cannot be determined a priori. The testing follows a methods-development sequencing, where a few tests will be run at first. The results will be analyzed and a decision made as to whether or not to run more tests, modify the testing procedure, adopt a different type of test, or that adequate and sufficient data has been acquired. Site Characterization Plan (SCP) Section 8.3.1.2.4 (page 8.3.1.2-445) identified the logical relationships between the various studies comprising the SCP's geohydrology program. Although the real-time dates indicated in the schedule section are now obsolete, the feeds from, and to, other studies preserve the relative logic.

Another concern pertained to integration among data collection and modeling studies; and relationship to iterative performance assessments. The exchange of information between these activities is inherent in the way studies have been planned. Much of this data is being gathered as input to specific physical system and subsystem models. Principal investigators and modelers potentially needing data from field studies are alerted to its availability through the Technical Data Catalog, which is published quarterly. Also, many of the same hydrologists at the USGS are involved in both data collection and modeling.

With respect to iterative performance assessments, how data obtained by Study 8.3.1.2.3.1 are used in a total-system performance assessment, or the frequency intended for these exercises, is beyond the scope of this study plan. These variables are being considered by DOE and the participant organizations, and as these plans mature, the NRC will be informed. In general, iteration between data collection as part of this study and modeling in Study Plan 8.3.1.2.3.3, "Site Saturated Zone Synthesis and Modeling," would take place through concurrent data collection and modeling. Groundwater flow and transport modeling would occur at the same time as data collection, and not only after all data has been collected.

Initial models would be set up and calibrated with available data. New data would serve to refine these models. Early modeling would identify the need for additional types of data or other locations where data should be acquired to improve the models and reduce uncertainties.

The NRC also requested a list of references that were cited in the study plan. These references are collected into one enclosure (enclosure 2).

If you have any questions, please contact Mr. Chris Einberg of my office at 202-586-8869.

Sincerely,

John P. Roberts
Acting Associate Director for
Systems and Compliance
Office of Civilian Radioactive
Waste Management

Enclosures:
1. Ltr, 12/6/91, Holonich to Roberts, w/encl.
2. Not-Readily-Available References Cited in Study Plan (Not Record Material)

cc: w\enclosures
Alice Cortinas, CNWRA, San Antonio, TX

cc: w\enclosures
C. Gertz, YMPO
R. Loux, State of Nevada
T. Hickey, Nevada Legislative Commission
M. Baughman, Lincoln County, NV
J. Bingham, Clark County, NV
B. Raper, Nye County, NV
P. Niedzielski-Eichner, Nye County, NV
G. Derby, Lander County, NV
P. Goicoechea, Eureka, NV
C. Schank, Churchill County, NV
F. Mariani, White Pine County, NV
V. Poe, Mineral County, NV
E. Wright, Lincoln County, NV
J. Pitts, Lincoln County, NV
R. Williams, Lander County, NV
J. Hayes, Esmeralda County, NV
M. Hayes, Esmeralda County, NV
B. Mettam, Inyo County, CA
C. Abrams, NRC

see enclosures on the shelf

UNITED STATES
NUCLEAR REGULATORY COMMISSION
WASHINGTON, D. C. 20555

DEC 06 1991

Mr. John P. Roberts, Acting Associate Director
for Systems and Compliance
Office of Civilian Radioactive Waste Management
U. S. Department of Energy, RW 30
Washington, D.C. 20585

Dear Mr. Roberts:

SUBJECT: PHASE I REVIEW OF U.S. DEPARTMENT OF ENERGY (DOE) STUDY PLAN FOR CHARACTERIZATION OF THE SITE SATURATED-ZONE GROUND-WATER FLOW SYSTEM AND OF DOE STUDY PLAN FOR TESTING OF THE C-HOLE SITES WITH REACTIVE TRACERS

On April 6, 1990, DOE transmitted the study plan entitled "Study Plan for Testing of the C-Hole Sites with Reactive Tracers" (Study Plan for Study 8.3.1.2.3.1.7) to the U.S. Nuclear Regulatory Commission (NRC) for review and comment. On March 7, 1991, DOE transmitted the study plan entitled "Study Plan for Characterization of the Site Saturated-Zone Ground-Water Flow System" (Study Plan for Study 8.3.1.2.3.1.1-6). These two study plans describe the activities for one study as presented in DOE's Site Characterization Plan (SCP) for the Yucca Mountain, Nevada site. Because the two documents describe one study, NRC chose to review them together. The NRC has completed its Phase I Review of these documents using the Review Plan for NRC Review of DOE Study Plans, Revision 1 (December 6, 1990).

The material submitted in the study plans was considered to be consistent, to the extent possible at this time, with the agreement on content resulting from the NRC-DOE agreements made at the May 7-8, 1986 NRC-DOE meeting on Level of Detail for Site Characterization Plans and Study Plans. Many procedures remain to be developed, including ones covering key work activities at the site, such as the drilling and coring of wells, equipment calibration, and methods for conducting cross-hole hydraulic tests, large-scale pumping tests, and tracer tests. The NRC staff did not consider that the lack of availability of such information compromised its ability to conduct its Phase I Review of the material provided. However, we request that DOE provide us with a schedule for the development of the procedures and that DOE notify us when the procedures are available for review.

Among the references listed for these study plans are several which have not been provided to NRC and are not readily available in the public domain. We therefore request that DOE provide us with the documents listed in the enclosure to this letter.

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ENCLOSURE 1

A major purpose of the Phase I Review is to identify concerns with studies, tests, or analyses that if started could cause significant and irreparable adverse effects on the site, the site characterization program, or the eventual usability of the data for licensing. Such concerns would constitute objections, as that term has been used in earlier NRC staff reviews of DOE's documents related to site characterization (Consultation Draft Site Characterization Plan and the Site Characterization Plan for the Yucca Mountain site). The Phase I Review of this study plan identified no objections with any of the activities proposed.

There are two technical matters related to the study plans that the NRC staff wishes to call to the attention of DOE. First, there is considerable potential for interferences among the tests proposed, especially given the large number of hydrologic and tracer tests planned at multi-well sites like the C-hole cluster. If such interferences occur, they could cause disruptions of the site characterization schedules, which could subsequently have an adverse impact on schedules for licensing. Therefore, it is important for DOE to develop a timeline that shows all proposed work activities under the site saturated-zone investigations, including durations and start and finish dates. If it is not yet possible to identify actual start dates, the timeline could still show the relative starting times and durations for all related tests. If such a timeline exists now or is developed at some future time, we request that it be provided to NRC when available.

In addition to this first matter, the NRC staff notes that the connection between this study and the study concerned with Saturated Zone Synthesis and Modeling Activities (Study 8.3.1.2.3.3) is described as an iterative process whereby modeling may lead to additional data collection and refinement of a conceptual model. The staff considers the proposed process to be a positive step but is unclear as to how DOE plans to implement it. Although the study plan describes general relationships between this and other studies, there is no explicit discussion of an overall program of iterative performance assessment, or discussion of the timing of this study relative to such a program.

After completion of the Phase I Review, based on the relationship of a given study plan to key site-specific issues or NRC open items, or its reliance on unique, state-of-the-art test or analysis methods, certain study plans are selected to receive a second level of review, called a Detailed Technical Review. The NRC staff considers the two subject study plans likely candidates

for a Detailed Technical Review. However, until the important procedures and the test and activity timeline mentioned earlier in this letter are completed and available to the NRC staff, we cannot make a final determination regarding the need for a Detailed Technical Review of these study plans.

If you have any questions concerning this letter, please contact King Stablein (FTS/[301]-492-0446) of my staff.

Sincerely,

Joseph J. Holonich, Director
Repository Licensing and Quality
Assurance Project Directorate
Division of High-Level Waste Management
Office of Nuclear Material Safety
and Safeguards

Enclosure: As Stated

cc: R. Loux, State of Nevada
C. Gertz, DOE/NV
S. Bradhurst, Nye County, NV
M. Baughman, Lincoln County, NV
D. Bechtel, Clark County, NV
D. Weigel, GAO
P. Niedzielski-Eichner, Nye County, NV
C. Thistlethwaite, Inyo County, CA
V. Poe, Mineral County, NV
F. Sperry, White Pine County, NV

References for Study Plan 8.3.1.2.3.1
(including Studies 8.3.1.2.3.1.1-6 and 8.3.1.2.3.1.7)
Requested by NRC

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- Barenblatt, G.E., Zheltov, I.P., and Kochina, I.N., 1960, Basic concepts in the theory of homogeneous liquids in fissured rocks: Journal of Applied Mathematics and Mechanics (USSR), v. 24, no. 5, p. 1286-1303.
- Cringarten, A.C., and Witherspoon, P.A., 1972, A method of analyzing pumping test data from fractured aquifers: Symposium on Percolation in Fissured Rock, Stuttgart, International Society of Rock Mechanics, Proceedings: v. 3, p. B1-B9.
- Hanson, J.M., 1984, Evaluation of subsurface fracture geometry using fluid pressure response to solid earth tidal strain: Lawrence Livermore National Laboratory, UCID-20156, 135 p.
- Karasaki, K., 1986, Well test analysis in fractured media: Lawrence Berkeley Laboratory Report LBL-21442, University of California, Berkeley, 239 p.
- Loeb, J., and Poupon, A., 1965, Temperature logs in production and injection wells: 27th Meeting of the European Association of Exploration Geophysicists, Madrid, May 5-7, 1965.
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ENCLOSURE

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Rec'd w/ letter dtd 9/14/92

References for Study Plan 8.3.1.2.3.1
(including Studies 8.3.1.2.3.1.1-6 and 8.3.1.2.3.1.7)
Requested by NRC

8.3.1.2.3.1.1-6

- Barenblatt, G.E., Zheltov, I.P., and Kochina, I.N., 1960, Basic concepts in the theory of homogeneous liquids in fissured rocks: Journal of Applied Mathematics and Mechanics (USSR), v. 24, no. 5, p. 1286-1303.
- Cringarten, A.C., and Witherspoon, P.A., 1972, A method of analyzing pumping test data from fractured aquifers: Symposium on Percolation in Fissured Rock, Stuttgart, International Society of Rock Mechanics, Proceedings: v. 3, p. B1-B9.
- Hanson, J.M., 1984, Evaluation of subsurface fracture geometry using fluid pressure response to solid earth tidal strain: Lawrence Livermore National Laboratory, UCID-20156, 135 p.
- Karasaki, K., 1986, Well test analysis in fractured media: Lawrence Berkeley Laboratory Report LBL-21442, University of California, Berkeley, 239 p.
- Loeb, J., and Poupon, A., 1965, Temperature logs in production and injection wells: 27th Meeting of the European Association of Exploration Geophysicists, Madrid, May 5-7, 1965.
- Papadopulos, S.S. 1965, Nonsteady flow to a well in an infinite anisotropic aquifer: Dubrovnik Symposium of Hydrology of Fractured Rocks, Proceedings, v. 1, p. 21-31.
- Rojstaczer, S.A., 1988, The response of the water level in a well to atmospheric loading and earth tides; Theory and application: Ph.D. Thesis, Stanford University, March 1988.
- Saad, K.F., 1967, Determination of the vertical and horizontal permeabilities of fractured water-bearing formations: Bulletin IASH, v. 3, p. 22.
- Snow, D.T., 1965, A parallel plate model of fractured permeable media (Ph.D. dissertation), Berkeley, University of California.

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ENCLOSURE 2

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BASIC CONCEPTS IN THE THEORY OF SEEPAGE OF HOMOGENEOUS LIQUIDS IN FISSURED ROCKS [STRATA]

(OB OSNOVNYKH PREDSTAVLENIIAKH TEORII FIL'TRATSII ODNORODNYKH ZHIDKOSTEI V TRESHCHINOVATYKH PORODAKH)

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PMM Vol.24, No.5, 1960, pp. 852-864

G.I. BARENBLATT, Iu.P. ZHELTOV and I.N. KOCHINA
(MOSCOW)

(Received 20 June 1960)

The modern theory of seepage [infiltration] is based on the concept of a porous medium consisting of impermeable grains separated by pore spaces. Comparison of the results of theoretical and laboratory investigations of non-steady-state flow of liquids with data for strata under natural conditions leads to the conclusion that current concepts of a porous medium are inadequate. In all natural strata, the development of some degree of fissuring is a characteristic feature. The description of non-steady-state flow of liquids in fissured strata by means of the usual equations of infiltration theory can lead, in some cases, to conflicting conclusions of qualitative nature.

At first glance, it appears that non-steady-state seepage in fissured rocks can be studied by assuming a system of fissures, which are regular to some extent, in the stratum. Apparently, for studying seepage in fissured rocks, this method is not promising. Even if it were possible to overcome the enormous mathematical difficulties involved in solving problems of non-steady-state flow in strata with a system of fissures of a sufficiently general type, it is not possible to determine the configuration of this system with any degree of reliability. Information obtained in the analysis of cores – specimens of the rock obtained by drilling from the surface – gives very incomplete data on the fissure system. The position is to some extent similar to that which occurs in investigating the flow of a liquid in an ordinary porous medium – even if it were possible to overcome all the difficulties involved in the integration of the equations of motion of a viscous liquid in the pore spaces, the method would not be suitable for investigating seepage, since the pore configuration remains unknown. Various models of a porous medium, which

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are based on one or another type of arrangement of the system of pores and grains and on the study of the motion of the liquid in such systems (ideal soil, fictitious soil, etc. [1]), proved suitable only for the qualitative investigation of seepage phenomena. Seepage theory has followed the trend which is characteristic of the mechanics of continuous media generally, namely, the introduction of mean characteristics of the media and flow (porosity, permeability, pressure, seepage velocity, etc.) and the formulation of basic laws in terms of these mean characteristics.

Such an approach, applied irrespective of whether or not the system of fissures is regular in the natural stratum, also proved most advantageous in investigating seepage in fissured rocks.

In this paper, the basic concepts of the motion of liquids in fissured rocks are presented. Mean-characteristics are introduced, whereby the averaging is carried out on a scale which is large compared to the dimensions of the individual blocks. The difference between the present scheme and the more usual scheme of seepage in a porous medium consists in the introduction at each point in space of two liquid pressures – liquid pressure in the pores and pressure of the liquid in the fissures α, and in taking into consideration the transfer of liquid between the fissures and the pores. Under certain assumptions, an expression is obtained for the intensity of this transfer. The basic equation of the seepage of a liquid in a fissured rock and the same general equations of the seepage of liquid in a porous medium with a double porosity are derived. These equations will obviously contain, as a particular case, the equations for the seepage of a liquid in an ordinary porous medium; in the paper an evaluation is made which indicates for which cases the latter equations are valid and when the more accurate expressions given in this paper have to be used. The formulation of the basic boundary-value problems for seepage equations in fissured rocks is considered. Some characteristic features of non-steady-state seepage in fissured rocks are discussed, particularly the possibility of the occurrence, under certain conditions, of a pressure jump [discontinuity] within the system and at the boundaries, similar to the "infiltration gap" in non-pressure seepage [2]. Conditions at jumps are derived, and the features pertaining to the formulation of boundary-value problems in the presence of jumps are pointed out. Solutions are given of certain specific problems of non-steady-state seepage in fissured rocks.

1. Basic physical concepts. A fissured rock consists of pores and permeable blocks, generally speaking blocks separated from each other by a system of fissures (Fig. 1). The dimensions of the blocks will vary for the various rocks within wide limits, depending on the extent to which fissures are developed in the rock. The widths of the fissures are considerably greater than the characteristic dimensions of the pores, so

that the permeability of the fissure system considerably exceeds the permeability of the system of pores in the individual blocks. At the same time, it is a characteristic feature of fissured rocks that the fissures occupy a much smaller volume than the pores, so that the coefficient of fissuring of the rock a_1 – the ratio of the volume of the cavity space occupied by the fissures to the total volume of the rock – is considerably smaller than the porosity of the individual blocks a_2. Much factual data on fissured rocks has been published in [3-9]; the paper by Pirson [4] is of particular interest, since it gives a qualitative description of the structure of a porous medium with double porosity, which is close to that considered in this paper.

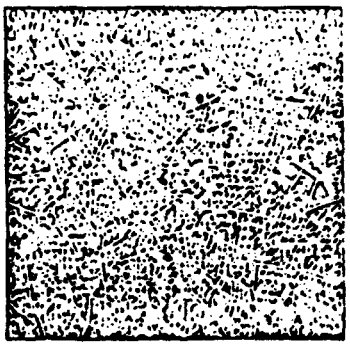

Fig. 1.

If the system of fissures is sufficiently well developed, the motion of the liquid in fissured rocks can be investigated by the following method. Unlike the classical seepage theory, for each point in space, not one liquid pressure but two, p_1 and p_2 are introduced. The pressure p_1 represents the average pressure of the liquid in the *fissures* in the neighborhood of the given point, while the pressure p_2 is the average pressure of the liquid in the *pores* in the neighborhood of the given point. For obtaining reliable averages, the scale of averaging should include a sufficiently large number of blocks. Therefore, it is necessary to take into consideration that any infinitely small volume includes not only a larger number of pores, as is assumed in the classical theory of seepage, but also that it contains a large number of blocks. This condition permits the use of the method of analysis of infinitesimals in investigating fissured rocks.

In a similar manner, two velocities of seepage of the liquid can be defined at each point in space: $\mathbf{V}_1$ and $\mathbf{V}_2$. Vector $\mathbf{V}_1$ of the seepage velocity of the liquid along the fissures is determined as follows: the projection of this vector in some particular direction is equal to the flow of the liquid through the cross-section of the fissures of a small zone passing through the given point in a direction perpendicular to the given direction, divided by the density of the liquid and the *total* area of this zone. In the same way, the projection of vector $\mathbf{V}_2$, the seepage velocity of the liquid through the pores in a given direction, will equal the flow of the liquid through the cross-section of the blocks of the small zone mentioned, also divided by the density of the liquid and the

total area of the zone.

It is a characteristic of fissured rocks that the flow of the liquid proceeds essentially along the fissures, so that the flow velocity of the liquid through the blocks is negligibly small as compared to seepage of liquid along the fissures.

If the boundary between the fissures and the blocks is imagined impermeable, the fissured rock can be considered as being a coarse-grained porous medium in which the fissures play the role of pores and the blocks play the role of grains. If, furthermore, the fissures are sufficiently narrow and the velocity of the liquid is sufficiently small, the motion of the liquid along the fissures will be inertialess and Darcy's Law is fulfilled:

$$\mathbf{V}_1 = -\frac{k_1}{\mu}\operatorname{grad} p_1 \tag{1.1}$$

where k_1 is the permeability of the system of fissures and μ is the viscosity of the liquid. Application of Darcy's Law to seepage along the system of fissures is not of principal importance; if desired, inertia of the motion can be taken into account, using thereby a more complicated nonlinear law.

A characteristic feature of the non-steady-state motion of a liquid in fissured rocks is the transfer of liquid between the blocks and the fissures. Therefore, in investigating the seepage of liquids in fissured rocks it is necessary, in contrast to the classical theory of seepage, to take into consideration the outflow of liquid from the "grains" – blocks into the "pores" – of the fissures.

The process of transfer of liquid from the pores and the blocks takes place essentially under a sufficiently smooth change of pressure, and, therefore, it can be assumed that this pressure is quasi-stationary, i.e. it is independent of time explicitly. It is obvious in such a case that during motion of a homogeneous liquid in the fissures of the rock, the volume of the liquid v, which flows from the blocks into the fissures per unit of time and unit of volume of the rock, depends on the following: (1) viscosity of the liquid μ; (2) pressure drop between the pores and the fissures $p_2 - p_1$; and (3) on certain characteristics of the rock, which can only be geometrical ones, i.e. they may have the dimension of length, area, volume, etc., or even be dimensionless. On the basis of dimensional analysis [10], we obtain for v an expression of the type

$$v = \frac{\alpha}{\mu}(p_2 - p_1) \tag{1.2}$$

where α is some new dimensionless characteristic of the fissured rock.

Thus, for the mass q of the liquid which flows from the pores into the fissures per unit of time, per unit of rock volume, the following equation is valid:

$$q = \frac{\rho\alpha}{\mu}(p_2 - p_1) \qquad (1.3)$$

where ρ is the density of the liquid.

It should be pointed out that in a somewhat different form relation (1.3) was applied for the integral estimate of the flow along the stratum as a whole [11].

2. Equation of motion of a uniform liquid in fissured rocks. In accordance with what has been said above, the law of conservation of mass of liquid in the presence of fissures can be written as follows:

$$\frac{\partial m_1\rho}{\partial t} + \operatorname{div}\rho V_1 - q = 0 \qquad (2.1)$$

In view of the smallness of the volume of the fissures, the first term, which expresses the change in mass of the liquid due to compression in the fissures and changes in the volume of the fissures in some element of the rock, is small as compared to the second term, which expresses changes in the mass of the liquid caused by the inflow of the liquid along the fissures through the boundary of this element. Therefore, relation (2.1) can be disregarded. Inserting Equation (1.1) (Darcy's Law) into Equation (2.1), taking into consideration the fact that the liquid is slightly compressible so that

$$\rho = \rho_0 + \beta\delta p \qquad (2.2)$$

(ρ_0 is the density of the liquid at some standard pressure, for instance, the initial pressure in the stratum, β is the coefficient of compressibility of the liquid, δp is the change in the pressure relative to the standard pressure), assuming that the medium is homogeneous and neglecting the small higher-order terms, we obtain

$$k_1\Delta p_1 + \alpha(p_2 - p_1) = 0 \qquad (\Delta \text{ is Laplace operator}) \qquad (2.3)$$

Further, the equations of conservation of mass of the liquid which is present in the pores can be written thus*:

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* Strictly speaking, in Equation (2.4), m_2 will not represent the porosity of the blocks but the ratio of the volume of the pores to

$$\frac{\partial m_2\rho}{\partial t} + \operatorname{div}\rho V_2 + q = 0 \tag{2.4}$$

so that the quantity of liquid which flows into the fissures equals the quantity of the liquid which flows out of the blocks.

In view of the low permeability of the blocks, the second term of Equation (2.4), which expresses changes in the mass of the liquid within the pores in some element of the rock, due to the inflow of liquid along the pores through the boundaries of the element, can be disregarded as compared to the first term which represents changes in the mass of the liquid in the pores due to its expansion, and also to changes in the volume of the pores. Therefore, Equation (2.4) can be re-written as

$$\frac{\partial m_2\rho}{\partial t} + q = 0 \tag{2.5}$$

Furthermore, the porosity of the blocks m_2 in the case of a constant pressure of the upper strata of the rocks on the roof of the stratum depends, generally speaking, on the pressure of the liquid in the fissures p_1 and the pressure of the liquid in the pores p_2. However, the volume of the fissures in the rock is considerably smaller than the volume of the pores. It can be assumed that, in contrast to the liquid located in the pores, the liquid located in the fissures does not participate in supporting the upper strata of the rock formations. Therefore, the influence of the pressure of the liquid in the fissures p_1 on the porosity of the blocks can be disregarded as compared to the influence of the pressures of the liquid in the pores p_2, and it can be assumed that

$$dm_2 = \beta_{c2} dp_2 \tag{2.6}$$

where β_{c2} is the coefficient of compressibility of the blocks. Taking into consideration, also, relations (1.3) and (2.2) and neglecting small terms of higher order, we obtain

$$(\beta_{c2} + m_0\beta)\frac{\partial p_2}{\partial t} + \frac{\alpha}{\mu}(p_2 - p_1) = 0 \tag{2.7}$$

where m_0 is the magnitude of the porosity of the blocks at standard pressure. Equations (2.3) and (2.7) describe the motion of the liquid in

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the entire volume of the rock, including the volume of the fissures. However, in view of the small relative volume of the fissures compared to the relative volume of the pores, m_2 can be considered as representing the porosity of the individual blocks.

fissured rocks. Eliminating from these equations p_2, we obtain for the pressure of the liquid in the fissures p_1 the equation

$$\frac{\partial p_1}{\partial t} - \eta \frac{\partial \Delta p_1}{\partial t} = \varkappa \Delta p_1 \qquad \left(\varkappa = \frac{k_1}{\mu(\beta_{c2} + m_0\beta)}, \quad \eta = \frac{k_1}{\alpha}\right) \tag{2.8}$$

The coefficient $\varkappa$ represents the coefficient of piezo-conductivity of the fissured rock; it is interesting that this does not correspond to the permeability of the system of fissures k_1 but to the porosity and compressibility of the blocks. The coefficient η represents a new specific characteristic of fissured rocks. If η tends to zero, it corresponds to a reduction of the block dimensions and an increase in the degree of fissuring, and Equation (2.8) will obviously tend to coincide with the ordinary equation for seepage of a liquid under elastic conditions.

An approximate estimate of the possible magnitudes of the coefficient η will be made. The dimensionless coefficient α, characterizing the intensity of the liquid transfer between the blocks and fissures, depends on the permeability of the blocks k_2 and the degree of fissuring of the rock, as a measure of which it is obvious to take the specific surface of the fissures σ, i.e. the surface of the fissures per unit of volume of the rock. The quantity σ has the dimension of the reciprocal of length. On the basis of dimensional analysis we obtain

$$\alpha \sim k_2\sigma^2 \tag{2.9}$$

From this and Equation (2.8) we obtain

$$\eta \sim \frac{k_1}{k_2\sigma^2} \sim \frac{k_1}{k_2} l^2$$

where l is the average dimension of a single block (the specific surface of the fissures is inversely proportional to the average dimension of a single block). Evaluations show that for various rocks the parameter η will assume values within wide limits – from a few cm^2 to values of the order of 10^{10} cm^2.

Determination of the parameter η should be carried out by means of data for the steady-state flow of liquids in fissured rocks. Thus, where natural strata are involved, determination of this parameter should be carried out only on the basis of investigations of the behavior of the stratum under non-steady-state conditions and not on the basis of tests carried out on rock specimens brought to the surface.

3. Equations of motion of a homogeneous liquid in a medium with double porosity. The system of equations (2.3), (2.7) represents a particular

case of the system of equations of motion of a homogeneous liquid in a medium with double porosity. In some cases the latter equations may be of interest and, therefore, we will deal briefly with their derivation.

The motion of a uniform liquid in a "double" porous medium will be considered: the first porous medium consists of relatively wide pores of the first order – fissures and blocks; the relative volume of the pores of the first order, the porosity of the first order, equals m_1. The blocks in themselves are porous, consisting of grains which are separated by fine pores of the second order; together they form the second porous medium. The porosity of this medium – the porosity of the second order – is designated by m_2. It is pointed out that, generally speaking, m_2 cannot be considered equal to the porosity of the blocks, since m_2 represents the ratio of the volume of the second-order pores to the total volume of the elements of the rock in which a known space is occupied by the fissures – pores of the first order. In the case where the upper strata exert a constant pressure on the roof of the stratum, both porosities, m_1 and m_2, will depend on the pressures of the liquid in the pores of the first and second order, p_1 and p_2, so that

$$dm_1 = \beta_{c1} dp_1 - \beta_e dp_2, \qquad dm_2 = \beta_{c2} dp_2 - \beta_{ee} dp_1 \tag{3.1}$$

where β_{c1}, β_{c2}, β_e, β_{ee} are positive constant coefficients.

The equations for conservation of mass of the liquid for both media are of the form (2.1) and (2.4), respectively. Assuming that the flow of the liquid in the first medium (and thus also in the second medium) is inertialess, the Darcy law for both media can be written as

$$\mathbf{V}_1 = -\frac{k_1}{\mu} \operatorname{grad} p_1, \qquad \mathbf{V}_2 = -\frac{k_2}{\mu} \operatorname{grad} p_2 \tag{3.2}$$

where k_1 is the porosity of the system of pores of the first order and k_2 the porosity of the system of pores of the second order.

By inserting into Equations (2.1) and (2.4) relations (3.2), Expression (1.3) for the liquid flow from one medium to the other (which will obviously remain valid even in this more general case), relation (2.2) for the density of the liquid and relation (3.1) for the (differentials of the porosity), and discarding small quantities of higher order for the pressures of the liquid in both media p_1 and p_2, the following system of equations is obtained:

$$\begin{aligned} \frac{k_1}{\mu} \Delta p_1 &= (\beta_{c1} + m_{10}\beta) \frac{\partial p_1}{\partial t} - \beta_e \frac{\partial p_2}{\partial t} - \frac{\alpha}{\mu}(p_2 - p_1) \\ \frac{k_2}{\mu} \Delta p_2 &= (\beta_{c2} + m_{20}\beta) \frac{\partial p_2}{\partial t} - \beta_{ee} \frac{\partial p_1}{\partial t} + \frac{\alpha}{\mu}(p_2 - p_1) \end{aligned} \tag{3.3}$$

where m_1 and m_2 are the values of the first and second order porosity at standard pressure.

If the pressure p_2 changes, say decreases, at a constant pressure of the upper strata on the roof, the porosity of the first order will increase, on the one hand, as a result of the compression of the blocks and, on the other hand, it will decrease as a result of compression by the overlying strata. These effects will apparently compensate each other to some extent. The situation is similar for the second-order porosity m_2 in the case of a change in the pressure p_1. It is, therefore, advisable to consider the model of the double porosity of the medium for which the porosity of each order depends only on the appropriate pressure, so that the coefficient β_c and β_{cc} in Equation (3.1) can be considered small and the appropriate terms in Equation (3.1) can be disregarded.

Equations (3.3) for such a model of a porous medium with double porosity will be of the form similar to the equations for heat transfer in a heterogeneous medium considered by Rubinshtein [12]:

$$\frac{k_1}{\mu}\Delta p_1 = (\beta_{c1} + m_{10}\beta)\frac{\partial p_1}{\partial t} - \frac{\alpha}{\mu}(p_2 - p_1)$$

$$\frac{k_2}{\mu}\Delta p_2 = (\beta_{c2} + m_{20}\beta)\frac{\partial p_2}{\partial t} + \frac{\alpha}{\mu}(p_2 - p_1)$$

Disregarding in Equations (3.3) the terms representing a change in the mass of the liquid due to the compressibility of the first medium and the compression of the liquid in the pores of the first order, and the changes in the mass of the liquid as a result of the seepage inflow along the pores of the second order, we again obtain the equations of motion of a liquid in a fissured porous medium (2.3) and (2.7).

4. Basic boundary-value problems of the theory of non-steady-state seepage in fissured rocks. Equation (2.8), to which corresponds the pressure distribution of the liquid in the pores p_1, can be written as

$$\beta_0\frac{\partial p_1}{\partial t} - \operatorname{div}\left[\frac{k_1}{\mu}\operatorname{grad} p_1 + \eta\beta_0\frac{\partial}{\partial t}\operatorname{grad} p_1\right] = 0 \qquad (4.1)$$

where β_0 is the total effect of compressibility equalling $\beta_{c2} + m_0\beta$. This form of writing the basic equation indicates that motion in the system of fissures can be considered as the motion of a liquid in a porous medium with a total compressibility coefficient β_0, and the expression for the velocity of seepage of the liquid can be written as

$$\mathbf{V} = -\frac{k_1}{\mu}\operatorname{grad} p_1 - \eta\beta_0\frac{\partial}{\partial t}\operatorname{grad} p_1 \qquad (4.2)$$

The initial and the boundary conditions have to be added to Equation (2.8). As in the theory of seepage in a porous medium, the steady-state initial conditions are of greatest interest in the given case (i.e. the harmonic initial distributions p_1, which satisfy Equation (4.1)). Among the possible types of boundary conditions the most important are the following:

1) The pressures p_1 at the boundary of the rock of volume s under consideration are given (first boundary-value problem):

$$p_1|_S = f(S, t) \tag{4.3}$$

2) At the boundary s the flow of the liquid is given (second boundary-value problem), the following quantity being given, in accordance with what was stated above, at the boundary of the surface S:

$$-\left\{\frac{k_1}{\mu}\frac{\partial p_1}{\partial n} + \eta\beta_0 \frac{\partial}{\partial t}\left(\frac{\partial p_1}{\partial n}\right)\right\}\Big|_S = f(S, t) \tag{4.4}$$

($\partial/\partial n$ is the derivative along the normal to the surface S), and, finally,

3) At the boundary a linear combination of the pressure and the flow of the liquid, generally speaking with variable coefficients A and B, is given (mixed problem):

$$\left\{Ap_1 + B\left[\frac{k_1}{\mu}\frac{\partial p_1}{\partial n} + \eta\beta_0 \frac{\partial}{\partial t}\left(\frac{\partial p_1}{\partial n}\right)\right]\right\}\Big|_S = f(S, t) \tag{4.5}$$

If the initial pressure-distribution is continuous and the boundary conditions are consistent with the initial ones (i.e. the boundary values of the initial distribution on approaching the boundary points equal to the boundary values of the corresponding functions at the initial instant of time), the solutions of the above-stated boundary-value problems will be the ordinary classical solutions (4.1). However, if the initial pressure distribution is discontinuous or if the initial and the boundary conditions are unrelated, then the derived distributions will also be discontinuous and there is no classical solution for the boundary-value problems formulated above; it is necessary to seek a generalized solution in the sense of Sobolev [13]. To proceed further it is necessary to derive the conditions at the discontinuities. It is sufficient to consider the one-dimensional case, since in the neighborhood of the given point the surface of discontinuity can be considered as being plane. Thus, it is assumed that within a sufficiently small vicinity on both sides of the isolated discontinuity surface $x = 0$ (x is the direction of the normal to the surface of the discontinuity), the function p_1 is continuous, has appropriate continuous derivatives and satisfies the equation

$$Lp_1 = \beta_0 \frac{\partial p_1}{\partial t} - \eta\beta_0 \frac{\partial^3 p_1}{\partial x^2 \partial t} - \frac{k_1}{\mu}\frac{\partial^2 p_1}{\partial x^2} = 0 \tag{4.6}$$

In the region $G(-h < x < h,\ 0 < t < T)$, where h is a small number, the terms in the expression Lp_1 are piece-wise continuous. By means of term-by-term integration of Lp_1 along the region G we obtain

$$\int_G Lp_1\,dx\,dt = \beta_0 \int_{-h}^{h} \left\{p_1(x,T) - p_1(x,0)\right\} dx - \int_0^T \left\{\eta\beta_0 \frac{\partial^2 p_1}{\partial x \partial t} + \frac{k_1}{\mu}\frac{\partial p_1}{\partial x}\right\}\Bigg|_{x=-h}^{x=h} dt = 0 \tag{4.7}$$

For $h \to 0$ the first integral tends to zero and the preceding equality yields

$$\int_0^T \left[\eta\beta_0 \frac{\partial^2 p_1}{\partial x \partial t} + \frac{k_1}{\mu}\frac{\partial p_1}{\partial x}\right] dt = 0 \tag{4.8}$$

where as usual the sign [ ] designates the difference between the values of the function on both sides of the discontinuity surface. Since T is arbitrary and the expression under the integral sign is a continuous function of time, it follows that the expression under the integral sign equals zero:

$$\left[\eta\beta_0 \frac{\partial}{\partial t}\left(\frac{\partial p_1}{\partial n}\right) + \frac{k_1}{\mu}\frac{\partial p_1}{\partial n}\right] = \eta\beta_0 \frac{\partial}{\partial t}\left[\frac{\partial p_1}{\partial n}\right] + \frac{k_1}{\mu}\left[\frac{\partial p_1}{\partial n}\right] = 0 \tag{4.9}$$

i.e. the condition of continuity at the surface of discontinuity of the total flow of the liquid ($\partial/\partial x$ was replaced by $\partial/\partial n$). To obtain the second condition, Equation (4.6) is multiplied by x and integration is carried out over the same region G:

$$\int_G xLp_1\,dx\,dt = \beta_0 \int_{-h}^{h} \left\{p_1(x,T) - p_1(x,0)\right\} x\,dx -$$

$$-\int_0^T \left\{\eta\beta_0 x \frac{\partial^2 p_1}{\partial x \partial t} + \frac{k_1}{\mu} x \frac{\partial p_1}{\partial x}\right\}\Bigg|_{-h}^{h} dt - \int_0^T \left\{\eta\beta_0 \frac{\partial p_1}{\partial t} + \frac{k_1}{\mu} p_1\right\}\Bigg|_{-h}^{h} dt = 0$$

As $h \to 0$, the first and second integrals will become zero, and thus we obtain

$$\int_0^T \left[\eta\beta_0 \frac{\partial p_1}{\partial t} + \frac{k_1}{\mu} p_1\right] dt = 0 \tag{4.10}$$

so that the second condition at the surface of discontinuity is obtained in the form

$$\left[\eta\beta_0 \frac{\partial p_1}{\partial t} + \frac{k_1}{\mu} p_1\right] = \eta\beta_0 \frac{\partial [p_1]}{\partial t} + \frac{k_1}{\mu}[p_1] = 0 \qquad (4.11)$$

For $\eta = 0$ the basic conditions at the surface of discontinuity (4.9) and (4.11) will change into the condition of continuity of the function and its derivative along the normal to any surface, i.e. the condition of absence of discontinuities*, which is well known in the theory of heat conduction and the theory of seepage in a porous medium.

Integrating (4.9) and (4.11), we obtain the conditions at the discontinuities in the form

$$[p_1] = [p_1]_{t=0}\, e^{-\varkappa t/\eta}, \qquad \left[\frac{\partial p_1}{\partial n}\right] = \left[\frac{\partial p_1}{\partial n}\right]_{t=0} e^{-\varkappa t/\eta}. \qquad (4.12)$$

so that the pressure jumps and normal derivative of pressure which occur, due to the discontinuity or due to inconsistent initial conditions, will not be eliminated instantaneously as in a porous medium (and as is the case for jumps in temperature and heat flow in the theory of heat conduction) but will decrease in accordance with the law $e^{-\varkappa t/\eta}$. This property is a characteristic qualitative feature of the mathematical description of non-steady-state flow in fissured rocks, which is comparable with flow through a porous medium.

5. Some specific problems of non-steady-state flow in fissured rocks. General qualitative conclusions. *1. Non-steady-state flow of liquid in a gallery.* From the literature the importance of the study of non-steady-state seepage in drainage galleries is well known. This problem is formulated as follows: at the initial instant the pressure of the liquid in a semi-infinite stratum $(0 < x < \infty)$ P_0 is constant; the pressure at the boundary $x = 0$ suddenly assumes the value P_1, differing from P_0, and then remains constant. The problem of determining seepage flow requires the solution of the equation

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* In obtaining the second law of conservation in a medium with a variable permeability coefficient k_1 it is necessary to multiply both parts of Equation (4.6) by

$$\int \frac{1}{k_1} dx$$

$$\frac{\partial p_1}{\partial t} - \eta \frac{\partial^3 p_1}{\partial x^2 \partial t} = \varkappa \frac{\partial^2 p_1}{\partial x^2} \tag{5.1}$$

for the uncoordinated initial and boundary conditions

$$p_1(x, 0) = P_0, \qquad p_1(-0, t) = P_1 \tag{5.2}$$

(the boundary pressure is given immediately to the left of the boundary $x = 0$). At the initial instant there will be a pressure jump at the boundary equal to $(P_0 - P_1)$; according to (4.12), at time t this jump will equal $(P_0 - P_1)e^{-\varkappa t/\eta}$, so that the pressure of the liquid immediately to the right of the boundary will equal

$$p_1(+0, t) = P_1 + (P_0 - P_1)\, e^{-\varkappa t/\eta} \tag{5.3}$$

To find the pressure distribution at any desired instant of time t, the Laplace transform with respect to time t is applied. We set

$$p_1(x, t) = P_0 - (P_0 - P_1)\, u(x, t) \tag{5.4}$$

Then, for determining $u(x, t)$ $(t \geqslant 0,\ 0 \leqslant x < \infty)$, we obtain the boundary-value problem

$$\frac{\partial u}{\partial t} - \eta \frac{\partial^3 u}{\partial x^2 \partial t} = \varkappa \frac{\partial^2 u}{\partial x^2}, \qquad u(+0, t) = 1 - e^{-\varkappa t/\eta}, \quad u(x, 0) = 0 \tag{5.5}$$

Let

$$U(x, \lambda) = \int_0^\infty e^{-\lambda t} u(x, t)\, dt$$

Applying the Laplace transform to Equation (5.5) we obtain

$$\frac{d^2 U}{dx^2} - \frac{\lambda}{\varkappa + \lambda\eta} U = 0, \qquad U(0, \lambda) = \frac{1}{\lambda(\varkappa + \lambda\eta)}, \qquad U(\infty, \lambda) = 0 \tag{5.6}$$

$$U = \frac{1}{\lambda(\varkappa + \lambda\eta)} \exp\left(-\sqrt{\frac{\lambda}{\varkappa + \lambda\eta}}\, x\right)$$

whence, on the basis of the known rule of inversion [14], we obtain

$$u(x, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \exp\left(-\sqrt{\frac{\lambda}{\varkappa + \lambda\eta}}\, x\right) \frac{d\lambda}{\lambda(\varkappa + \lambda\eta)} \tag{5.7}$$

The evolution of the integral on the right-hand side of the previous equation yields

$$2\pi i - 2i \int_0^1 e^{-\sigma t} \sin\left(\sqrt{\frac{\sigma}{1-\sigma}}\, x\right) \frac{d\sigma}{\sigma(1-\sigma)} \tag{5.8}$$

Inserting this expression into Equation (5.7) and substituting the variables $\sigma/(1-\sigma) = \nu^2$, we obtain

$$u(x,t) = 1 - \frac{2}{\pi}\int_0^\infty \frac{1}{\nu} \sin \nu x \exp\left(-\frac{\nu^2 \varkappa t}{1+\nu^2\eta}\right) d\nu = 1 - \exp\left(-\frac{\varkappa t}{\eta}\right) -$$

$$- \frac{2}{\pi}\int_0^\infty \frac{1}{\nu} \sin \nu x \left\{\exp\left(-\frac{\nu^2 \varkappa t}{1+\nu^2\eta}\right) - \exp\left(-\frac{\varkappa t}{\eta}\right)\right\} d\nu \tag{5.9}$$

From this and from Equation (5.4) we finally obtain

$$p_1(x,t) = P_1 + \frac{2(P_0 - P_1)}{\pi}\int_0^\infty \frac{\sin \nu x}{\nu} \left\{\exp\left(-\frac{\nu^2 \varkappa t}{1+\nu^2\eta}\right) - \exp\left(-\frac{\varkappa t}{\eta}\right)\right\} d\nu + \tag{5.10}$$

$$+ (P_0 - P_1) \exp\left(-\frac{\varkappa t}{\eta}\right)$$

Since the integral in Equation (5.10) and the integral obtained after differentiation with respect to x under the integral sign are both uniformly convergent, the expression given by Equation (5.10) can be differentiated with respect to x. Hence, from Equation (5.10) we obtain the expression for the flow of the liquid through the boundary of the stratum $x = 0$

$$q = -\frac{k_1}{\mu}\left(\frac{\partial p_1}{\partial x}\right)_{x=+0} = -\frac{2(P_0 - P_1)k_1}{\pi\mu}\int_0^\infty \left\{\exp\left(-\frac{\nu^2 \varkappa t}{1+\eta\nu^2}\right) -\right.$$

$$\left. - \exp\left(-\frac{\varkappa t}{\eta}\right)\right\} d\nu = -\frac{2(P_0 - P_1)k_1}{\pi\mu\sqrt{\eta}} \exp\left(-\frac{\varkappa t}{\eta}\right) \times$$

$$\times \int_0^\infty \left\{\exp\left(\frac{\varkappa t}{\eta(1+\zeta^2)}\right) - 1\right\} d\zeta \tag{5.11}$$

It is interesting to compare the derived solutions with the appropriate solution of the problem of the theory of seepage in a porous medium. This well-known self-similar solution is obtained from Equation (5.10) for the case when $\eta = 0$

$$p_1(x,t) = P_1 + \frac{2}{\pi}(P_0 - P_1)\int_0^{\frac{1}{2}\xi} e^{-\beta^2} d\beta = P_1 + (P_0 - P_1)\,\Phi\left(\frac{1}{2}\xi\right)$$

$$\left(\xi = \frac{x}{\sqrt{\varkappa t}}\right) \tag{5.12}$$

where Φ is the symbol of the Kramp function. For comparing the pressures, Equations (5.10) and (5.12) which correspond to fissured and ordinary porous media, respectively, the distribution quantities $u(x, t) = (p_1 - P_1)/(P_0 - P_1)$ for various values of the parameter $\kappa t/\eta$ have been plotted as a function of the self-similar variable ξ in Fig. 2. (The calculations were carried out by A. L. Dyshko at the Computing Center of the Academy of Sciences, USSR). It can be seen that with increasing $\kappa t/\eta$ the pressure distribution in a fissured rock tends to the self-similar distribution which is obtained in an ordinary porous medium.

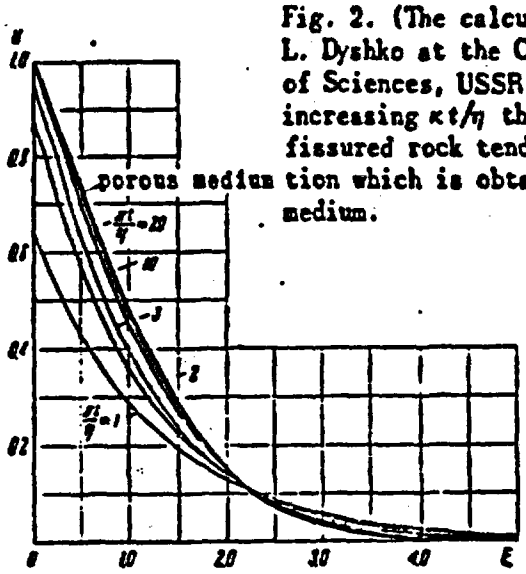

Fig. 2.

2. *Non-steady-state infiltration of a liquid from a well discharging at a constant rate.* In addition to the problem considered previously, the problem of non-steady-state infiltration of a liquid from a well of infinitely small radius with a constant yield is of considerable interest. It is formulated as follows: an infinite horizontal stratum of constant thickness h is penetrated by a vertical well of negligibly small radius. At the initial instant, the pressure of the liquid in the stratum is constant and equal to P. Then, a liquid begins to flow in or out at a constant volume rate Q.

The pressure of the liquid in the fissures $p_1(r, t)$ (r is the distance from the axis of the well) satisfies the equation

$$\frac{\partial p_1}{\partial t} - \eta \frac{\partial}{\partial t} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial p_1}{\partial r} = \varkappa \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial p_1}{\partial r} \tag{5.13}$$

with the initial conditions

$$p_1(r, 0) = P \tag{5.14}$$

In accordance with Equation (4.4) the boundary conditions can be expressed as

$$Q = -2\pi h \left\{ \frac{k_1}{\mu} \left(r \frac{\partial p_1}{\partial r} \right) + \eta \beta_0 \frac{\partial}{\partial t} \left(r \frac{\partial p_1}{\partial r} \right) \right\}_{r=0}$$

We obtain therefrom

$$-\frac{Q\mu}{2\pi k_1 h} = \left(r\frac{\partial p_1}{\partial r}\right)_{r=+0} + \frac{\eta}{\varkappa}\frac{\partial}{\partial t}\left(r\frac{\partial p_1}{\partial r}\right)_{r=+0}$$

Integrating the last relation and applying the condition

$$(r\partial p_1/\partial r)_{r=+0} = 0 \qquad \text{for } t = 0$$

we obtain the final formula for the boundary conditions

$$\left(r\frac{\partial p_1}{\partial r}\right)_{r=+0} = -\frac{Q\mu}{2\pi k_1 h}(1 - e^{-\varkappa t/\tau}) \tag{5.15}$$

Setting

$$p_1(r, t) = P + \frac{Q\mu}{2\pi k_1 h} u(r, t) \tag{5.16}$$

we obtain the following boundary-value problem for determining the function $u(r, t)$:

$$\begin{gathered} \frac{\partial u}{\partial t} - \eta\frac{\partial}{\partial t}\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial u}{\partial r} = \varkappa\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial u}{\partial r} \\ u(r, 0) = 0, \qquad \left(r\frac{\partial u}{\partial r}\right)_{r=+0} = -(1 - e^{-\varkappa t/\tau}) \end{gathered} \tag{5.17}$$

To solve the boundary-value problem (5.17) we again revert to the Laplace transform, the relations (5.17) being reduced to the following form:

$$\frac{1}{r}\frac{d}{dr} r\frac{dU}{dr} - \frac{\lambda}{\varkappa + \lambda\eta} U = 0, \quad \left(r\frac{dU}{dr}\right)_{r=0} = -\frac{\varkappa}{\lambda(\varkappa + \lambda\eta)} \tag{5.18}$$

Taking into consideration, also, the condition $U(\infty, \lambda) = 0$, we obtain

$$U(r, \lambda) = \frac{\varkappa}{\lambda(\varkappa + \lambda\eta)} K_0\left(\sqrt{\frac{\lambda}{\varkappa + \lambda\eta}}\, r\right) \tag{5.19}$$

(K_0 is the symbol of the Macdonald function), and by the general rule of inversion [14] we obtain

$$u(r, t) = \frac{\varkappa}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda(\varkappa + \lambda\eta)} K_0\left(\sqrt{\frac{\lambda}{\varkappa + \lambda\eta}}\, r\right) d\lambda \tag{5.20}$$

In an entirely analogous manner, after calculating the integral and reverting to the variable p_1, we obtain the pressure distribution in the form

$$p_1(r, t) = P + \frac{Q\mu}{2\pi k_1 h}\int_0^\infty \frac{J_0(\nu r)}{\nu}\left[1 - \exp\left(-\frac{\nu^2\varkappa t}{1 + \nu^2\eta}\right)\right] d\nu \tag{5.21}$$

The known self-similar solution of the corresponding problem of the theory of seepage in a porous medium is obtained from Equation (5.21) for $\eta = 0$:

$$p_1(r,t) = P + \frac{Q\mu}{2\pi k_1 h}\int_0^\infty \frac{J_0(\nu r)}{\nu}(1 - e^{-\nu^2 \varkappa t})\,d\nu = P - \frac{Q\mu}{4\pi k_1 h}\,\mathrm{Ei}\left(-\frac{r^2}{4\varkappa t}\right) \qquad (5.22)$$

It can be seen, in the same way as in the previous problem, that for increasing values of $\varkappa t/\eta$, the solution (5.21) of the problem of seepage in a fissured rock tends asymptotically to the solution (5.22) of the problem of seepage in a porous medium.

It can also be seen from the examples investigated that the most characteristic property of the non-steady-state flow of liquids in fissured rocks is the occurrence of some delay in the transient processes; the characteristic time of this delay is

$$\tau = \eta/\varkappa \qquad (5.23)$$

Thus, the following general conclusion can be advanced: in considering processes of non-steady-state infiltration in fissured rocks, the ordinary equations of non-steady-state flow in a porous medium can be applied only if the characteristic times of the process under consideration are long compared to the delay times τ. However, if the characteristic times of the process are comparable to τ, it is necessary to apply the model and the basic equations presented in this paper. The estimates which have been carried out have shown that fissuring must be taken into consideration in many cases when investigating such processes as the restoration of pressure in shut-down wells and, generally, transient processes during changes of the operating conditions of the well.

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Unsteady-State Pressure Distributions Created by a Well With a Single Horizontal Fracture, Partial Penetration, or Restricted Entry

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ABSTRACT

Although there have been many studies on steady behavior of wells with vertical fractures, and although there was at one time a controversy concerning the occurrence of horizontal or vertical fractures as a result of hydraulic fracturing, to date there has been published no study of the unsteady behavior of a well containing a horizontal fracture. This is particularly surprising because such a study might have indicated significant differences between the performance of wells with horizontal fractures and those with vertical fractures. The purpose of this study was to fill that existing gap in knowledge of fractured-well behavior.

An analytical solution was developed by means of the concept of instantaneous sources and Green's functions. The analytical solution modeled the behavior of constant-rate production from a well containing a single, horizontal fracture of finite thickness at any position within a producing interval in an infinitely large reservoir with impermeable upper and lower boundaries. This general solution also contained solutions for the cases of (1) a single, plane (zero thickness) horizontal fracture, (2) partial penetration of the producing formation, and (3) limited flow entry throughout a producing interval. Although those are interesting solutions, the main purpose of this study was to investigate the horizontal fracture case. The analytical solution for this case was evaluated by computer to produce tables of dimensionless pressures vs dimensionless times sufficient for well-test analysis purposes.

A careful analysis of the general solution for a horizontal fracture indicated the existence of four different flow periods. It appears that during the first period all production originates within the fracture, causing a typical storage-controlled period. This period is followed by a period of vertical, linear flow. There then follows a transitional period, after which flow appears essentially radial. During the last period, the pressure is the same as that created by a line-source well with a skin effect. The skin effect is independent of time, but does depend upon the position of the pressure point. It was found that there is a radius of influence beyond which flow is essentially radial for all times. Approximating solutions and appropriate time limits for approximate solutions were derived.

INTRODUCTION

Hydraulic fracturing has been used for improving well productivity for the last 20 years and is generally recognized as a major development in well-completion technology. There was considerable discussion in the early 1950's about the orientation and the number of fractures created by this type of well stimulation. It is now generally agreed[1] that a vertical fracture will result if the least principal stress in the formation is horizontal, whereas a horizontal fracture will be created if the least principal stress is vertical. Further, data collected and reported by Zemanek *et al.*[2] shows that hydraulic fracturing usually results in one vertical fracture, the plane of which includes the axis of the wellbore. This conclusion appears widely held today. Thus, most studies of the flow behavior for fractured wells consider vertical fractures only.

However, the existence of horizontal fractures has been proved in some cases, and various authors have considered them. The steady-state behavior of horizontally fractured wells has been studied numerically by Hartsock and Warren.[3] Their model assumed the reservoir to be homogeneous, of constant thickness, of anisotropic permeabilities, and completely penetrated by a well of small radius. A single, horizontal, symmetrical fracture of negligible thickness and finite conductivity was located at the center of the formation. Radial flow was assumed beyond a critical radius four times as large as the fracture radius, and there was no flow across the drainage radius. The only flow into the well itself was through the fracture. With this model,

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[1]References listed at end of paper.

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skin effect factors that were independent of the drainage radius could be calculated for various combinations of parameters. Hartsock and Warren concluded that a poorly designed fracture treatment could reduce productivity, and that the apparent skin factor curves could be used in fracture treatment design.

Further knowledge of the transient pressure behavior for horizontal fractures with a uniform flux distribution located at the center, top, or bottom of the formation was presented by Warren[4] in his discussion of Coats' mathematical model[5] for water movement about bottom-water-drive reservoirs. Warren gave an expression for the skin factor that depended upon the fracture radius, the well radius, the reservoir thickness, the ratio of the horizontal permeability to the vertical permeability, and a geometric constant obtainable from Coats' results. No study of the unsteady-state behavior of wells with horizontal fractures has yet appeared.

It is our purpose here to develop a mathematical model for investigating the unsteady-state behavior of horizontally fractured wells. The solution has been found to be applicable to plane horizontal fractures and to wells with partial penetration or limited flow entry. For each case, the dimensionless pressure drop function has been analyzed, and different flow periods have been recognized.

MATHEMATICAL MODEL

A cross-section of the idealized reservoir-fracture model is shown in Fig. 1. The following assumptions are made:

1. The reservoir is horizontal, homogeneous, and has anisotropic radial and vertical permeabilities, k_r and k_z, respectively. It is of thickness h, porosity ϕ, of infinite extent, and completely penetrated by a well of radius r_w, which will be represented as a line source.

2. A single, horizontal, symmetrical fracture of radius r_f and thickness h_f is centered at the well, and the horizontal plane of symmetry of the fracture is at an altitude z_f.

3. A single-phase, slightly compressible liquid flows from the reservoir into the fracture at a constant rate q_f, which is uniform over the entire volume of the fracture.

4. There is no flow across the upper and lower boundaries of the reservoir, and the pressure remains unchanged and equal to the initial pressure as radial distance approaches infinity.

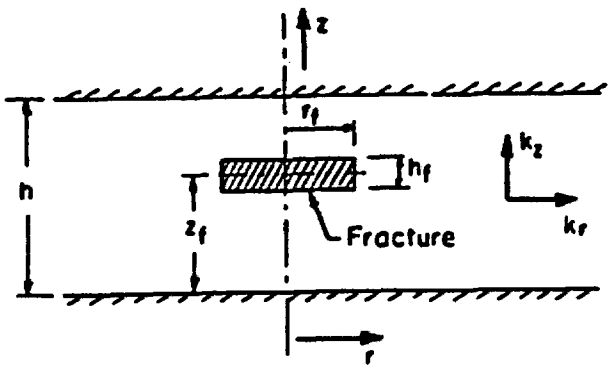

FIG. 1—CROSS-SECTION OF HORIZONTAL FRACTURE MODEL.

An analytical expression for the pressure distribution created by the fracture may be obtained by means of the Green's function and product solution method presented by Gringarten and Ramey in Ref. 6. In this method, the idealized reservoir system is visualized as the intersection of an infinite solid cylinder source, with an infinite horizontal slab source in an infinite horizontal slab reservoir with impermeable upper and lower boundaries, as sketched in Fig. 2. The *instantaneous* source function for such a system is equal to the product of the *instantaneous* source function for the slab source in the slab reservoir (Ref. 6, Table 2):

$$\frac{h_f}{h}\left[1+\frac{4h}{\pi h_f}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{n^2\pi^2\eta_z t}{h^2}\right)\sin n\pi\frac{h_f}{2h}\cos n\pi\frac{z_f}{h}\cos n\pi\frac{z}{h}\right], \quad \ldots\ldots (1a)$$

with the *instantaneous* source function for the infinite solid cylinder in an infinite reservoir (Ref. 6, Table 1, thick-wall cylinder with $r_1 = 0$):

$$\left[\frac{\exp(-r^2/4\eta_r t)}{2\eta_r t}\int_0^{r_f} I_0\left(\frac{rr'}{2\eta_r t}\right)\exp\left(-\frac{r'^2}{4\eta_r t}\right)r'dr'\right]. \quad \ldots\ldots (1b)$$

The pressure distribution in the system is then given by[6]

$$\Delta p(r,z,t) = \frac{1}{\phi c}\int_0^t q(t')\frac{h_f}{h}\left\{1+\frac{4h}{\pi h_f}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left[-\frac{n^2\pi^2\eta_z(t-t')}{h^2}\right]\sin n\pi\frac{h_f}{2h}\cos n\pi\frac{z_f}{h}\cos n\pi\frac{z}{h}\right\}$$

$$\sin n\pi \frac{z_f}{h} \cos n\pi \frac{z}{h}\Bigg\} \Bigg\{ \frac{e^{-\frac{r^2}{4\eta_r(t-t')}}}{2\eta_r(t-t')} \int_0^{r_f} I_0\left[\frac{rr'}{2\eta_r(t-t')}\right] e^{-\frac{r'^2}{4\eta_r(t-t')}} r'dr' \Bigg\} dt' \quad \ldots\ldots (2)$$

where $q(t)$ is the withdrawal rate per unit volume of source. In this study $q(t)$ is assumed constant, and uniform over the fracture. Thus, Eq. 2 can be written as

$$\Delta p(r,z,t) = \frac{q}{\phi c}\frac{h_f}{h} \int_0^t \Bigg[1 + \frac{4h}{\pi h_f} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2\eta_z t'}{h^2}} \sin\frac{n\pi h_f}{2h} \cos n\pi \frac{z_f}{h} \cos n\pi \frac{z}{h} \Bigg] \Bigg[\int_0^{r_f} I_0\left(\frac{r_m r}{2\eta_r t'}\right) e^{-\frac{r^2}{4\eta_r t'}} rdr \Bigg] \frac{e^{-\frac{r_m^2}{4\eta_r t'}}}{2\eta_r t'} dt' \quad \ldots\ldots (3)$$

Substituting the total withdrawal rate from the fracture,

$$q_f = \pi r_f^2 h_f q \quad \ldots\ldots (4)$$

and the dimensionless variables

$$r_D = r/r_f \quad \ldots\ldots (5)$$

$$z_D = \frac{z}{r_f}\sqrt{\frac{k_r}{k_z}} \quad \ldots\ldots (6)$$

$$t_D = \frac{k_r t}{\phi \mu c r_f^2} \quad \ldots\ldots (7)$$

$$p_D(r_D, z_D, t_D) = \frac{2\pi k_r h}{q_f \mu} \Delta p(r,z,t) \quad \ldots\ldots (8)$$

into Eq. 3, we obtain the dimensionless pressure drop at any point in the reservoir:

$$p_D(r_D, z_D, t_D) = \int_0^{t_D} \frac{\exp\left(-\frac{r_D^2}{4t_D'}\right)}{t_D'} \left[\int_0^1 I_0\left(\frac{r_D r_D'}{2t_D'}\right) \exp\left(-\frac{r_D'^2}{4t_D'}\right) r_D' dr_D' \right]$$

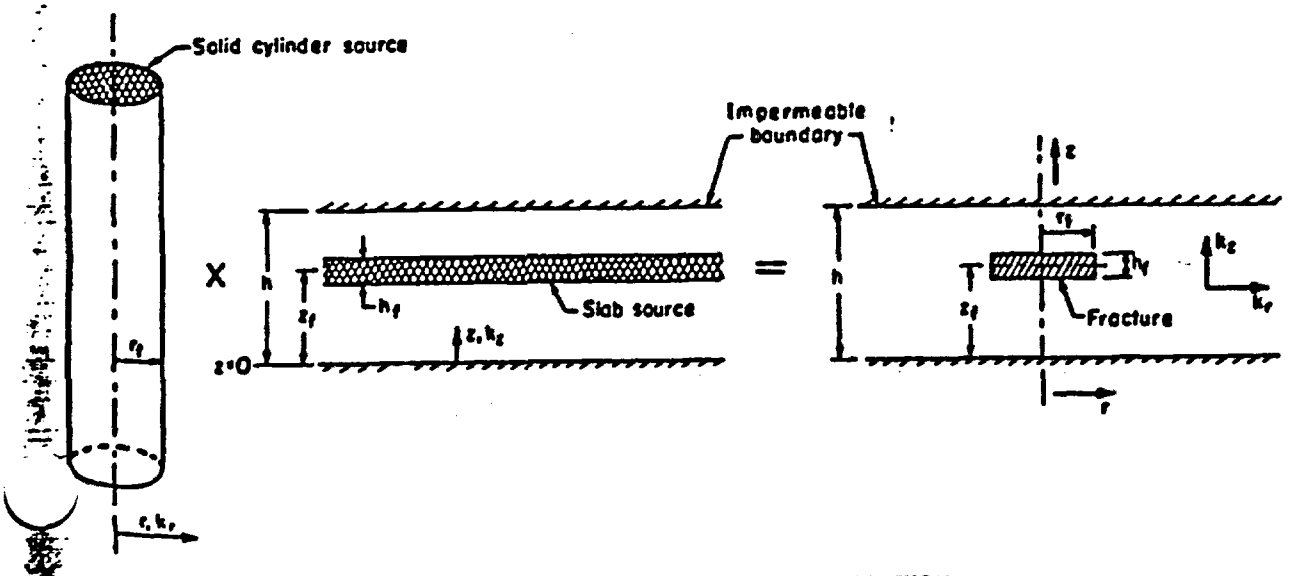

FIG. 2 — PRODUCT SOLUTION VISUALIZATION.

$$\cdot \left[1 + \frac{4h}{\pi h_f} \sum_{n=1}^{\infty} \frac{\exp\left(-\frac{n^2\pi^2 t'_D}{h_D^2}\right)}{n} \cdot \sin n\pi \frac{h_f}{2h} \cos n\pi \frac{z_f}{h} \cos n\pi \frac{z}{h} \right] dt'_D \quad \ldots\ldots (9)$$

Eqs. 5 and 7 define r_D and t_D in terms of r_f, rather than in terms of the conventional r_w. In Eq. 9, the ratios h_f/h, z_f/h, and z/h, already dimensionless, have not been modified; t'_D and r'_D refer to the dimensionless variables of integration. h_D represents the dimensionless reservoir thickness and is derived from Eq. 6.

The study of the behavior of Eq. 9 is simplified by the introduction of the following functions, which have been described elsewhere.[6-9]

1. The P function:

$$P(r_D, t_D) = \frac{\exp\left(-\frac{r_D^2}{4t_D}\right)}{2t_D} \int_0^1 I_0\left(\frac{r_D r'_D}{2t_D}\right) \exp\left(-\frac{r'^2_D}{4t_D}\right) r'_D \, dr'_D, \quad \ldots\ldots (10)$$

is proportional to the instantaneous source function for a solid cylinder source in an infinite reservoir. The pressure distribution created by a continuous solid cylinder source is given by[7,9]

TABLE 1 — $p_D(r_D, t_D)$ VALUES FOR A CONTINUOUS UNIFORM FLUX SOLID CYLINDER SOURCE IN AN INFINITE POROUS MEDIUM[9]

t_D \ r_D	0	0.25	0.50	0.75	1	2	3	5	10
1.10^{-2}	0.0200	0.0200	0.0200	0.0197	0.0096				
2.10^{-2}	0.0400	0.0400	0.0399	0.0380	0.0189				
3.10^{-2}	0.0600	0.0600	0.0595	0.0547	0.0280				
4.10^{-2}	0.0800	0.0799	0.0787	0.0705	0.0370				
6.10^{-2}	0.1197	0.1191	0.1151	0.0996	0.0544				
8.10^{-2}	0.1585	0.1570	0.1494	0.1265	0.0714	0.0001			
1.10^{-1}	0.1960	0.1932	0.1817	0.1517	0.0879	0.0004			
2.10^{-1}	0.3586	0.3494	0.3194	0.2614	0.1651	0.0047			
3.10^{-1}	0.4855	0.4718	0.4289	0.3521	0.2343	0.0141	0.0003		
4.10^{-1}	0.5879	0.5711	0.5194	0.4296	0.2965	0.0272	0.0015		
6.10^{-1}	0.7466	0.7259	0.6633	0.5568	0.4036	0.0597	0.0055		
8.10^{-1}	0.8673	0.8444	0.7753	0.6587	0.4930	0.0958	0.0134		
1	0.9646	0.9402	0.8668	0.7437	0.5693	0.1327	0.0243	0.0003	
2	1.2820	1.2544	1.1715	1.0332	0.8389	0.2986	0.0994	0.0070	
3	1.4747	1.4459	1.3596	1.2156	1.0136	0.4292	0.1799	0.0244	
4	1.6135	1.5841	1.4960	1.3491	1.1432	0.5342	0.2541	0.0489	0.0002
6	1.8111	1.7811	1.6912	1.5412	1.3312	0.6960	0.3802	0.1060	0.0017
8	1.9524	1.9221	1.8312	1.6797	1.4675	0.8186	0.4828	0.1642	0.0059
10	2.0624	2.0319	1.9405	1.7880	1.5745	0.9171	0.5686	0.2195	0.0130
20	2.4062	2.3753	2.2827	2.1283	1.9122	1.2372	0.8614	0.4401	0.0741
30	2.6080	2.5769	2.4839	2.3289	2.1119	1.4308	1.0454	0.5964	0.1471
40	2.7513	2.7202	2.6270	2.4717	2.2542	1.5701	1.1798	0.7161	0.2169
60	2.9535	2.9224	2.8290	2.6733	2.4554	1.7682	1.3729	0.8940	0.3383
80	3.0971	3.0659	2.9724	2.8166	2.5985	1.9097	1.5119	1.0252	0.4383
1.10^{2}	3.2085	3.1773	3.0838	2.9277	2.7096	2.0199	1.6205	1.1291	0.5226
2.10^{2}	3.5551	3.5238	3.3402	3.2741	3.0555	2.3640	1.9615	1.4604	0.8121
3.10^{2}	3.7578	3.7265	3.6328	3.4767	3.2580	2.5658	2.1623	1.6580	0.9950
4.10^{2}	3.9016	3.8703	3.7766	3.6204	3.4017	2.7092	2.3052	1.7992	1.1288
6.10^{2}	4.1043	4.0730	3.9793	3.8230	3.6043	2.9115	2.5070	1.9993	1.3213
8.10^{2}	4.2481	4.2168	4.1231	3.9668	3.7481	3.0551	2.6503	2.1418	1.4600
1.10^{3}	4.3596	4.3284	4.2346	4.0784	3.8596	3.1666	2.7616	2.2526	1.5685

$$p_D(r_D,t_D) = 2\int_0^{t_D} P(r_D,t'_D)\,dt'_D \quad \ldots\ldots (11)$$

and is listed in Table 1.

2. The Z function[6]

$$Z\left(\frac{t_D}{h_D^2}, \frac{h_f}{2h}, \frac{z_f}{h}, \frac{z}{h}\right) = 1 + \frac{4h}{\pi h_f}\sum_{n=1}^{\infty} \frac{\exp\left(-\frac{n^2\pi^2 t_D}{h_D^2}\right)}{n} \sin n\pi\frac{h_f}{2h} \cos n\pi\frac{z_f}{h} \cos n\pi\frac{z}{h} \quad \ldots (12)$$

is proportional to the instantaneous source function for an infinite horizontal slab source in an infinite horizontal slab reservoir[6] with impermeable boundaries. It accounts for the partial penetration of the solid cylinder source in the reservoir.

The pressure-drop distribution created by the horizontal fracture is thus given by

$$p_D(r_D,z_D,t_D) = 2\int_0^{t_D} P(r_D,t'_D)\, Z\left(\frac{t'_D}{h_D^2}, \frac{h_f}{2h}, \frac{z_f}{h}, \frac{z}{h}\right) dt'_D \quad \ldots\ldots (13)$$

Any variation in the fracture radius, r_f, affects only the P function; whereas any variation in the fracture thickness, h_f, affects only the Z function. Two important applications follow immediately. First, the pressure distribution created by a plane (zero thickness) horizontal fracture is obtained from Eq. 13 by taking the limit of the Z function as $h_f \to 0$. The result is[6]

$$p_D(r_D,z_D,t_D) = 2\int_0^{t_D} P(r_D,t'_D)\, Z\left(\frac{t'_D}{h_D^2}, 0, \frac{z_f}{h}, \frac{z}{h}\right) dt'_D, \quad \ldots\ldots (14)$$

where

$$Z\left(\frac{t_D}{h_D^2}, 0, \frac{z_f}{h}, \frac{z}{h}\right) = 1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{n^2\pi^2 t_D}{h_D^2}\right) \cos n\pi\frac{z_f}{h} \cos n\pi\frac{z}{h} \quad \ldots\ldots (15)$$

Second, the pressure distribution created by a line-source well with partial penetration or limited entry may be obtained from Eq. 13 by taking the limit of the P function as $r_f \to 0$. The result is[7]

$$p_D(r_D,z_D,t_D) = \int_0^{t_D} \frac{e^{-\frac{r_D^2}{4t_D}}}{2t_D}\, Z\left(\frac{t'_D}{h_D^2}, \frac{h_f}{2h}, \frac{z_f}{h}, \frac{z}{h}\right) dt'_D, \quad \ldots\ldots (16)$$

with

$$r \geq r_f \quad \ldots\ldots (17)$$

This can also be written as

$$p_D(r_D,z_D,t_D) = -\frac{1}{2}\,\mathrm{Ei}\left(-\frac{r_D^2}{4t_D}\right) + \frac{4h}{\pi h_f}\sum_{n=1}^{\infty} \frac{1}{n}\sin n\pi\frac{h_f}{2h}\cos n\pi\frac{z_f}{h}\cos n\pi\frac{z}{h} \int_{\frac{r_D^2}{4t_D}}^{\infty} \exp\left[-u - \frac{\left(\frac{n\pi r_D}{h_D}\right)^2}{4u}\right]\frac{du}{u}, \quad \ldots (18)$$

which is identical with the solution derived by Hantush[10] by integrating Lord Kelvin's instantaneous point source solution. The pressure distribution is thus the same as that created by a totally penetrating line-source well plus a "skin." This result can be extended to the fracture case by writing Eq. 13 as

$$p_D(r_D,z_D,t_D) = p_D(r_D,t_D) + \sigma(r_D,z_D,t_D), \quad \ldots\ldots (19)$$

where $p_D(r_D, t_D)$, given by Eq. 11, represents the pressure-drop distribution created by an infinite

solid cylinder source. The function $\sigma(r_D, z_D, t_D)$ is given by

$$\sigma(r_D, z_D, t_D) = 2\int_0^{t_D} P(r_D, t_D') \left[Z\left(\frac{t_D'}{h_D^2}, \frac{h_f}{2h}, \frac{z_f}{h}, \frac{z}{h}\right) - 1\right] dt_D' \quad \ldots\ldots (20)$$

and will be called a "pseudo skin function" by analogy with the skin factor introduced by van Everdingen[11] and by Hurst.[12] The skin factor is defined as a constant additional pressure drop across a damaged "skin" at the wellbore. The pseudo skin function, $\sigma(r_D, z_D, t_D)$, represents an additional time-dependent pressure drop in a zone of finite radial extent, as will be proved later. The pseudo skin function becomes a skin function in the case of a well with partial penetration or limited entry. Brons and Marting[13] actually coined the term "pseudo-skin" in their study of partial penetration and limited entry. There is a difference in the use of the term "pseudo-skin" factor by Brons and Marting and the term "pseudo skin function" in this study.

LONG-TIME PRESSURE BEHAVIOR

In well-test analysis, the pressure drop at the well is generally graphed vs the logarithm of time. In the case of a slightly compressible fluid, a straight line of slope ½ per ln cycle (1.1515 per log base 10 cycle) is obtained after a certain time, which defines the beginning of the transient flow period.[1]

If we make such a graph in the case of a horizontal fracture, a curve results, the slope of which varies with time and is given by

$$\frac{\partial p_D(r_D, z_D, t_D)}{\partial(\ln t_D)} = 2t_D\, P(r_D, t_D)\, Z\left(\frac{t_D}{h_D^2}, \frac{h_f}{2h}, \frac{z_f}{h}, \frac{z}{h}\right), \quad h_f > 0 \quad \ldots\ldots (21)$$

As time increases, the Z function approaches unity whithin 1 percent, when

$$t_D \geq \frac{5}{\pi^2} h_D^2 \quad \ldots\ldots (22)$$

or, in terms of real variables,

$$\frac{k_z t}{\phi \mu c h^2} \geq \frac{5}{\pi^2} \quad \ldots\ldots (23)^6$$

and the P function is equivalent to $1/4t_D$ when

$$t_D \geq 12.5(2r_D^2 + 1) \quad \ldots\ldots (24)^6$$

for the fracture case, or when

$$t_D \geq 25\, r_D^2 \quad \ldots\ldots (25)$$

for a well with partial penetration or limited entry ($r_D \geq 1$ and r_f infinitely small).

Therefore, the slope of the pressure curve becomes equal to ½ within 1 percent as time increases, when Eqs. 22 through 25 are satisfied, and the flow becomes radial. The form of the pressure-drop function itself is obtained from Eq. 19: at long times, defined by Eqs. 24 and 25, $p_D(r_D, t_D)$ can be approximated by[6]

$$\underline{0 \leq r_D \leq 1}: \quad p_D(r_D, t_D) = \frac{1}{2}(\ln t_D + 1.80907 - r_D^2), \quad r_f > 0 \quad \ldots\ldots (26)$$

(note that "1.80907" *is* correct in Eq. 26) for a solid cylinder of finite radius (r_f large)

$$\underline{r_D \geq 1}: \quad p_D(r_D, t_D) = \frac{1}{2}\left(\ln \frac{t_D}{r_D^2} + 0.80907\right), \quad r_f \geq 0 \quad \ldots\ldots (27)$$

for a solid cylinder, and a line source well (r_f infinitely small).

The integrand in Eq. 20, which defines the pseudo skin function, σ, approaches zero as the Z function approaches unity. Therefore, the pseudo skin function becomes constant when $t_D \geq (5/\pi^2)h_D^2$. This constant, defined as the pseudo skin factor, is the maximum value of the pseudo skin function and is given by

$$\sigma\left(r_D, z_D, \frac{5h_D^2}{\pi^2}\right) = 2\int_0^{\frac{5}{\pi^2}h_D^2} P(r_D, t_D') \left[Z\left(\frac{t_D'}{h_D^2}, \frac{h_f}{2h}, \frac{z_f}{h}, \frac{z}{h}\right) - 1\right] dt_D' \quad \ldots\ldots$$

The pseudo skin factor cannot be expressed in terms of tabulated functions, except for the case of a well with partial penetration or limited entry, in which case,

$$\sigma\left(r_D, z_D, \frac{5h_D^2}{\pi^2}\right) = \frac{4h}{\pi h_f} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi \frac{h_f}{2h}$$

$$\cos n\pi \frac{z_f}{h} \cos n\pi \frac{z}{h} K_o\left(\frac{n\pi r_D}{h_D}\right), \quad r_D \geq 1 \quad \ldots\ldots (29)$$

as given in Ref. 10.

SHORT-TIME PRESSURE BEHAVIOR

The short-time behavior of the pressure distribution function is obtained by means of Eq. 13 from the short-time behavior of the Z and P functions, which have been described elsewhere.[6,8] The results are summarized in the following paragraphs.

HORIZONTAL FRACTURE

The P function becomes constant as time decreases, within 1 percent when

$$t_D \leq \frac{(1 - r_D)^2}{20}, \qquad r_D \neq 1 \quad \ldots (30)$$

$$t_D \leq 10^{-4}\pi, \qquad r_D = 1 \quad \ldots (31)$$

the constant is unity for $0 \leq r_D < 1$; one half for $r_D = 1$; and zero for $r_D > 1$.

Therefore, at very early times, flow occurs only in the $0 \leq r_D \leq 1$ region. The pressure-drop function, Eq. 13, then becomes

$$p_D(0 \leq r_D < 1, z_D, t_D) =$$

$$2\int_0^{t_D} Z\left(\frac{t_D'}{h_D^2}, \frac{h_f}{h}, \frac{z_f}{h}, \frac{z}{h}\right) dt_D', \quad \ldots (32)$$

which is independent of r_D and indicates vertical linear flow into the fracture. The early-time behavior of Eq. 32 depends upon the form of the Z function, which in turn depends upon h_f.

Horizontal fracture of Finite Thickness ($h_f \neq 0$)

Above or below the fracture, at very early times, the pressure-drop distribution function is equivalent to[7]

$$p_D(0 \leq r_D < 1, z_D, t_D) = \frac{h}{h_f}\left[\left(t_D + \frac{\delta_D^2}{2}\right) \operatorname{erfc} \frac{\delta_D}{2\sqrt{t_D}} - \delta_D\sqrt{\frac{t_D}{\pi}} \exp\left(-\frac{\delta_D^2}{4t_D}\right)\right] \quad \ldots\ldots (33)$$

where δ_D represents the dimensionless vertical distance from the pressure point to the closest (upper or lower) horizontal face of the fracture. Eq. 33 represents vertical linear flow into the fracture with a storage effect caused by the finite thickness of the fracture. The storage constant is equal to h_{f_D}. The pressure-drop function becomes negligible when

$$t_D \leq \frac{\delta_D^2}{20}; \quad \ldots\ldots (34)$$

or, in terms of real variables,

$$\frac{k_z t}{\phi\mu c\, \delta^2} \leq \frac{1}{20} \quad \ldots\ldots (35)$$

Inside the fracture, the Z function becomes constant at very early times and the pressure-drop function can be approximated within 1 percent by

$$p_D\left(0 \leq r_D < 1;\ z_f - \frac{h_f}{2} < z < z_f + \frac{h_f}{2};\right.$$

$$t_D) = 2\frac{h}{h_f} t_D \quad \ldots\ldots (36)$$

when Eq. 34 is satisfied.

Substitution of real variables into Eq. 36 yields

$$p_i - p(r, z, t) = \frac{q_f t}{\pi r_f^2 h_f \phi c}, \quad \ldots (37)$$

which is a volumetric balance assuming all production originates within the volume of the fracture. The rate of decline is given by

$$\frac{dp}{dt} = -\frac{q_f}{\pi r_f^2 h_f \phi c} \quad \ldots\ldots (38)$$

and is constant and inversely proportional to the fluid-filled pore volume of the fracture.

On the horizontal faces of the fracture ($\delta = 0$), the pressure drop is one-half that within the fracture at very early times, when

$$\frac{k_z t}{\phi\mu c h_f^2} \leq \frac{1}{20} \quad \ldots\ldots (39)$$

Therefore, at very early times, the only flow is within the fracture, and is of a storage type. A straight line of unit slope is obtained when the pressure drop is plotted vs time on log-log coordinates. As time increases, the linear vertical flow into the fracture becomes important, and a straight line of slope ½ is obtained on log-log coordinates. The length of this last straight line is limited by the distance from the pressure point to the closest upper or lower reservoir boundary

(boundary effects), and by the distance from the pressure point to r_f, because the P function does not remain constant when Eq. 30 is no longer satisfied.

Plane Horizontal Fracture ($h_f = 0$)

At very early times, the pressure-drop function is given by[7]

$$p_D(0 \leq r_D < 1, z_D, t_D) = 2h_D\sqrt{\frac{t_D}{\pi}}\, e^{-\frac{(z-z_f)_D^2}{4t_D}} - h_D\left|z-z_f\right|_D \operatorname{erf}\frac{\left|z-z_f\right|_D}{2\sqrt{t_D}}, \quad \ldots\ldots (40)$$

which represents linear vertical flow without storage into the fracture, and p_D becomes negligible when

$$\frac{k_z t}{\phi\mu c(z-z_f)^2} \leq \frac{1}{20}, \qquad z \neq z_f. \quad (41)$$

Graphing the pressure drop vs time on log-log coordinates yields a straight line of slope ½ whose length is limited as in the case of a finite-thickness fracture.

WELL WITH PARTIAL PENETRATION OR LIMITED ENTRY

In this case, the P function is no longer constant at early times, and the early-time behavior of the pressure is solely determined by that of the Z function. Above or below the well opening, the Z function is negligible when Eq. 35 is satisfied and there is no flow in that region of the reservoir.

At the level of the well opening, the pressure-drop function can be approximated by

$$p_D\left(r_D, z_f - \frac{h_f}{2} < z < z_f + \frac{h_f}{2}, t_D\right) = -\frac{1}{2}\frac{h}{h_f} E_i\left(-\frac{r_D^2}{4t_D}\right) \ldots\ldots\ldots (42)$$

when Eq. 34 is satisfied.

In terms of real variables, Eq. 42 can be written as

$$\frac{2\pi k_r h_f}{q_f \mu}\Delta p = -\frac{1}{2} Ei\left(-\frac{r_D^2}{4t_D}\right), \ldots (43)$$

which indicates that the well behaves as if the total sand thickness were equal to h_f.[13]

At the top or bottom of the well opening, the pressure drop is one-half that given by Eq. 1[illegible] when Eq. 39 is satisfied.

RADIUS OF INFLUENCE

The pseudo skin factor, $\sigma(r_D, z_D, 5h_D^2/\pi^2)$, defined by Eq. 28 is independent of time, but does depend upon the position of the pressure point. It is possible to define a radius of influence, r_i, so that $\sigma(r_D, z_D, 5h_D^2/\pi^2)$ is zero for $r > r_i$. That is, at points beyond the radius of influence, the pseudo skin factor due to the fracture (or the skin factor due to the limited entry or to the partial penetration) is essentially zero, and the effect of the fracture (or of the restricted entry or partial penetration) has disappeared. The pressure drop is the same as that created by a solid cylinder source of radius [illegible] or even a line source.

In the fracture case (r_f large), the P function is negligible at all times at distances so that

$$(r_D - 1)^2 > 20\, t_D \ldots\ldots\ldots (44)$$

from Eq. 30.

If Eq. 22 is also satisfied, that is, if

$$t_D \geq \frac{5}{\pi^2} h_D^2 \ldots\ldots\ldots\ldots (22)$$

the pseudo skin factor $\sigma(r_D, z_D, 5h_D^2/\pi^2)$ is zero. Because $\sigma(r_D, z_D, 5h_D^2/\pi^2)$ is the maximum of the pseudo skin function (which is always positive), the pseudo skin function itself is zero, and the pressure drop at all times is the same as that created by a solid cylinder.

The radius of influence is obtained by combining Eqs. 44 and 22, which yields

$$r_{iD} = 1 + \frac{10}{\pi} h_D, \ldots\ldots\ldots (4[illegible]$$

or, in terms of real variables,

$$r_i = r_f + \frac{10h}{\pi}\sqrt{\frac{k_r}{k_z}} \ldots\ldots (4[illegible]$$

Therefore, the distance $(r_i - r_f)$ depends only on the reservoir thickness, h.

This result is valid if $r_f \to 0$ for a well with partial penetration or restricted entry for which

$$r_i = \frac{10}{\pi} h\sqrt{\frac{k_r}{k_z}} \ldots\ldots\ldots (4[illegible]$$

This result can be verified from the analytical expression of the skin factor equation (Eq. [illegible]) which is negligible when the Bessel function [illegible] negligible.

WELLBORE PRESSURE FOR HORIZONTALLY FRACTURED WELLS

Clearly, Eq. 9 does not yield a uniform potential [illegible]

within the fracture (for $0 \le r_D \le 1$ and $z_f - h_f/2 \le z \le z_f + h_f/2$), except at very early times. However, the potential variation is small, and the fracture can be considered as similar to one having a very high permeability.

An *infinite conductivity* fracture can be obtained as indicated in Ref. 6, by dividing the fracture into a number of elements, each with uniform flux, and obtained as the intersection of an elementary thick-wall cylinder with an elementary slab source. The condition of infinite conductivity and the flux distribution are obtained from the condition of equal pressure drops in the source elements, and that of constant production rate at all times.

The solution for a *uniform flux* fracture presented in this paper is useful, however. In the reservoir, *away from the fracture*, there is little difference between the behavior of an infinite conductivity fracture and that of a uniform flux fracture. Furthermore, our solution appears to match the behavior of a well intersecting a *natural* horizontal fracture *when computed at the center of the source* ($r = 0$, $z = z_f$). In that case, the analytical expression for pressure drop is much simplified.

Along the fracture axis ($r = 0$), the P function can be expressed in terms of tabulated functions, and a simple analytic expression can be obtained for the pseudo skin factor as a function of the following reservoir parameters:

$$P(0,t_D) = 1 - e^{-\frac{1}{4t_D}} \quad \ldots\ldots (48)$$

$$P_D(0,t_D) = -\frac{1}{2}\,\mathrm{Ei}\left(-\frac{1}{4t_D}\right) + 2t_D\left(1 - e^{-\frac{1}{4t_D}}\right) \quad \ldots\ldots (49)$$

$$\sigma(0,z_D,t_D) = \frac{8h}{h_f}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\frac{h_D^2}{\pi^2 n^2}\left(1 - e^{-\frac{n^2\pi^2 t_D}{h_D^2}}\right) - \frac{h_D}{\pi}K_1\left(\frac{n\pi}{h_D}\right) + \int_0^{\frac{1}{4t_D}} \exp\left[-u - \frac{\left(\frac{n\pi}{h_D}\right)^2}{4u}\right]\frac{du}{u}\right\}\sin n\pi\frac{h_f}{2h}\cos n\pi\frac{z_f}{h}\cos n\pi\frac{z}{h},$$

$$h_f \neq 0 \quad \ldots\ldots (50)$$

The transient pressure along the *fracture axis*, obtained when Eqs. 21 and 23 are satisfied, is equal to

$$p_{wD} = \frac{1}{2}\left[\ln t_D + 1.80907 + 2\sigma\left(0, z_D, \frac{5t_D^2}{\pi^2}\right)\right], \quad \ldots\ldots (51)$$

(note "1.80907" is correct)

where z_D is such that

$$z_f - \frac{h_f}{2} < z < z_f + \frac{h_f}{2},$$

and the corresponding analytical expression of the pseudo skin factor at the wellbore for a horizontal fracture of finite thickness is

$$\sigma = h_D^2\left[\frac{2}{3} - \frac{\left(\frac{z_f}{h} - \frac{z}{h}\right)^2}{\frac{h_f}{h}} - \left(\frac{z_f}{h} + \frac{z}{h}\right) - \frac{1}{4}\frac{h_f}{h} + \frac{1}{12}\left(\frac{h_f}{h}\right)^2 + \left(\frac{z_f}{h}\right)^2 + \left(\frac{z}{h}\right)^2\right] - \frac{8h_D}{\pi^2}\frac{h}{h_f}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin n\pi\frac{h_f}{2h}\cos n\pi\frac{z_f}{h}\cos n\pi\frac{z}{h}\,K_1\left(\frac{n\pi}{h_D}\right) \quad \ldots\ldots (52)$$

The pseudo skin factor, σ, is thus a function of h_D, h_f/h, z_f/h, and z/h only. The Bessel function K_1 decreases rapidly when the argument increases, and the infinite sum can be neglected for $h_D < 1$.

Taking the limit of Eq. 52 as $h_f \to 0$ yields *the pseudo skin factor at the wellbore* for a plane horizontal natural fracture:

$$\sigma = \frac{2}{3}h_D^2\left[1 - 3\frac{z_f}{h}\left(1 - \frac{z_f}{h}\right)\right] - \frac{4}{\pi}h_D\sum_{n=1}^{\infty}\frac{1}{n}\cos^2 n\pi\frac{z_f}{h}K_1\left(\frac{n\pi}{h_D}\right). \quad \ldots\ldots (53)$$

σ is a function of h_D and z_f/h only, which is symmetrical with respect to $z_f/h = ½$. The pseudo skin factor is graphed in Fig. 3 vs h_D for different values of the parameter z_f/h.

Only one other expression appears in the literature for the transient pressure drop created by a plane horizontal uniform flux fracture. It has been derived by Warren[4] from a study of bottom-water-drive reservoirs by Coats[5] and represents an average value over the area of the fracture, when the fracture is at the center, top, or bottom of the formation. Defining the average pressure in the fracture as

$$p_{av_D} = \frac{1}{\pi} \int_0^1 p_D(r_D, z_D, t_D)\ 2\pi\ r_D\ dr_D\ , \qquad (54)$$

and the average pseudo skin factor as

$$\sigma_{av} = \frac{1}{\pi} \int_0^1 \sigma\left(r_D,\ z_{fD},\ \frac{5h_D^2}{\pi^2}\right)\ 2\pi\ r_D\ dr_D\ , \qquad (55)$$

it follows from Eq. 26 that the average transient pressure in the fracture is given by

$$p_{av_D} = \frac{1}{2}\ (\ln\ t_D + 0.80907) + \frac{1}{4} + \sigma_{av}\ . \qquad (56)$$

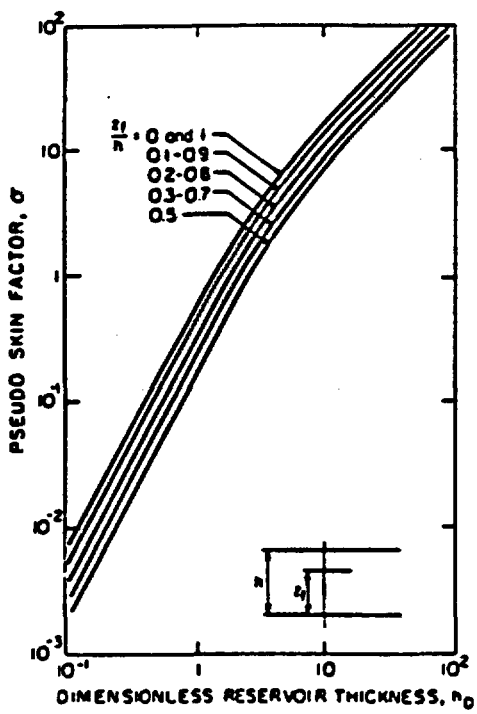

FIG. 3 — PSEUDO SKIN FACTOR, σ, AT WELLBORE (PLANE HORIZONTAL FRACTURE).

Warren's results[4] can be written in a similar way as

$$(p_{av})_W = \frac{1}{2}\ (\ln\ t_D + 0.80907) + \frac{1}{4} + (\sigma_{av})_W\ , \qquad (57)$$

where

$$(\sigma_{av})_W = \begin{cases} h_D\ A\left(\frac{h_D}{2}\right) - 0.6545, & \text{if } z_f = \frac{h}{2} \\ 2h_D\ A(h_D) - 0.6545, & \text{if } z_f = 0 \\ \text{or } z_f = h \end{cases} \qquad (58)$$

in which $A(h_D)$ is a geometrical constant given in Table 2 of Coats' paper. A comparison has been made in Fig. 4 between the numerical values of the average pseudo skin factors from this study and those in Ref. 4, for different values of the dimensionless reservoir thickness h_D, in the case of a fracture at the center of the formation; there is excellent agreement.

For the steady-state case, values of the usual skin factor have been presented by Hartsock and Warren[3] as a function of the following dimensionless parameters:

$$\frac{h}{r_w}\sqrt{\frac{k_r}{k_z}}\ ;\quad \frac{r_f}{r_w}\ ;\quad (kh)_f / r_w\sqrt{k_r k_z}\ .$$

$(kh)_f$ being the fracture flow capacity. Hartsock and Warren mentioned that there was an infinite number of combinations of these parameters that would yield the same value for the skin factor.

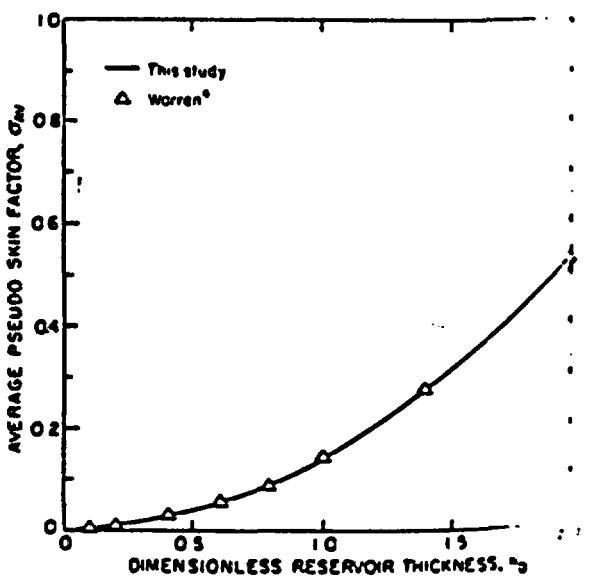

FIG. 4 — AVERAGE PSEUDO SKIN FACTOR, σ_{av}, VS h_D FOR A SINGLE PLANE HORIZONTAL FRACTURE AT CENTER OF FORMATION.

Actually, this should not be so, and Hartsock and Warren's calculated skin effect should be a unique function of h_D and the fracture/formation capacity ratio, $(kh)_f/(kh)_r$, because the well has apparently has no significant influence on the pressure distribution. As a check, Hartsock and Warren's values of skin factor, $s_{H,W}$, were used to compute a pseudo skin factor, $\sigma_{H,W}$, as defined in this study:

$$\sigma_{H,W} = s_{H,W} + \ln \frac{r_f}{r_w} - \frac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$\sigma_{H,W}$ has been plotted in Fig. 5 as a function of h_D, for two values of the fracture-formation flow capacity ratio — 10 and infinity. Except for some scattering of the data, the relationship for the pseudo skin factor is indeed unique for each $(kh)_f/(kh)_r$ value.

These relationships, as obtained from Hartsock and Warren's paper, are not quite correct, however. The pseudo skin factor, σ, from Eq. 53 has also been plotted in Fig. 5, for a fracture at the center of the formation ($z_f = h/2$). It is difficult to reach a conclusion from the comparison of the σ and $\sigma_{H,W}$ curves for h_D values less than 2 because of scattering of the data. The fact that there is no agreement for h_D values greater than 2 may result from Hartsock and Warren's assumption of a radius of influence equal to four times the fracture radius, $r_e = 4r_f$, according to the present study (Eq. 45), implies implicitly that $h_D = 0.3\pi$. Because the value of the radius of influence is determined by the precision required for the numerical computations, one can conclude that Hartsock and Warren's results are valid for h_D values between unity and 2. In particular, it is apparent that Hartsock and Warren's data leading to negative $\sigma_{H,W}$ values in Fig. 5 are not accurate, the pseudo skin function being essentially positive.

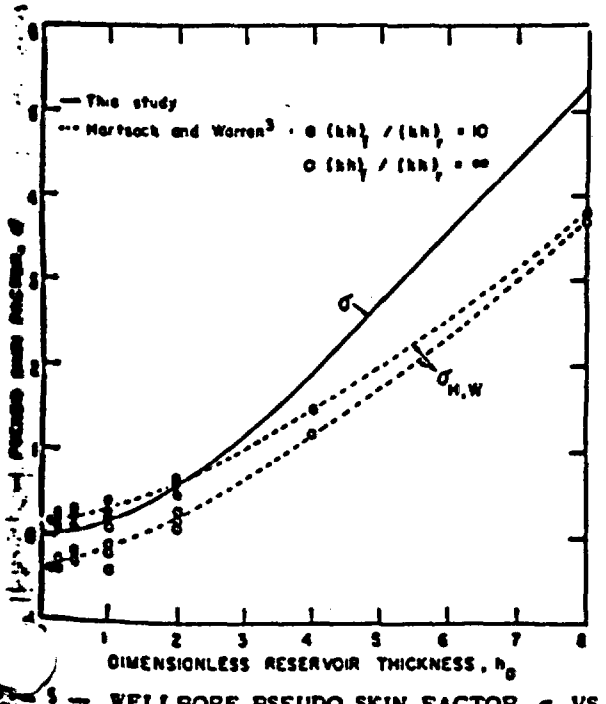

FIG. 5 — WELLBORE PSEUDO SKIN FACTOR, σ, VS h_D FOR A SINGLE PLANE HORIZONTAL FRACTURE AT CENTER OF FORMATION.

The dimensionless wellbore pressure for a plane, horizontal, uniform flux fracture at the center of the formation, given as a function of dimensionless time for an extensive range of values of the dimensionless formation thickness h_D in Table 2, has been graphed in Fig. 6 on log-log coordinates to show the early-time behavior, and in Fig. 7 on semilog coordinates to show the transient flow period.

Variation of the pressure in the fracture for a given value of dimensionless formation thickness is illustrated in Figs. 8 and 9. As can be seen, pressure is essentially constant for locations from $r = 0$ to $r = 0.25\, r_f$.

CONCLUSIONS

An analytical study of the pressure distribution created by a single *uniform flux* horizontal fracture in an infinite reservoir with impermeable upper and lower boundaries has shown that there exist four

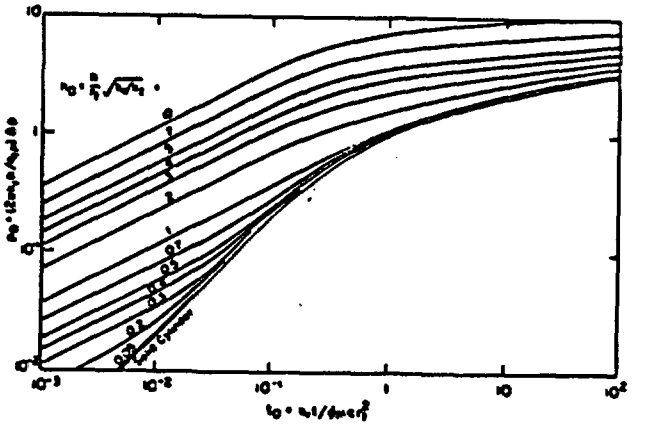

FIG. 6 — DIMENSIONLESS PRESSURE IN THE FRACTURE VS DIMENSIONLESS TIME FOR A SINGLE PLANE UNIFORM FLUX HORIZONTAL FRACTURE AT CENTER OF FORMATION ($r_D = 0$).

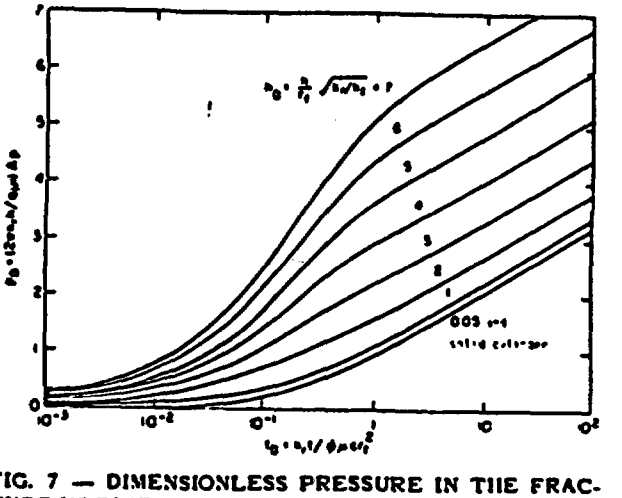

FIG. 7 — DIMENSIONLESS PRESSURE IN THE FRACTURE VS DIMENSIONLESS TIME FOR A SINGLE PLANE UNIFORM FLUX HORIZONTAL FRACTURE AT CENTER OF FORMATION ($r_D = 0$).

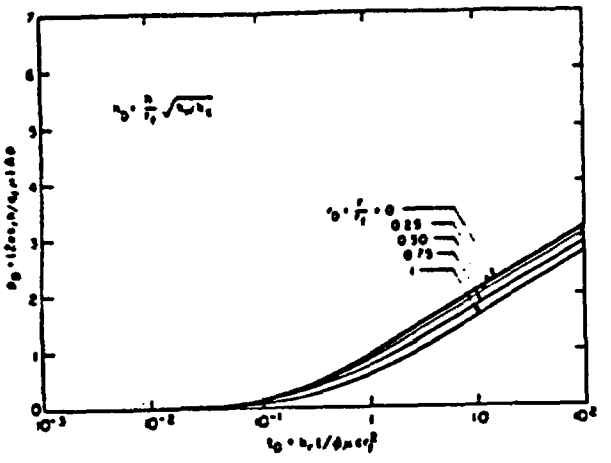

FIG. 8 — DIMENSIONLESS PRESSURE IN THE FRACTURE VS DIMENSIONLESS TIME FOR A SINGLE PLANE UNIFORM FLUX HORIZONTAL FRACTURE AT CENTER OF FORMATION (h_D = 0.05).

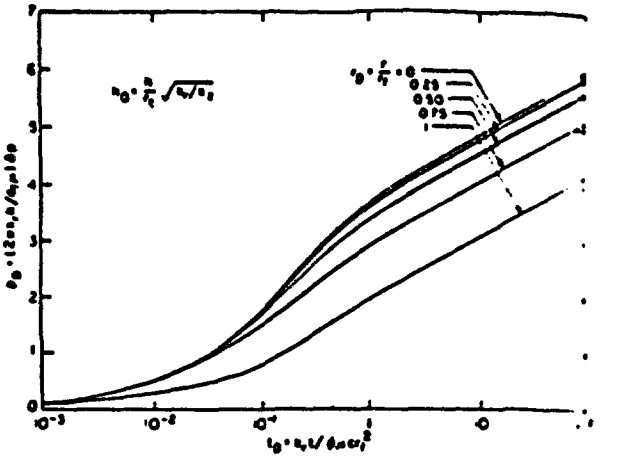

FIG. 9 — DIMENSIONLESS PRESSURE IN THE FRACTURE VS DIMENSIONLESS TIME FOR A SINGLE PLANE UNIFORM FLUX HORIZONTAL FRACTURE AT CENTER OF FORMATION (h_D = 5).

different flow periods.

A storage-type flow occurs first, the duration of which is limited by the fracture thickness and by the shortest vertical distance from the point at which the pressure is measured to the fracture boundaries.

A vertical linear-type flow from the reservoir into the fracture follows, and the duration of this period is limited by the shortest vertical distance from the fracture to the reservoir boundaries, and by the radial distance from the point to the fracture outer boundary.

After a period of transition, the transient pseudoradial flow period starts at a time depending upon the radial distance from the point of measurement to the wellbore axis, and upon the reservoir thickness. The flow is radial and the pressure is the same as that created by a line-source well with a constant skin. The skin effect is independent of time, but does depend upon the position of the pressure point, and is negligible at distances that are greater than three times the reservoir thickness.

If the fracture thickness is small compared with

TABLE 2 — $p_D(r_D, z_D, t_D)$ VALUES FOR A SINGLE PLANE UNIFORM FLUX HORIZONTAL FRACTURE AT CENTER OF FORMATION FOR VARIOUS DIMENSIONLESS RESERVOIR THICKNESSES, h_D (r_D = 0; z_D = 0.5)

t_D \ h_D	0.05	0.2	0.3	0.4	0.5	0.7	1	2	3	4	5	6	7	10
1.10^{-2}	$[p_D(0,t_D)]$* $=0.0001]$	$[p_D(0,t_D)$ $=0.0047]$	0.0349	0.0453	0.0564	0.0798	0.1129	0.2237	0.3386	0.4514	0.5643	0.6771	0.7906	[illegible]
2.10^{-2}			$[p_D(0,t_D)$ $=0.0156]$	0.0664	0.0806	0.1118	0.1596	0.3192	0.4788	0.6384	0.7980	0.9576	1.1172	[illegible]
3.10^{-2}				$[p_D(0,t_D)$ $=0.0287]$	0.1015	0.1373	0.1953	0.3909	0.5864	0.7819	0.9774	1.1728	1.3683	[illegible]
4.10^{-2}					0.1216	0.1597	0.2257	0.4514	0.6770	0.9027	1.1284	1.3541	1.5798	[illegible]
6.10^{-2}					$[p_D(0,t_D)$ $=0.0417]$	0.2018	0.2768	0.5521	0.8281	1.1042	1.3802	1.6562	1.9323	[illegible]
8.10^{-2}						0.2410	0.3207	0.6352	0.9528	1.2704	1.5879	1.9055	2.2231	[illegible]
1.10^{-1}						$[p_D(0,t_D)$ $=0.0517]$	0.3604	0.7016	1.0587	1.4112	1.7644	2.1175	2.4704	[illegible]
2.10^{-1}							$[p_D(0,t_D)$ $=0.1661]$	0.9488	1.4228	1.8960	2.3700	2.8439	3.3179	[illegible]
3.10^{-1}								1.0986	1.6488	2.1851	2.7313	3.2775	3.8238	[illegible]
4.10^{-1}								1.2079	1.7866	2.3614	2.9767	3.5720	4.1674	[illegible]
6.10^{-1}								1.3693	1.9826	2.6379	3.2944	3.9557	4.6169	[illegible]
8.10^{-1}								1.4903	2.1151	2.8021	3.5010	4.2012	4.9014	[illegible]
1								$[p_D(0,t_D)$ $=0.6222]$	2.2164	2.9210	3.6465	4.3754	5.1044	[illegible]
2									$[p_D(0,t_D)$ $=1.2546]$	3.2616	4.0356	4.8115	5.6347	[illegible]
3										3.6548	4.2370	5.0123	5.8216	[illegible]
4										$[p_D(0,t_D)$ $=1.9602]$	4.3771	5.1945	6.0175	[illegible]
6											$[p_D(0,t_D)$ $=2.7638]$	5.3943	6.2436	[illegible]
8												5.5377	6.3658	[illegible]
10												$[p_D(0,t_D)$ $=3.5834]$	6.4960	[illegible]
20													$[p_D(0,t_D)$ $=6.5117]$	[illegible]
30														

*$p_D(0,t_D)$ is given in Table 1.

formation thickness, the fracture behaves like a single-plane horizontal fracture; the storage-type flow no longer exists, and only the last three flow periods are left,[15] with the same time limits.

The same mathematical model can be applied to derive the pressure distribution created by a well with restricted flow entry or created by a well that only partially penetrates the formation. Again, three different flow periods can be characterized: (1) an early radial flow into the open zone, the duration of which is limited by the height of the opening, and by the altitude of the point of measurement in the open interval; (2) a transition period; and (3) a transient radial flow period as in the fracture case. These results are summarized in Fig. 10.

The time at which the transient radial flow period starts is the same in all cases, but the skin is different in each case. Analysis of pressure buildup data can thus be made in a conventional manner if the straight line has been obtained,[1] or — as recommended by Ramey[14] — if early time data are available.

Finally, we should consider whether important differences exist between the behavior of wells with horizontal fractures and that of wells with vertical fractures. Russell and Truitt[16] in 1964 presented results for a well containing a single vertical fracture in a closed square reservoir. Comparison of log-log graphs of p_D vs t_D for the two types of fractures discloses both similarities and differences. Both possess an early period of linear flow from the fracture surface. This is characterized by a slope of ½ on log-log coordinates (for examples, see our Fig. 6, and Fig. 5 in Ref. 10). But the behavior of the horizontal-fracture case differs from that of the vertical-fracture case for small values of h_D, say less than 0.7. There is an increase in slope from ½ toward unity that has no counterpart for a vertical-fracture case. Because it appears that h_D would be less than 0.7 for practical cases, it should be possible to identify horizontal fractures from well-test data.

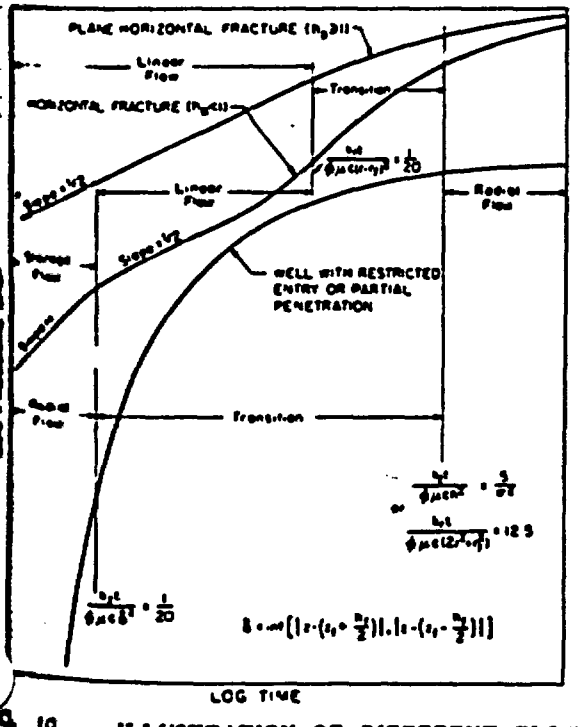

FIG. 10 — ILLUSTRATION OF DIFFERENT FLOW PERIODS FOR HORIZONTAL FRACTURES, LIMITED FLOW ENTRY, AND PARTIAL PENETRATION.

NOMENCLATURE

The Darcy system of units[1] has been used.

c = compressibility
h = formation thickness
h_f = fracture thickness
k = formation permeability
$(kh)_f$ = flow capacity of fracture
$(kh)_r$ = radial flow capacity of formation
Δp = pressure drop below initial pressure
p_i = initial reservoir pressure
p_{wD} = dimensionless wellbore pressure drop
p_D = dimensionless pressure drop
q = withdrawal rate per unit volume of source
q_f = production rate of fracture
r_D = dimensionless radius
r_w = wellbore radius
s = skin factor
t = time of flowing
t_D = dimensionless time of flowing based on r_f
z = vertical distance to the reservoir lower boundary
δ = vertical distance from the pressure point to the closest horizontal face of the fracture
η = diffusivity constant
μ = viscosity
σ = pseudo skin factor
ϕ = porosity

SUBSCRIPTS

av = areal average
D = dimensionless
f = fracture
i = initial, influence
w = well

SPECIAL FUNCTIONS

Error function.

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

Complementary error function,

$$\mathrm{erfc}\,(x) = 1 - \mathrm{erf}\,(x)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2}\, du$$

Exponential integral,

$$-\mathrm{Ei}\,(-x) = \int_x^{\infty} \frac{e^{-u}}{u}\, du$$

$I_0(x)$ = modified Bessel function of the first kind of order 0

$K_0(x)$ = modified Bessel function of the second kind of order 0

$K_1(x)$ = modified Bessel function of the second kind of order 1

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