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DATE OF MEETING
05/08/2000

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Docket Number(s)	**PROJECT NO. 669**
Plant/Facility Name	**EPRI**
TAC Number(s) (if available)	
Reference Meeting Notice	**4/27/00, CORRECTED ISSUED 5/2/00**
Purpose of Meeting (copy from meeting notice)	**TO DISCUSS THE REVIEW OF RETRAN-3D**

NAME OF PERSON WHO ISSUED MEETING NOTICE	TITLE
L. N. OLSHAN	**PROJECT MANAGER**

OFFICE
NRR

DIVISION
DLPM

BRANCH
PD II-1

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Agenda

Purpose of Meeting

Obtain a detailed list of the specific open issues / unfavorable review comments from the NRC review of RETRAN-3D. This list will be used to focus on closure of open issues and lead to near-term conclusion of the NRC review.

Discussion of Review Status and Plan for Closure

1. RETRAN Maintenance Group comments on the review process
    - Our approach was based on past experiences with submitting the EPRI RETRAN and VIPRE codes and licensee topical reports for NRC review
    - RETRAN-3D is viewed as a better code that will improve safety
    - We admit to the two errors and documentation problems that were discovered.
    - We have responded to the RAIs and have provided what we thought was requested.
    - We do not have a good idea of the specifics of the remaining issues. That is why we asked for this meeting.
2. Review responses to RAI #1 dated 5/21/99
3. Review responses to RAI #2 dated 10/26/99
4. Discussion of NRC reaction to supplemental information letter dated 3/6/2000
    - Is the NRC planning a response or followup questions?
5. NRC to provide current status of the review / expected SER conclusions
6. Discussion of response to ACRS T-H Phenomena Subcommittee comments
7. Discussion of how to move forward to conclusion of the review
8. NRC comments on the review process

RECD APR 25 2000

Scalar Macroscopic Momentum Balances for Multi-dimensional Fluid Flow

T. A. Porsching

April 18, 2000

Notation

Ω: Three dimensional control volume

V_Ω: Volume of Ω

$\partial\Omega$: Boundary of Ω

$\mathbf{N}$: Unit exterior normal vector to $\partial\Omega$

$\mathbf{q}$: Fluid velocity vector

ρ: Fluid density

p: Fluid pressure

$\mathbf{f}$: Body force vector

S_i, $i = 0, 1, 2$: Plane sections of Ω

A_i: Area of S_i

$\mathbf{n}_i$: Unit normal vectors to S_i

Ω_i, $i = 1, 2$: Parts of Ω lying on either side of S_0

$\partial\Omega_i$: Boundary of Ω_i

$\partial\omega_i$: Part of $\partial\Omega_i$ disjoint from S_0

Σ_i: Surface area of $\partial\omega_i$

σ_i: Sidewall of Ω_i

$\bar{P}_i$, $i = 1, 2$: Average pressure over Σ_i

$\bar{p}_i$, $i = 1, 2$: Average pressure over S_i

L_0: Length of Ω with respect to S_0

W_0: Mass flow rate a point in Ω in the direction of $\mathbf{n}_0$

W_i, $i = 1, 2$: Mass flow rate at a point on S_i in the direction of $\mathbf{n}_i$

$W_{i,0}$, $i = 1, 2$: Mass flow rate at a point on S_i in the direction of $\mathbf{n}_0$

$\mathbf{q}_i$, $i = 1, 2$: Fluid velocity vector at a point on S_i

θ_i, $i = 1, 2$: Angle between $\mathbf{q}_i$ and $\mathbf{n}_i$

$\theta_{i,0}$, $i = 1, 2$: Angle between $\mathbf{q}_i$ and $\mathbf{n}_0$

$\gamma_{i,0}$, $i = 1, 2$: Angle between $\mathbf{n}_i$ and $\mathbf{n}_0$

1 Introduction

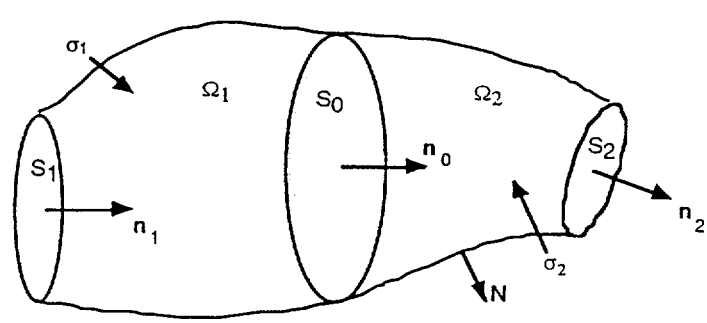
The purpose of this note is to give a systematic and rational derivation of several **scalar** momentum balances from the basic **vector** form of the multi-dimensional equations of motion of an inviscid fluid. Since no use is made of the continuity equation, the derivation is applicable to both compressible and incompressible flows. Furthermore, while it has not been done here for the sake of simplicity, it appears possible to use an analogous approach to include viscous effects by starting with the Navier-Stokes instead of the Euler equations.

The regularity assumption made about the fluid variables is that they are continuously differentiable functions of time and space. Thus, we have in mind **classical solutions** of the equations of motion instead of the more theoretically convenient **weak** or **generalized** solutions [2].

It is important to note that the mass flow rates appearing in the scalar momentum balances hold at particular (and in general unknown) **points** in a control volume at each instant of time. No assertion is made that these quantities are **averages** over specified control volume surfaces. The only averages introduced in the sequel are the pressures $\bar{P}_i$ and $\bar{p}_i$ defined by (22) and (27) respectively.

The remainder of this note is as follows. In Section 2 we derive equations (24) and (25), two forms of a scalar momentum balance that are consistent with the Euler equations. In Section 3 we make some remarks about how to close the computational models implied by (24) and (25). Any such modifications necessarily lead to **approximations** of these equations, and we present two such approximations as (36) and (37). Finally, in the Appendix we list some basic mathematical results that are used in our earlier development.

2 Scalar Momentum Models



Consider the control volume Ω shown in the above figure. We assign unit normal vectors $\mathbf{n}_i$ to the plane sections S_i, $i = 0, 1, 2$. The sections S_1 and S_2 are part of $\partial\Omega$, the boundary of Ω, whereas S_0 is in the interior of Ω. That part of $\partial\Omega$ not corresponding to S_1 and S_2 is assumed to be a rigid no-slip wall.

Note that S_0 partitions Ω into two subvolumes, Ω_1 and Ω_2 lying on opposite sides of S_0. The boundary of Ω_i, $i = 1, 2$, which we denote by $\partial\Omega_i$, consists of S_0 and $\partial\omega_i$, where $\partial\omega_1$ and $\partial\omega_2$ are those portions of $\partial\Omega$ on opposite sides of S_0. Furthermore, $\partial\omega_i = S_i \cup \sigma_i$, where σ_i is the sidewall of Ω_i.

To obtain a scalar form of the momentum balance we start with the multi-dimensional Euler equations written in vector form as

$$(\rho\mathbf{q})_t + \nabla \cdot (\rho\mathbf{q}\mathbf{q}^T) + \nabla p = \mathbf{f}, \tag{1}$$

where $\mathbf{q}\mathbf{q}^T$ is a dyadic product. We take the dot product of (1) with $\mathbf{n}_0$ and integrate the result over Ω to get

$$\int_\Omega (\rho\mathbf{q} \cdot \mathbf{n}_0)_t \, dV + \int_\Omega \nabla \cdot [\rho\mathbf{q}(\mathbf{q} \cdot \mathbf{n}_0)] \, dV + \int_\Omega \nabla \cdot (p\mathbf{n}_0) \, dV = \int_\Omega \mathbf{f} \cdot \mathbf{n}_0 \, dV. \tag{2}$$

Since $\mathbf{n}_0$ is a constant vector, the operations of differentiation and projection (i.e., "dotting") can be interchanged. Thus equation (2) is exact but, unlike the vector equation (1), it is a scalar equation.

Let us denote the four terms of (2) respectively by I_i, $i = 1, \ldots, 4$.

For the first term the operations of space integration and time differentiation may be interchanged so that

$$I_1 \equiv \int_\Omega (\rho\mathbf{q} \cdot \mathbf{n}_0)_t \, dV = \frac{d}{dt}\left[\int_\Omega (\rho\mathbf{q} \cdot \mathbf{n}_0) \, dV\right].$$

Next we apply the three dimensional version of the Mean Value Theorem for Integrals (see Appendix A) with $f(\mathbf{x}) = (\rho\mathbf{q})(\mathbf{x}) \cdot \mathbf{n}_0$ and $g(\mathbf{x}) = 1$ to the integral. This gives

$$I_1 = \frac{d}{dt}[(\rho\mathbf{q})(\mathbf{x}_0) \cdot \mathbf{n}_0] V_\Omega, \tag{3}$$

where $\mathbf{x}_0$ is some (generally unknown) point in Ω. If we define the mass flow rate

$$W_0 \equiv (\rho\mathbf{q})(\mathbf{x}_0) \cdot \mathbf{n}_0 A_0,$$

then we can write (3) in the form

$$I_1 = L_0 \frac{dW_0}{dt}, \tag{4}$$

where $L_0 \equiv V_\Omega / A_0$ can be interpreted as the "length" of Ω with respect to the cross section S_0.

The last term of (2) is handled in the same way. That is, we apply the Mean Value Theorem to

$$I_4 \equiv \int_\Omega \mathbf{f} \cdot \mathbf{n}_0 \, dV$$

with $f(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{n}_0$, $g(\mathbf{x}) = 1$. In view of the definition of L_0 this allows us to write

$$I_4 = \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0, \tag{5}$$

where $\mathbf{f}_0$ denotes $\mathbf{f}(\mathbf{x}^*)$ for some point $\mathbf{x}^* \in \Omega$.

Regarding the second term of (2), by the Divergence Theorem we have

$$I_2 \equiv \int_\Omega \nabla \cdot [\rho \mathbf{q}(\mathbf{q} \cdot \mathbf{n}_0)] \, dV = \int_{\partial\Omega} (\rho \mathbf{q} \cdot \mathbf{N})(\mathbf{q} \cdot \mathbf{n}_0) \, dS. \tag{6}$$

Now $\mathbf{q}|_{\partial\Omega} = \mathbf{0}$ except possibly on S_1 or S_2. So the domain of integration reduces to $S_1 \cup S_2$. Moreover, $\mathbf{N} = \pm \mathbf{n}_i$ on S_i, the $+$ or $-$ sign holding as $\mathbf{n}_i$ is an exterior unit normal or not. Throughout the rest of this note we assume that these vectors are oriented as shown in the figure. In this case

$$\mathbf{N} = \begin{cases} -\mathbf{n}_1 & \text{on } S_1 \\ \mathbf{n}_2 & \text{on } S_2. \end{cases} \tag{7}$$

Therefore,

$$\int_{\partial\Omega} (\rho \mathbf{q} \cdot \mathbf{N})(\mathbf{q} \cdot \mathbf{n}_0) \, dS = \int_{S_2} (\rho \mathbf{q} \cdot \mathbf{n}_2)(\mathbf{q} \cdot \mathbf{n}_0) \, dS - \int_{S_1} (\rho \mathbf{q} \cdot \mathbf{n}_1)(\mathbf{q} \cdot \mathbf{n}_0) \, dS. \tag{8}$$

If we apply the Mean Value Theorem of the Appendix to the integral

$$\int_{S_2} (\rho \mathbf{q} \cdot \mathbf{n}_2)(\mathbf{q} \cdot \mathbf{n}_0) \, dS$$

with $f(\mathbf{x}) = ((\rho \mathbf{q})(\mathbf{x}) \cdot \mathbf{n}_2)(\mathbf{q}(\mathbf{x}) \cdot \mathbf{n}_0)$, $g(\mathbf{x}) = 1$, we obtain

$$\int_{S_2} (\rho \mathbf{q} \cdot \mathbf{n}_2)(\mathbf{q} \cdot \mathbf{n}_0) \, dS = ((\rho \mathbf{q})(\mathbf{x}_2) \cdot \mathbf{n}_2)(\mathbf{q}(\mathbf{x}_2) \cdot \mathbf{n}_0) A_2$$

for some point $\mathbf{x}_2 \in S_2$. Now define

$$\begin{aligned} W_2 &\equiv (\rho \mathbf{q})(\mathbf{x}_2) \cdot \mathbf{n}_2 A_2, \\ W_{2,0} &\equiv (\rho \mathbf{q})(\mathbf{x}_2) \cdot \mathbf{n}_0 A_2, \\ \rho_2 &\equiv \rho(\mathbf{x}_2). \end{aligned}$$

Then

$$\int_{S_2} (\rho \mathbf{q} \cdot \mathbf{n}_2)(\mathbf{q} \cdot \mathbf{n}_0) \, dS = \frac{W_2 W_{2,0}}{A_2 \rho_2}. \tag{9}$$

A similar argument applied to the integral

$$\int_{S_1} (\rho \mathbf{q} \cdot \mathbf{n}_1)(\mathbf{q} \cdot \mathbf{n}_0)\, dS$$

leads to

$$\int_{S_1} (\rho \mathbf{q} \cdot \mathbf{n}_1)(\mathbf{q} \cdot \mathbf{n}_0)\, dS = \frac{W_1 W_{1.0}}{A_1 \rho_1}. \tag{10}$$

Combining (8), (9) and (10), we see that (6) can be written as

$$I_2 = \frac{W_2 W_{2.0}}{A_2 \rho_2} - \frac{W_1 W_{1.0}}{A_1 \rho_1}. \tag{11}$$

To treat the third term in (2), we start with the Divergence Theorem and use the fact that $\partial\Omega = \partial\omega_1 \cup \partial\omega_2$ to write

$$I_3 \equiv \int_\Omega \nabla \cdot (p\mathbf{n}_0)\, dV = \int_{\partial\Omega} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = \int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS + \int_{\partial\omega_2} p\mathbf{n}_0 \cdot \mathrm{N}\, dS. \tag{12}$$

We distinguish two options.

Option 1

We assume that S_0 can be chosen so that $\mathbf{n}_0 \cdot \mathrm{N}$ does not change sign on each of $\partial\omega_1$ and $\partial\omega_2$. (It will then necessarily be of opposite sign on these two surfaces.) For definiteness, suppose that as shown in the figure $\mathbf{n}_0 \cdot \mathrm{N} \leq 0$ on $\partial\omega_1$ and $\mathbf{n}_0 \cdot \mathrm{N} \geq 0$ on $\partial\omega_2$. We apply the Mean Value Theorem to the integral

$$\int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS \tag{13}$$

with $f(\mathbf{x}) = p(\mathbf{x})$, $g(\mathbf{x}) = -\mathbf{n}_0 \cdot \mathrm{N}(\mathbf{x})$ to get

$$\int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = -p(\boldsymbol{\xi}_1) \int_{\partial\omega_1} -\mathbf{n}_0 \cdot \mathrm{N}\, dS \tag{14}$$

for some point $\boldsymbol{\xi}_1 \in \partial\omega_1$. Following a suggestion of S. Clerc, we note that since $\mathbf{n}_0$ is a constant vector, $\nabla \cdot \mathbf{n}_0 = 0$. Therefore, the Divergence Theorem shows that

$$0 = \int_{\Omega_1} \nabla \cdot \mathbf{n}_0\, dV = \int_{\partial\omega_1} \mathbf{n}_0 \cdot \mathrm{N}\, dS + \int_{S_0} \mathbf{n}_0 \cdot \mathrm{N}\, dS.$$

That is,

$$\int_{\partial\omega_1} -\mathbf{n}_0 \cdot \mathrm{N}\, dS = \int_{S_0} \mathbf{n}_0 \cdot \mathrm{N}\, dS. \tag{15}$$

Furthermore, $\mathrm{N} = \mathbf{n}_0$ on S_0 so that

$$\int_{S_0} \mathbf{n}_0 \cdot \mathrm{N}\, dS = \int_{S_0} dS = A_0, \tag{16}$$

the area of S_0. In view of (14), (15) and (16) we have

$$\int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = -p(\boldsymbol{\xi}_1)A_0. \tag{17}$$

Proceeding in an entirely similar manner, we also find that

$$\int_{\partial\omega_2} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = p(\boldsymbol{\xi}_2)A_0 \tag{18}$$

for some point $\boldsymbol{\xi}_2 \in \partial\omega_2$.

Finally, we define $p_i \equiv p(\boldsymbol{\xi}_i)$, $i = 1, 2$ and substitute (17) and (18) into (12) to obtain

$$I_3 = (p_2 - p_1)A_0. \tag{19}$$

Option 2

We assume that the pressure is nonnegative over $\partial\Omega$. Then we can apply the Mean Value Theorem to (13) with $f(\mathbf{x}) = \mathbf{n}_0 \cdot \mathrm{N}(\mathbf{x})$, $g(\mathbf{x}) = p(\mathbf{x})$ to deduce that there is a point $\boldsymbol{\xi}_1 \in \partial\omega_1$ such that

$$\int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = \mathbf{n}_0 \cdot \mathrm{N}(\boldsymbol{\xi}_1) \int_{\partial\omega_1} p\, dS. \tag{20}$$

Similarly,

$$\int_{\partial\omega_2} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = \mathbf{n}_0 \cdot \mathrm{N}(\boldsymbol{\xi}_2) \int_{\partial\omega_2} p\, dS \tag{21}$$

for some $\boldsymbol{\xi}_2 \in \partial\omega_2$. If we define the average pressure over $\partial\omega_i$ as

$$\bar{P}_i \equiv \frac{1}{\Sigma_i} \int_{\partial\omega_i} p\, dS, \tag{22}$$

where Σ_i denotes the surface area of $\partial\omega_i$, and let the angle γ_i be the angle between $\mathbf{n}_0$ and $\mathrm{N}(\boldsymbol{\xi}_i)$. Then $\cos\gamma_i = \mathbf{n}_0 \cdot \mathrm{N}(\boldsymbol{\xi}_i)$ and it follows from (12), (20) and (21) that

$$I_3 = (\Sigma_1 \cos\gamma_1)\bar{P}_1 + (\Sigma_2 \cos\gamma_2)\bar{P}_2. \tag{23}$$

We can now replace the terms in (2), by their respective equivalent forms (4), (11), (19) or (23), and (5) to obtain a scalar momentum balance for the control volume Ω. We have

$$L_0 \frac{dW_0}{dt} + \frac{W_2 W_{2.0}}{A_2 \rho_2} - \frac{W_1 W_{1.0}}{A_1 \rho_1} + A_0(p_2 - p_1) = \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0, \tag{24}$$

under Option 1, and

$$L_0 \frac{dW_0}{dt} + \frac{W_2 W_{2.0}}{A_2 \rho_2} - \frac{W_1 W_{1.0}}{A_1 \rho_1} + (\Sigma_1 \cos\gamma_1)\bar{P}_1 + (\Sigma_2 \cos\gamma_2)\bar{P}_2 = \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0, \tag{25}$$

under Option 2.

3 Remarks

1. The validity of (24) obtained under Option 1 depends on the assumption that S_0 can be oriented in Ω so that $\mathbf{n}_0 \cdot \mathrm{N}$ does not change sign on each of the two surfaces $\partial\omega_1$ and $\partial\omega_2$. Since $\mathbf{n}_0 \cdot \mathrm{N} = \cos\gamma$, where γ is the angle between the constant vector $\mathbf{n}_0$ and the surface normal N, this assumption is satisfied if at each point on $\partial\omega_i$ no tangent line is parallel to $\mathbf{n}_0$. (At points where the tangent line has a jump, we also need to include the directions of all "intermediate tangent" lines.) Thus, while one can verify the assumption for some surfaces (e.g., if $\partial\omega_i$ is any surface that is cut in exactly one point by a line parallel to $\mathbf{n}_0$), there are other simple surfaces for which it is not obvious how to choose S_0 so that it holds (e.g., if $\partial\omega_i$ is a portion of a torus, i.e., an elbow).

   This drawback is not present with (25) which results from Option 2. However, there is a practical difficulty with (25), namely, the presence of the (generally) unknown angles γ_i. One way to deal with this problem is to recall that since $\partial\omega_i = S_i \cup \sigma_i$, we can write

$$\int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = \int_{S_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS + \int_{\sigma_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS. \tag{26}$$

   Utilizing (7), we see that

$$\int_{S_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = -\int_{S_1} p\mathbf{n}_0 \cdot \mathbf{n}_1\, dS = -\mathbf{n}_0 \cdot \mathbf{n}_1 \int_{S_1} p\, dS.$$

   However, $\mathbf{n}_0 \cdot \mathbf{n}_1 = \cos\gamma_{1,0}$, where $\gamma_{1,0}$ is the **known** angle between $\mathbf{n}_1$ and $\mathbf{n}_0$. Therefore, if we define the average pressure over S_1 as

$$\bar{p}_1 \equiv \frac{1}{A_1}\int_{S_1} p dS, \tag{27}$$

   then (26) becomes

$$\int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS = -A_1\bar{p}_1 \cos\gamma_{1,0} + \int_{\sigma_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS. \tag{28}$$

   The integral over the sidewall σ_1 in (28) will be small if either of the following conditions hold.

   (a) The sidewall σ_1 is nearly parallel to the vector $\mathbf{n}_0$. Indeed, if σ_1 is a cylinder whose generators are parallel to $\mathbf{n}_0$, then the integral vanishes.

   (b) The area of σ_1 is small. This condition can be met by creating control volumes whose lengths L_0 are sufficiently small.

In either of these cases we can neglect the second term on the right side of (28) to obtain the approximation

$$\int_{\partial\omega_1} p\mathbf{n}_0 \cdot \mathrm{N}\, dS \approx -A_1\bar{p}_1 \cos\gamma_{1,0}. \tag{29}$$

A similar line of reasoning leads to

$$\int_{\partial\omega_2} p\mathbf{n}_0 \cdot \mathrm{N}\, dS \approx A_2\bar{p}_2 \cos\gamma_{2,0}. \tag{30}$$

Combining (29) and (30) with (12), we see that

$$I_3 \approx A_2\bar{p}_2 \cos\gamma_{2,0} - A_1\bar{p}_1 \cos\gamma_{1,0}. \tag{31}$$

2. A second issue that must be addressed before (24) and (25) become suitable for computation is the need to relate the mass flow rates W_0, W_1, W_2, $W_{1,0}$, and $W_{2,0}$. In this connection let us observe that by definition

$$\begin{aligned} W_i &= |(\rho\mathbf{q})(\mathbf{x}_i)|A_i \cos\theta_i \\ W_{i,0} &= |(\rho\mathbf{q})(\mathbf{x}_i)|A_i \cos\theta_{i,0}, \quad i = 1, 2, \end{aligned}$$

where θ_i is the angle between $\mathbf{q}(\mathbf{x}_i)$ and $\mathbf{n}_i$ and $\theta_{i,0}$ is the angle between $\mathbf{q}(\mathbf{x}_i)$ and $\mathbf{n}_0$. Eliminating the quantity $|(\rho\mathbf{q})(\mathbf{x}_i)|A_i$ between these relationships and substituting the result for $W_{i,0}$ into (24) and (25) gives

$$L_0\frac{dW_0}{dt} + \frac{W_2^2}{A_2\rho_2}\frac{\cos\theta_{2,0}}{\cos\theta_2} - \frac{W_1^2}{A_1\rho_1}\frac{\cos\theta_{1,0}}{\cos\theta_1} + A_0(p_2 - p_1) = \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0 \tag{32}$$

and

$$L_0\frac{dW_0}{dt} + \frac{W_2^2}{A_2\rho_2}\frac{\cos\theta_{2,0}}{\cos\theta_2} - \frac{W_1^2}{A_1\rho_1}\frac{\cos\theta_{1,0}}{\cos\theta_1} + (\Sigma_1\cos\gamma_1)\bar{P}_1 + (\Sigma_2\cos\gamma_2)\bar{P}_2 = \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0 \tag{33}$$

respectively.

Consider first the case where Ω is a cylindrical region whose sidewall is parallel to $\mathbf{n}_0$. If we choose S_1 and S_2 parallel to S_0, then Option 1 is valid. Also $\mathbf{n}_0 = \mathbf{n}_1 = \mathbf{n}_2$ so that the ratio of cosines in (32) and (33) is unity. Therefore, (32) reduces to the more familiar "one dimensional" momentum balance

$$L_0\frac{dW_0}{dt} + \frac{W_2^2}{A_2\rho_2} - \frac{W_1^2}{A_1\rho_1} + A_0(p_2 - p_1) = \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0. \tag{34}$$

We get a similar result under Option 2 because in this special case (31) is an equality, $A_0 = A_1 = A_2$, and $\gamma_{1,0} = \gamma_{2,0} = 0$. The resulting equation is

$$L_0 \frac{dW_0}{dt} + \frac{W_2^2}{A_2\rho_2} - \frac{W_1^2}{A_1\rho_1} + A_0(\bar{p}_2 - \bar{p}_1) = \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0. \tag{35}$$

In the general case if we assume that $\mathbf{q}_i$ is parallel to $\mathbf{n}_i$, $i = 1, 2$ (i.e., the fluid velocity on S_i is essentially the normal velocity), then $\theta_i = k_i\pi$ and $\theta_{i,0} = \gamma_{i,0} + k_i\pi$, where the (known) angle $\gamma_{i,0}$ has been previously defined and k_i is 0 if $\mathbf{q}_i$ is in the same direction as $\mathbf{n}_i$ and 1 if it is in the opposite direction. It follows that

$$\frac{\cos\theta_{i,0}}{\cos\theta_i} = \frac{\cos(\gamma_{i,0} + k_i\pi)}{\cos(k_i\pi)} = \cos\gamma_{i,0}.$$

So in place of (32) we obtain the **approximate** balance equation

$$L_0 \frac{dW_0}{dt} + \frac{W_2^2}{A_2\rho_2}\cos\gamma_{2,0} - \frac{W_1^2}{A_1\rho_1}\cos\gamma_{1,0} + A_0(p_2 - p_1) \approx \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0. \tag{36}$$

We can proceed in the same way with (33) and in addition replace its fourth and fifth terms by the right side of (31). This gives us the approximation

$$L_0 \frac{dW_0}{dt} + \frac{W_2^2}{A_2\rho_2}\cos\gamma_{2,0} - \frac{W_1^2}{A_1\rho_1}\cos\gamma_{1,0} + A_2\bar{p}_2\cos\gamma_{2,0} - A_1\bar{p}_1\cos\gamma_{1,0} \approx \mathbf{f}_0 \cdot \mathbf{n}_0 L_0 A_0. \tag{37}$$

While (32) and (33) are exact, they are not very useful computationally. On the other hand the approximations (36) and (37) can be incorporated into larger computational algorithms. A common approach is to supplement the set of "momentum" control volumes Ω with a second set of interlaced "continuity" control volumes over which mass is conserved. The continuity control volumes typically have as part of their surfaces the **interior** cross sections of the momentum control volumes (e.g., S_0). This allows the mass flow rates W_1 and W_2 appearing in (36) and (37) to be expressed (say by averaging or interpolation) in terms of W_0 and the analogous flow rates of the momentum control volumes contiguous to Ω. With this understanding, the discrete flow variables of the overall system are then W_0 and its analogs.

A Basic Mathematical Results

In Section 2 we have made use of certain fundamental mathematical results. For completeness, we include formal statements of these in this Appendix. Proofs may be found in any reasonably complete text on mathematical analysis, for example [1].

Divergence Theorem: Let $\mathbf{F}$ be a continuously differentiable, vector valued function in a three dimensional, closed, bounded region Ω whose boundary surface $\partial\Omega$ is such that its exterior unit normal $\mathbf{N}$ is sectionally continuous. Then

$$\int_{\Omega} \nabla \cdot \mathbf{F}\, dV = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{N}\, dS,$$

where dV and dS are respectively volume and surfaces differentials.

Mean Value Theorem for Integrals: Let D be a closed, bounded, connected, two dimensional region. Let f and g be continuous, scalar functions in D with $g(\mathbf{x}) \geq 0$ there. Then there is a point $\mathbf{x}^* \in D$ such that

$$\int_D f\, g\, dA = f(\mathbf{x}^*) \int_D g\, dA,$$

where dA is an area differential.

We observe that since surface integrals are special kinds of two dimensional integrals, the theorem also applies to them (with obvious modifications to the notation). Furthermore, the Theorem is valid for three dimensional regions.

References

[1] R. C. Buck, *Advanced Calculus*, McGraw-Hill, 1978.

[2] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, 1969.