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REGULATORY GUIDE

DIRECTORATE OF REGULATORY STANDARDS

REGULATORY GUIDE 4.4

REPORTING PROCEDURE FOR MATHEMATICAL MODELS SELECTED TO PREDICT HEATED EFFLUENT DISPERSION IN NATURAL WATER BODIES

A. INTRODUCTION

In accordance with Appendix D, "Interim Statement of General Policy and Procedure: Implementation of the National Environmental Act of 1969," of 10 CFR Part 50, "Licensing of Production and Utilization Facilities," and proposed Section 51.5 of 10 CFR Part 51, the Commission prepares an environmental impact statement for consideration in the licensing actions of certain nuclear facilities. In addition, each applicant for a permit to construct such a facility must submit an "Applicant's Environmental Report–Construction Permit Stage" (§51.20, 10 CFR Part 51), which discusses the probable environmental impact of the proposed facility. Should the proposed facility be a nuclear power plant, thermal effects from the release of condenser cooling water or closed cycle blowdown to a natural water body can have a significant impact.

Direct, quantitative measurements of environmental impact due to thermal discharges in natural water bodies are not possible during the preconstruction stages of proposed plants. Consequently, reasonable approximations to the interactions of thermal discharges with the environment must be adopted to provide a basis for impact assessment. The applicant may select mathematical models as one such means of approximation. Section 5.1 of Regulatory Guide 4.2, "Preparation of Environmental Reports for Nuclear Power Plants," suggests that details of mathematical modeling methods should be given in an appendix to the Environmental Report. This guide describes a procedure acceptable to the Regulatory staff for completing such an appendix.

B. DISCUSSION

As applied to nuclear power plants, thermal discharge mathematical modeling attempts to accurately simulate the dispersion of cooling water effluent within the receiving water body. Because of the unique properties of each plant site, no single available model is universally applicable to all site conditions. Therefore, a need exists to differentiate among existing models and to identify those that will yield an optimum simulation of effluent/receiving water interactions at a particular site. For background purposes, a qualitative account of the basic physical principles and critical site factors upon which such interactions are founded is presented in Appendix A. The fundamental differential equations, mathematical approximations, and solution techniques for simulating turbulent transport processes are discussed in Appendix B. The Regulatory staff regards Appendix B as representative of acceptable mathematical procedures used to model the dispersion of heated effluent in aquatic systems.

C. REGULATORY POSITION

To aid in the assessment of proposed thermal discharge mathematical models by the Regulatory staff, a uniform reporting format is desirable. Consequently, an itemized table of relevant modeling factors, such as that shown in Exhibit 1, should accompany descriptive material for the one or more models submitted by an applicant. The table is a logical extension of that presented in an existing model review,[1] essentially differing only in the level of detail specified. Upon completion of pertinent entries, the table functions as a comparative tool, enabling an analyst to assess the

[1] Policastro, A. J., "Heated Effluent Dispersion in Large Lakes, State-of-the-Art of Analytical Modeling, Surface and Submerged Discharges," presented at Conference on Water Quality Considerations: Siting and operating nuclear power plants, Proceedings Atomic Industrial Forum, New York, 1972.

proposed model (or models) against prototypical conditions. A tool of this type facilitates the model evaluation process, but completion of the table does not assure that the optimum simulation model of those available will be selected, nor that the model (or models) ultimately chosen by the applicant will be acceptable to the Regulatory staff. Alternatively, the applicant may devise other acceptable approximations to the interaction of a thermal discharge with the environment; this guide should not be construed as advocating the exclusive use of mathematical models.

The Model Assessment Table (MAT) (Exhibit 1) is organized relative to the four principal types of characteristics pertinent to thermal discharge mathematical modeling:

- Discharge Characteristics
- Receiving Water Characteristics
- Discharge/Receiving Water Interactions
- Model Characteristics

Instructions for preparing each section of the MAT are presented below.

1. General Instructions

a. The basic intent of the table is to provide a simple and direct means of comparison between prototypical conditions and those phenomena being modeled. *The MAT is not intended to supplant the descriptive material that normally accompanies submission of a model.* On the contrary, such descriptive material should comprise the textual portions of the appendix to the Environmental Report and serve as a vehicle for justifying or elaborating entries made in the table.

b. A secondary objective of the MAT is to elucidate unique properties and ranges of applicability of each itemized model in order to establish a basis for differentiating between models. As a result, certain portions of the Discharge Characteristics and Receiving Water Characteristics sections are irrelevant to the models being assessed, and the Model Characteristics section and portions of the Discharge/Receiving Water Interactions section are irrelevant to the prototype. In such parts of the table entries need not be made.

c. As a rule, models are applied to a set of environmental conditions that may be characterized as worst probable cases. *All table entries should be based upon these conditions and the consequences stemming from them.* In addition, the descriptive material should indicate the environmental conditions to which the table applies. If more than one set of conditions is required to cover the cases of interest, more than one MAT may be necessary.

d. As presented in Exhibit 1, the MAT allows entries for four models as well as the prototype. This structuring does not imply that four different models should be assessed; any number would be acceptable.

e. Individual entries to the MAT usually consist of either a YES response or a numerical value. In cases where a numerical entry possesses units of measure, the International System of Units (SI) is preferred. Under circumstances in which the entry is not applicable to the conditions of the prototype or model, a response of N/A may be used.

f. The values of a number of MAT entries depend on which coordinate system is applied to a model. The terms "longitudinal," "lateral," and "vertical" have been proposed to define the orthogonal axes of a horizontal discharge that propagates longitudinally in the discharge direction, laterally in the horizontal transverse discharge direction, and vertically in the transverse discharge direction. The applicant should specify his choice of model coordinate system relative to local true North at the prototype.

2. Discharge Characteristics

Most descriptive properties of the discharge are supplied by a simple binary choice of entries (Exhibit 1). Except where otherwise stated, a YES response in the appropriate MAT location is sufficient.

a. Type. Discharge type should be identified as either single port or multiport; a discharge canal is regarded as a single port. If the prototype discharge is a multistructure and/or multiport variety, the total number and spacing of exit ports for each structure should be given in the descriptive text.

b. Shape. The exit port shape should be identified as either round or rectangular; a slot jet is considered rectangular. If a model uses a slot jet approximation for a multiport diffuser, the entry should indicate a rectangular shape for the model. The linear dimensions of each exit port should be specified, and irregularly shaped discharge ports should be described in the text.

c. Location. The discharge location is either shoreline or offshore. Shoreline is defined as the boundary between the land surface and the receiving water body. The decision for discharge location depends upon the point of entry to the receiving water body.

d. Position. The discharge position is either surface or submerged. A surface discharge is defined as one for which the water surface is a boundary at the point of entry to the receiving water body. All other positions are regarded as submerged. For a prototype with a submerged discharge, the text should state the vertical distances from the discharge centerline to (1) the bottom and (2) the surface of the receiving water body.

e. **Direction.** In the case of the prototype, the horizontal discharge direction should be specified as the compass angle (degrees) of the initial horizontal component of discharge relative to local true North; the nonhorizontal discharge direction should be specified as the angle (degrees) produced by the initial discharge direction vector and the horizontal. If a model can simulate both horizontal and nonhorizontal discharge directions, this should be indicated in the MAT.

f. **Volume Flow Rate.** The anticipated operating range of volumetric flow rates for the prototype should be given.

g. **Discharge Velocity.** The normal operating range of discharge velocities at the outfall should be specified.

h. **Effluent Excess Temperature.** Excess temperature is defined as the difference between effluent temperature and ambient water temperature at the discharge point. The measured or expected value of each temperature parameter (intake, discharge, ambient) should be given in the text.

3. **Receiving Water Characteristics**

a. **Type.** The type of receiving water body should be identified in the MAT by the following number code:

(1) lake
(2) river, tidal
(3) river, nontidal
(4) estuary
(5) ocean
(6) cooling pond
(7) reservoir

The entry for a model should include those water bodies to which the model is applicable.

b. **Depth at Outfall.** Depth at outfall is defined as the naturally occurring extreme range of water depths, such as might be caused by tides, likely to be encountered at the discharge point.

c. **Bottom Slope.** The near-field spatially averaged angle of the receiving water bottom in the direction of discharge should be specified for the prototype; the validity range of bottom slope angles should be given for each model. Also, a bathymetric map of the discharge area should be included with the text. Such a map would be a convenient medium on which to display the geometric properties of the discharge and the orientations of model coordinate systems.

d. **Natural Stratification.** Some receiving water bodies, especially lakes, exhibit natural thermal and/or saline stratification at some time during the annual thermal cycle. If such is the case for the prototype, the fact should be noted on the MAT, and the extent to which stratification affects the prototype should be described in the text along with a discussion of the ability of each model to simulate this situation.

e. **Natural Current.** The predominant natural current in the immediate vicinity of the discharge should be specified; model entries should include the validity range of current speed and direction. Direction is defined as the clockwise angle between the horizontal flow component and local true North. For systematic natural current variations in more than one direction, multiple entries may be made. The detailed current structure within that portion of the water body likely to be affected by the discharge should be discussed relative to its impact on model applicability.

4. **Discharge/Receiving Water Interactions**

a. **Jet Entrainment.** The MAT should indicate whether jet mixing is the dominant thermal dispersion mechanism in the near-field. The principal criterion for jet entrainment is the *initial densimetric Froude number*, the extreme operating range of which should be specified in the table. Model entries should cite the densimetric Froude number limits of applicability for each model.

A number of models utilize empirical *entrainment coefficients* to simulate jet mixing. In such cases, the numerical value(s) of coefficients applicable to prototype conditions should be given for each orthogonal direction in which entrainment is modeled, and the text should contain the rationale for selection of each coefficient.

b. **Cross Flow.** In the presence of an ambient current, all discharges, regardless of initial flow direction, ultimately flow in the ambient current direction. If the current interacts in the near-field with a noncoincidental discharge, cross flow conditions prevail, and the MAT should reflect this fact. The principal criterion for cross flow is the *velocity ratio* defined as the ratio of initial discharge velocity to ambient current velocity. The MAT should contain the operating range of velocity ratios for the prototype and the range of applicability for each model. If the initial velocity ratio is sufficiently small, *pressure drag* effects on the discharge by the ambient current can be significant. In order to simulate the pressure force exerted on the discharge, many models employ an empirical relation containing a *drag coefficient*. When such is the case, the coefficient should be specified as a MAT entry, and the functional form of the relation should be presented in the text.

c. **Natural Turbulence.** Natural water bodies possess a varying degree of natural turbulence. Consequently, the effects of turbulence should be included in any thermal dispersion model which purports to be valid outside the near-field region. When dealing with natural turbulence, many models employ an *eddy diffusivity* or dispersion coefficient to simulate transport by turbulent diffusion. Under this circumstance, the assumed initial values of the coefficient in the three orthogonal directions of the model's coordinate system should be specified, and the coefficient's functional form(s) and conditions of application should be discussed in the text.

Although *wind stress* is typically responsible for considerable induced turbulence occurring in water bodies, this phenomenon can also directly affect the surface velocity distribution. The average and extreme range of observed wind vectors for the worst probable conditions at the prototype should be included. A MAT entry should appear for those models that include wind stress.

d. **Buoyancy.** Buoyant forces accelerate flow in the direction of discharge while enhancing lateral spreading. A detailed discussion of the importance of buoyancy relative to other thermal dispersion mechanisms in the prototype should be presented; the means by which the effects of buoyancy are simulated in each model should also be discussed.

e. **Recirculation.** When only a limited amount of entrainment water is available for mixing, such as in some shallow water discharges, partly diluted effluent may be re-entrained or recirculated into the discharge plume. The inclusion of nonambient dilution water decreases thermal dispersion while increasing plume size. The MAT should indicate if recirculation is feasible for the prototype and whether the models can simulate this process. Substantiation for the entry should be presented in the text.

f. **Surface Heat Transfer.** Every thermal discharge plume which is bounded by the water surface experiences heat loss to the atmosphere. Since the quantity of heat dissipated in such a manner varies directly with the surface area of the plume, surface heat transfer is largely a far-field phenomenon. The simplest, most direct approach for estimating surface heat transfer employs the excess temperature concept, which requires the adoption of an empirical surface *heat transfer coefficient* and often an *equilibrium temperature*. Under such circumstances, the range of pertinent coefficients and temperatures applicable to the prototype should be specified in the MAT, and the coefficient's functional form, if any, should appear in the text. An alternative method for modeling surface heat transfer may be adopted, provided the method is described in detail.

5. Model Characteristics

This section of the MAT supplies no additional information for understanding thermal dispersion by the prototype. However, in order to expand the basis for differentiation among models, the functional properties of each model should be included for completeness.

a. **Field.** The dispersion field to which the model applies should be specified in the MAT by the following number code:

(1) Near-field
(2) Intermediate-field
(3) Far-field
(4) Complete-field

b. **Dimension.** This entry identifies the spatial properties of the model solution. The directions are defined consistent with those previously utilized in the table. The Regulatory staff puts no constraint on the number of dimensions to which a model applies.

c. **Mathematical Approach.** The solution technique utilized by each model should be specified by the following number code:

(1) Phenomenological
(2) Analytical
(3) Finite Difference
(4) Integral
(5) Finite Element
(6) Stochastic
(7) Other (state)

d. **Approximations.** Simplifying approximations to the mathematical formulation of the model should be indicated by one or more of the following:

(1) Steady State
(2) Boussinesq
(3) Hydrostatic Pressure
(4) Buoyancy Decoupling
(5) Other (state)

"Other" approximations applied to each model should be discussed in the text.

e. **Model Verification.** Efforts to verify the model by field and/or laboratory measurements should be indicated in the MAT and described in the text.

f. **Computer Program.** The availability of a computerized version of the model should be specified by one of the following:

(1) Proprietary
(2) Available on request
(3) Available in open literature
(4) No program exists.

MODEL ASSESSMENT TABLE

			PROTOTYPE	MODEL	MODEL	MODEL	MODEL
DISCHARGE	TYPE	SINGLE PORT					
		MULTIPORT					
	SHAPE	ROUND					
		RECTANGULAR					
	LOCATION	SHORELINE					
		OFFSHORE					
	POSITION	SURFACE					
		SUBMERGED					
	DIRECTION	HORIZONTAL					
		NON–HORIZONTAL					
	VOLUME FLOW RATE (M^3/SEC)						
	DISCHARGE VELOCITY (M/SEC)						
	EXCESS TEMPERATURE (deg C)						
RECEIVING WATER	TYPE						
	DEPTH AT OUTFALL (M)						
	BOTTOM SLOPE (deg)						
	NATURAL STRATIFICATION						
	CURRENT	SPEED (M/SEC)					
		DIRECTION (deg)					
DISCHARGE/RECEIVING WATER INTERACTIONS	JET ENTRAINMENT						
	FROUDE NUMBER						
	ENTRAINMENT COEFFICIENT	LONGITUDINAL					
		LATERAL					
		VERTICAL					
	CROSS FLOW						
	VELOCITY RATIO						
	PRESSURE DRAG						
	DRAG COEFFICIENT						
	NATURAL TURBULENCE						
	EDDY DIFFUSIVITY (M^2/SEC)	LONGITUDINAL					
		LATERAL					
		VERTICAL					
	WIND STRESS						
	BUOYANCY						
	RECIRCULATION						
	SURFACE HEAT TRANSFER						
	HEAT TRANSFER COEF (CAL/M^2 SEC-deg C)						
	EQUILIBRIUM TEMPERATURE (deg C)						
MODEL	FIELD						
	DIMENSION	LONGITUDINAL					
		LATERAL					
		VERTICAL					
	MATHEMATICAL APPROACH						
	APPROXIMATIONS						
	MODEL VERIFICATION						
	COMPUTER PROGRAM						

EXHIBIT 1

APPENDIX A

BEHAVIOR OF THERMAL DISCHARGES

1. Nomenclature

From a phenomenological viewpoint, a thermal discharge can be partitioned into three spatial fields, each characterized by a different set of dominant processes. The "near-field" is marked by the interaction between the kinematic heated effluent and the receiving water body. Since the effluent velocity usually exceeds the receiving water velocity, the discharge is referred to as a "jet." By definition, jet momentum forces predominate in the near-field; however, buoyancy forces may also be important. In the "far-field," ambient flow determines the shape and position of the thermal discharge, which typically is called a "plume." The ill-defined region joining near- and far-fields has been designated variously as the "intermediate-field" or transition zone. In this region the heated discharge changes from an active jet to a passive plume as the effluent comes increasingly under the influence of the receiving water body. In the transition region both the discharge flow and the ambient flow are important.

This overview of discharge/receiving water interactions serves to introduce the fundamental terminology of thermal discharge analysis, as well as to identify the gross behavior of heated effluents.

2. Engineering Design Factors Governing Thermal Dispersion

Outfall designs for aqueous thermal effluent from power plants have changed drastically in recent years. Early structures usually consisted of simple shoreline canals or pipes that discharged their contents at or just below the water surface. Heated releases from structures of this type tend to produce thin surface plumes of large areal extent, which are efficient in transferring heat to the atmosphere. Currently, many discharge structures are being designed to maximize dilution near the outfall while minimizing detectable surface plume area in the far-field. The result has been submerged structures with single or multiple high-velocity exit ports. Outfalls of this type are called diffusers, and they utilize a high discharge velocity to produce intensive mechanical mixing in the near-field.

Details of engineering design for outfall structures are generally unnecessary for thermal discharge modeling problems. However, consideration must be given to outfall geometry, that is, the size and shape of the exit port(s) or canal and its location and orientation relative to some fixed coordinate system. The choice of coordinate system is arbitrary, but a system is usually selected that simplifies the mathematics of the model.

In addition to outfall geometry, several other plant design factors must be known for modeling purposes. These include the heat rejection rate, temperature rise of the discharge relative to the intake, the volumetric flow rate of cooling water, and the initial discharge velocity. If not directly measurable, the discharge velocity can be estimated from the volumetric flow rate and the exit port cross-sectional area and orientation.

3. Environmental Factors Governing Thermal Dispersion

Three factors or physical processes govern the thermal dispersion of heated effluents in natural environments:

- entrainment
- turbulent diffusion
- surface heat exchange

Advection is another major process that directly influences the size, shape, and distribution of heated effluents. The interactive role played by advective phenomena such as ambient currents is discussed below, but only to the extent that advection affects dispersion processes.

a. Entrainment

Consider first the hypothetical case of a nonbuoyant discharge into a stagnant homogeneous environment. (In the context of this discussion, a stagnant environment is one in which the magnitude of the initial discharge velocity is much greater than any local ambient velocity.) The injection of a fluid as a jet into another fluid results in the generation of turbulent eddies due to shearing stresses caused by the velocity difference between the two fields. Shearing stresses so produced represent the lateral flux of momentum, and they are directly proportional to the velocity vector difference.

Eddy motion along the jet boundary yields a net mixing of jet fluid with ambient fluid, in effect broadening and diluting the jet at increasing centerline distances from the outfall. This mixing and dilution is called entrainment, and the constant of proportionality relating jet volume flux to velocity is referred to as the entrainment coefficient. Note that for a submerged jet discharging into an infinite medium, entrainment largely propagates transversely to the discharge direction. If the jet exits at the surface, entrainment is constrained by the air-water interface; if the vertical extent of the jet is equivalent to the receiving water depth, vertical entrainment is nonexistent.

The entrainment-induced transfer of jet momentum to the ambient medium progresses outward from the jet boundary, in effect altering the transverse velocity profile from a top-hat shape at the discharge point to a normal or Gaussian distribution. Laboratory experimental data have indicated that suitably time-averaged Gaussian velocity profiles are similar:[1] that is, each transverse profile along the jet beyond the point at which the centerline velocity begins to decay has the general form

$$U_n = Ue^{-n^2/2\sigma_u^2} \qquad \text{(A-1)}$$

in which U_n is the velocity at distance n normal to the jet centerline, U is the jet centerline velocity, and σ_u^2 is the jet velocity variance along n. This finding has permitted the simplification of many near-field thermal discharge models, although the general applicability of the Gaussian approximation to real discharges remains open to question.

For the case of a heated jet, the Gaussian approximation of the transverse temperature profile along the jet takes the form,

$$T_n = Te^{-n^2/2\lambda^2\sigma_T^2} \qquad \text{(A-2)}$$

where T_n and T are the temperature in the transverse direction and the jet centerline temperature, respectively, σ_T^2 is the temperature variance along n, and λ is an adjustable constant. According to Taylor's theory,[2] heat diffuses more rapidly than momentum in the transverse flow direction. Therefore, the value of λ is always greater than unity.

Under most circumstances, heated effluent is less dense than the receiving water in the immediate vicinity of the outfall. As a result of the density disparity, there is a buoyant force on the jet acting both vertically and horizontally. A submerged buoyant jet tends to rise to the surface; a buoyant jet at the surface forms a stable density layer. For either case entrainment is reduced, especially in the vertical direction, and the dilution rate decreases.

The degree to which buoyancy influences a heated jet is suggested by the densimetric Froude number, defined as

$$F_d = \frac{U_o}{\left(gh\frac{\Delta\rho}{\rho}\right)^{1/2}} \qquad \text{(A-3)}$$

where U_o is the initial jet velocity, g the gravitational acceleration, h the vertical thickness of the jet, $\Delta\rho$ the density difference between effluent and ambient water, and ρ the ambient water density. The nondimensional Froude number represents the ratio of inertial forces to buoyant forces in the jet.

By definition, for Froude numbers greater than unity, inertial forces (i.e., jet momentum) dominate; for Froude numbers less than unity, buoyancy effects (i.e., density stability) are predominant. This principle has been corroborated by laboratory experiments, although field data[3] show that buoyancy effects at the discharge can be significant for initial densimetric Froude numbers as large as about 3. Obviously, in order to maximize thermal dispersion by entrainment, the preferred approach would be to optimize plant engineering characteristics: increase discharge velocity and/or decrease the vertical dimension of the outfall (i.e., increase the densimetric Froude number).

Regardless of initial conditions at the discharge port, jet velocity eventually decreases to a point where the local Froude number falls below one. When this happens, vertical mixing is suppressed, and the jet ceases to thicken. However, buoyancy also tends to induce lateral spreading due to the horizontal density gradient, in effect enhancing dilution while broadening the plume. With decreasing thermal plume temperature the normalized density difference, $\Delta\rho/\rho$, approaches zero, the Froude number increases above unity, and vertical mixing can become significant again.

In the presence of an ambient current, the jet discharge and natural flow interact to cause the jet to be deflected towards the direction of natural flow. This cross-flow effect is important if the jet and current are initially perpendicular, becoming less so as the two flow directions approach coincidence. At some distance from the discharge point, the magnitude of which depends upon the ratio of discharge velocity to ambient velocity, the motion of the effluent completely follows that of the ambient current. If the initial velocity ratio is less than about 10, cross-flow effects on the discharge jet are significant in the near-field.[4]

Ambient current motion around the discharge jet creates a pressure drag analogous to that induced on a solid object inserted into a uniform flow field. A fraction of initial discharge momentum is expended to maintain the integrity of the jet in response to the external pressure field. If the initial velocity ratio is below 2, pressure drag effects on the discharge jet by the ambient current can become important.[5]

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[1] G. N. Abramovich, "The Theory of Turbulent Jets," Massachusetts Institute of Technology Press, Cambridge, Massachusetts, 1963.

[2] ibid.

[3] R. A. Paddock, A. J. Policastro, A. A. Frigo, D. E. Frye, and J. V. Tokar, "Temperature and Velocity Measurements and Predictive Model Comparisons in the Near-Field Region of Surface Thermal Discharges," Center for Environmental Studies, Argonne National Laboratory, ANL/ES-25, 1973.

[4] H. H. Carter, "A Preliminary Report on the Characteristics of a Heated Jet Discharged Horizontally into a Transverse Current; Part I–Constant Depth," Chesapeake Bay Institute Technical Report No. 61, November 1969.

[5] B. A. Benedict, J. L. Anderson, and E. L. Yandell, "Analytical Modeling of Thermal Discharges: A Review of the State of the Art," Center for Environmental Studies, Argonne National Laboratory, ANL/ES-18, 1974.

In the case of a shallow-water, nearshore jet in which the discharge extends from the water surface to the bottom, entrainment is effectively confined to the upstream, offshore side of the jet. On the nearshore side in the downstream direction, the quantity of ambient dilution water can be severely limited, and recirculation of partially diluted effluent becomes a definite possibility. The situation is less serious for a discharge in deep water. In this case, ambient water can move under as well as around the jet; pressure drag is reduced, and fresh dilution water is available on the jet's downstream side.

The influence of ambient stratification on thermal discharges depends directly upon the density of the heated effluent relative to the vertical density profile of the ambient water. As a result, the heated effluent can assume any of a number of different configurations.

If the heated effluent is less dense than the ambient water, the effluent forms a surface layer analogous to the unstratified case. Moreover, natural stratification can enhance effects due to buoyancy and can inhibit vertical mixing, effectively thinning the surface layer while increasing its horizontal extent. If the effluent has an intermediate density relative to the density range of the stratified ambient, the discharge jet tends to rise or fall to the point of neutral density, depending on whether the exit port is submerged or at the surface. Usually, when the effluent sinks to the level of density compensation, the phenomenon is referred to as a "sinking plume." Note that effluent dilution by entrainment continually influences the density along the discharge trajectory. Dilution may be sufficiently pronounced to affect the level at which the plume equilibrates. Once the neutral density level has been achieved, the effluent is free to disperse horizontally, regulated by natural diffusion and any residual jet momentum. Under certain conditions the effluent may be more dense than the receiving water and, as a result, flow to the bottom. During the winter months this can occur if the ambient water has equilibrated at a temperature below the point of maximum water density. Should the effluent form a bottom layer, dispersion can be inhibited by bathymetry and bottom friction.

b. Turbulent Diffusion

With the decay of jet momentum, heated effluent increasingly becomes subject to external perturbations of the ambient water body. The influence of advective motions such as currents has been discussed in the preceding section, but of greater importance to heat dispersion is the effect of ambient turbulent diffusion.

Essentially all water motions or transport processes in natural water bodies can be regarded as turbulent; that is, inertial forces in the water dominate over viscous shear forces. The transformation from laminar flow to turbulent flow in a fluid is defined by Reynold's criterion,

$$Re = \frac{\rho v l}{\mu} \qquad \text{(A-4)}$$

where v is fluid velocity, l represents a characteristic dimension (typically the water depth), and μ is dynamic viscosity. The flow is turbulent if Re, the Reynolds number, exceeds some critical value, which in most natural water bodies proves to be small.

Turbulent flow is irregular in time, and any dependent variable of that flow can be characterized by the sum of a mean component and an unsteady component. If the dependent variable happens to be temperature,

$$T = \bar{T} + T' \qquad \text{(A-5)}$$

where T is the instantaneous temperature, $\bar{T}$ corresponds to the mean temperature, and T' is the unsteady component.

Turbulence is manifested by eddy motions, the size of which may vary up to the characteristic dimension of the turbulent medium. Turbulent diffusion by eddies can be expressed as a variation in the unsteady component of the diffusing property. If the diffusing property is heat,

$$\overline{U'T'} = - D_s \frac{d\bar{T}}{ds} \qquad \text{(A-6)}$$

where $\overline{U'T'}$ represents the time-averaged product of the unsteady components of velocity and temperature, $d\bar{T}/ds$ is the mean temperature gradient, and D_s is the eddy diffusion coefficient. All quantities are defined relative to the s direction. The definition for eddy diffusion given in Eq. (A-6) is analogous to that for molecular diffusion in a Fickian substance.

Because turbulent diffusion is scaled to eddies, the magnitude of the eddy diffusion coefficient depends directly upon eddy size, which in turn determines the size of a diffusing plume. As a result, an empirical 4/3 power law of plume width is often applied to estimate horizontal eddy diffusion. The 4/3 law apparently holds only for a semi-infinite water body such as an ocean; for finite systems such as lakes and rivers, a constant horizontal eddy diffusion coefficient may be preferable at large distances from the effluent source.[6] Furthermore, since the eddy spectrum is limited by the size of the system, boundary effects may appreciably diminish horizontal dispersion in near-shore or shallow areas. Similarly, the typical vertical turbulence structure,

[6] G. T. Csanady, "Dispersal of Effluents in the Great Lakes," Water Research, Vol. 4, No. 1, 1970.

with a maximum near the water surface due to wind stress, usually results in decreasing horizontal diffusion with depth.

Whereas horizontal eddy diffusion varies with the scale of horizontal turbulence, eddy diffusion in the vertical direction can be constrained by shallowness of the water body and the interaction between turbulence and buoyancy. Buoyant forces result from ambient thermal stratification and/or heated discharge, and they impose a density stability on the water column which wind-induced turbulence must overcome. The interaction between turbulence and buoyancy is often expressed by the Richardson number,

$$Ri = \frac{\frac{g}{\rho}\frac{d\rho}{dz}}{\left(\frac{d\bar{U}}{dz}\right)^2} \qquad \text{(A-7)}$$

where g represents the acceleration of gravity, $d\rho/dz$ is the vertical density gradient, and $d\bar{U}/dz$ is the vertical gradient of the horizontal mean velocity. The quantity $(1/\rho)(d\rho/dz)$ represents buoyancy or density stratification and is often referred to as the stability, while $d\bar{U}/dz$ can be used as a measure of turbulence. As Eq. (A-7) indicates, the Richardson number increases with increasing stability, implying that vertical eddy diffusion varies inversely with Richardson number. A number of formulations for vertical diffusion as a function of Richardson number have been proposed, but no single relationship has received universal acceptance.

Natural turbulent diffusion occurs over all parts of a heated discharge. However, in the near-field region, mechanically induced turbulence due to jet momentum entrainment is the major source of heat dispersal; in the far-field, surface heat exchange actively transfers heat from surface plumes. Therefore, natural turbulence is often masked by other dispersion mechanisms except in submerged plumes or perhaps the transition regions of surface discharges.

c. **Surface Heat Exchange**

The third factor influencing the thermal dispersion of heated effluents is heat dissipation to the atmosphere from the surface of the water body. Surface cooling is an inherent property of air-water coupling and therefore acts over all portions of a surface effluent. However, since surface heat exchange varies directly with the surface area affected, the process becomes significant only over large plume areas. This condition is satisfied at some distance from the discharge point, after the effluent has been cooled appreciably by jet entrainment and natural turbulent diffusion. Hence, the effects of surface heat exchange need be considered only for the far-field.

The processes that determine the amount of surface cooling from a heated plume are identical to those that prevail under typical ambient conditions. As a result, heating by a discharge can be regarded as a perturbation on the normal thermal regime. Adoption of this viewpoint leads to the concept of excess temperature, T_e, the difference between the observed plume temperature and the natural or ambient water temperature. On the basis of this definition, a heat budget formulation may be applied to excess temperature to yield the rate of heat transfer across the air-water interface due to the plume:

$$H = KT_e \qquad \text{(A-8)}$$

where the heat transfer coefficient, K, is primarily a function of wind speed, ambient temperature, and excess temperature. Tables are available from which K may be estimated, given the appropriate meteorological parameters.[7]

Note that Eq. (A-8) is based upon the premise that the ambient water temperature is equal to the equilibrium value (i.e., the temperature at which the net rate of heat transfer across the air-water interface is zero). This condition is rarely achieved in nature, but the equation offers a simplified means of expressing surface heat exchange independent of the full heat budget equation. Since the error introduced by assuming an equilibrium condition is believed to be inconsequential, any disparities are ignored.

Surface cooling, whereby the excess heat is passed to the atmosphere, is ultimately responsible for limiting the areal extent of heated plumes. If surface cooling were not utilized in far-field modeling, the isotherms of excess temperature would never achieve closure.

[7] D. W. Pritchard and H. H. Carter, "Design and Siting Criteria for Once-Through Cooling Systems Based on a First-Order Thermal Plume Model," Chesapeake Bay Institute Technical Report No. 75, 1972.

APPENDIX B

MATHEMATICAL FORMULATION OF THERMAL DISPERSION

1. Fundamental Equations

All thermal dispersion phenomena are governed by the basic laws of mass, momentum, and energy conservation and an equation of state. Individual models differ in formulation to the extent that approximations and simplifying assumptions are applied to the set of equations expressing these laws.

If one considers a fluid having velocity components u_j ($j = 1, 2, 3$) with density ρ a function of position x_j ($j = 1, 2, 3$), the basic hydrodynamic and thermodynamic equations governing thermal dispersion may be written in Cartesian tensor notation as follows:[1]

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \qquad \text{(B-1)}$$

Conservation of momentum

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} - 2\epsilon_{ijk}\rho u_j \Omega_k$$

$$= \rho X_i \quad - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \qquad \text{(B-2)}$$

where

μ = coefficient of viscosity
ϵ_{ijk} = permutation (cyclic) tensor
Ω_k = the component of the earth's rotation vector in the k direction
p = pressure
X_i = the i component of any external force

Conservation of heat energy (enthalpy)

$$\rho \frac{\partial}{\partial t}(c_v T) + \rho u_j \frac{\partial}{\partial x_j}(c_v T)$$

$$= \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) - p\frac{\partial u_j}{\partial x_j} + \Phi \qquad \text{(B-3)}$$

where

T = temperature
c_v = specific heat at constant volume
k = coefficient of thermal conductivity
Φ = viscous energy dissipation function

The term $p(\partial u_j/\partial x_j)$ represents the increase in internal energy due to compression of the fluid.

Equation of state

If one ignores the effects on density of dissolved solids[2] and restricts attention to temperatures above that of maximum density, an approximate equation of state may be written as

$$\rho = \rho_o\,[1 - \alpha(T - T_o)] \qquad \text{(B-4)}$$

where

T_o = reference temperature at which $\rho = \rho_o$
α = coefficient of thermal expansion.

2. Conservation Equations with the Boussinesq Approximation

The derivation of the above equations was quite general in that the coefficients μ, c_v, α, k, and the density ρ were not assumed to be constant. In any practical situation, however, these quantities are only very slightly temperature dependent and, with one exception, may be treated as constants. The one exception is the external force term, ρX_i, in the momentum equation. Here, the density cannot be treated as a constant. Its variation, when multiplied by X_i, can produce an acceleration comparable in magnitude to that of the inertial term. Hence, μ, c_v, α, and k may be treated as constants wherever they appear, and ρ may be treated as a constant *except when it appears in the external force term*. This treatment of the density variation is the Boussinesq approximation. Its physical significance becomes clear upon consideration that, in practical situations, the external force term is the acceleration of gravity.

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[1] S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability," Oxford University, Oxford at the Clarendon Press, 1961.

[2] Effects of salinity have been ignored for the sake of brevity of discussion. The simplest equation of state, including the salinity, would be of the form $\rho = \rho_o[1 - \alpha(T - T_o) + \beta(S - S_o)]$, where S = salinity, β = coefficient of saline contraction.

Under the Boussinesq approximation, the equation for mass conservation becomes

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad \text{(B-5)}$$

which is the familiar expression of incompressibility.

In the light of Eq. (B-5), the momentum equation becomes

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - 2\varepsilon_{ijk} u_j \Omega_k = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_i} + \left(1 + \frac{\Delta\rho}{\rho_o}\right) X_i + \nu\nabla^2 u_i \qquad \text{(B-6)}$$

where ∇^2 is the Laplacian operator

$$= \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}$$

ν is the coefficient of kinematic viscosity $= \dfrac{\mu}{\rho_o}$

and

$$\Delta\rho = -\rho_o \, \alpha (T - T_o) \qquad \text{(B-7)}$$

The energy equation reduces to

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T \qquad \text{(B-8)}$$

where $\kappa = \dfrac{k}{\rho_o c_v}$ is the thermometric conductivity.

Note that the viscous energy dissipation function, Φ, has been dropped under the Boussinesq approximation. For an incompressible fluid, the dissipation function is given by

$$\Phi = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]^2$$

From Eqs. (B-6) and (B-7), the velocity scale is of the order $(\alpha\Delta T|X|d)^{1/2}$ where d is a characteristic length scale and $|X|$ becomes the acceleration of gravity in any practical problem. Consequently, the scale of Φ is of the order $\mu\alpha\Delta Tg/d$. From Eq. (B-3), the scale of Φ relative to that of diffusion is $\mu\alpha g d/k$, which for typical length scales (d~1 cm) is much smaller than unity. Hence, dropping the dissipation function has negligible effect on the heat energy equation.

Equations (B-5) through (B-8) express the basic conservation laws subject to the Boussinesq approximation.

3. Conservation Equations for Turbulent Flow

The above equations express the basic conservation principles in terms of the instantaneous fields of velocity, temperature, pressure, and density. In any natural system, the flow is turbulent. Because turbulence is inherently random, solution of the above equations is impractical for any real problem. The equations must be subjected to an averaging operation that separates the deterministic and stochastic components of the quantities in question.

Accordingly, the instantaneous quantities, u_i, p, and T, are each written as the sum of an average component (denoted by a bar over the symbol) and an instantaneous fluctuation about the average (denoted by a primed symbol):

$$u_i = \bar{u}_i + u_i'$$

$$p = \bar{p} + p'$$

$$T = \bar{T} + T'$$

where the average is taken over a suitably chosen time period that is small compared with the time scales of interest. Pritchard[3] discusses certain advantages in performing ensemble rather than time averaging. However, the more conventional concept of time averaging is followed here. Consistent with the Boussinesq approximation, the above equations do not contain a term representing fluctuations in density.

If the above expressions are substituted into Eqs. (B-4), (B-5), and (B-6) and the system is averaged, the following conservation equations in mean quantities result:

Equation of state

$$\bar{\rho} = \rho_o \, [1 - \alpha(\bar{T} - T_o)] \qquad \text{(B-9)}$$

Conservation of mass

$$\frac{\partial \bar{u}_j}{\partial x_j} = \frac{\partial u_j'}{\partial x_j} = 0 \qquad \text{(B-10)}$$

Conservation of momentum

$$\frac{\partial \bar{u}_1}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_1}{\partial x_j} - 2\Omega_3 \sin(\phi)\bar{u}_2 = -\frac{1}{\rho_o}\frac{\partial \bar{p}}{\partial x_1} + \nu\nabla^2 \bar{u}_1 - \frac{\partial}{\partial x_j}\overline{(u_1' u_j')} \qquad \text{(B-11)}$$

[3] D. W. Pritchard, "Three-dimensional Models," in Estuarine Modeling: An Assessment, Environmental Protection Agency, Water Pollution Control Research Series, 16070 DZV, 1971.

$$\frac{\partial \bar{u}_2}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_2}{\partial x_j} + 2\Omega_3 \sin(\phi)\bar{u}_1$$

$$= -\frac{1}{\rho_o}\frac{\partial \bar{p}}{\partial x_2} + \nu \nabla^2 \bar{u}_2 - \frac{\partial}{\partial x_j}(\overline{u_2' u_j'}) \quad \text{(B-12)}$$

$$\frac{\partial \bar{u}_3}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_3}{\partial x_j}$$

$$= -\frac{1}{\rho_o}\frac{\partial \bar{p}}{\partial x_3} - \frac{\rho}{\rho_o} g + \nu \nabla^2 \bar{u}_3 - \frac{\partial}{\partial x_j}(\overline{u_3' u_j'}) \quad \text{(B-13)}$$

The momentum equations have been written in component form for a Cartesian system with x_3 positive upward. The quantities ϕ and Ω_3 are, respectively, the latitude and the locally vertical component of the earth's rotation vector. Rotational effects are negligible in the vertical momentum equation, and the corresponding terms have been dropped for simplicity.

Conservation of energy

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_j \frac{\partial \bar{T}}{\partial x_j} = \kappa \nabla^2 \bar{T} - \frac{\partial}{\partial x_j}(\overline{u_j' T'}) \quad \text{(B-14)}$$

Equations (B-9) through (B-14) differ in form from their instantaneous counterparts only through inclusion of the terms representing time averages of products of fluctuating quantities. These terms represent turbulent diffusion of heat and momentum and arise from those stochastic components of the motion that have time scales shorter than the averaging period.

These equations, together with appropriate boundary conditions, are the fundamental relationships governing thermal dispersion in a turbulent incompressible medium and form the basis for any subsequent deterministic mathematical models.

4. Approximations to the Basic Equations

The conservation equations cannot be solved in their complete forms as shown. The actual formulation of any given model depends upon the simplifications and assumptions invoked consistent with the time and space scales of interest in the dispersion process, plant design parameters, and the flow field and geometry of the receiving waters. For highly simplified geometry and flow conditions, analytical solutions are possible in some instances. In general, however, more realistic models require solution by numerical methods.

The more common simplifying assumptions are enumerated and discussed briefly below.

a. Eddy Coefficients

The velocity correlation terms on the right-hand side of Eqs. (B-11) through (B-13) may be interpreted as the nine components of the turbulent stress tensor R_{ij}

$$R_{ij} = -\rho\overline{u_i' u_j'} \quad \text{(B-15)}$$

The classical approximation is to relate the turbulent momentum flux in Eq. (B-15) to gradients of the mean velocity field, giving rise to the concept of eddy viscosity coefficients. These may be introduced through expressions of the form

$$R_{ij} = -\rho\overline{u_i' u_j'} = A(i)\frac{\partial \bar{u}_j}{\partial x_i} + A(j)\frac{\partial \bar{u}_i}{\partial x_j} \quad \text{(B-16)}$$

where A(i) and A(j) are interpreted as the lateral eddy viscosity coefficient A_h for $j \neq 3$, and the vertical eddy viscosity coefficient A_v for $j = 3$. The anisotropy of turbulent diffusion, discussed in Appendix A, leads to the use of separate coefficients for the horizontal and vertical directions.

Similarly, in Eq. (B-14) it is customary to relate the turbulent diffusion of heat energy to gradients of the mean temperature, giving rise to eddy diffusion coefficients D_j for heat:

$$\overline{u_j' T'} = -D_j \frac{\partial \bar{T}}{\partial x_j} \quad \text{(B-17)}$$

The turbulent momentum and heat transports shown above are much larger than their molecular counterparts, and the terms representing the latter may be dropped from Eqs. (B-11) through (B-14).

If the above approximations are used, the velocity and temperature appear explicitly in the conservation equations only as mean values. The effects of turbulence are concealed in the eddy coefficients A_v, A_h, D_j. This simplification is to a certain extent illusory since the eddy coefficients are functions of the stochastic part of the motion and cannot be determined *a priori*. The usefulness of any predictive model using eddy coefficients is severely limited by the reliability with which the magnitudes and spatial variations of these parameters can be determined beforehand.

b. Steady State

For analytical simplicity, thermal dispersion models often assume that the velocity and temperature fields are in steady state, i.e., that

$$\frac{\partial \bar{u}_i}{\partial t} = \frac{\partial \bar{T}}{\partial t} = 0$$

The validity of this assumption depends upon a careful assessment of the relative magnitudes of the averaging period, the important time scales of variability in the velocity and temperature fields, and the time period over which the model is intended to apply. For example, the assumption of a steady state clearly cannot be made for a model intended to describe the behavior of a thermal plume in a tidal river or estuary if the averaging period is short compared with the tidal period. If, on the other hand, the average is taken over several tidal cycles, and if the time period of interest is less than the scale of seasonal fluctuations in fresh water flow (river flow), a steady state assumption is reasonable. It is important to realize, however, that in these two examples, the magnitudes of the eddy coefficients differ greatly. For a longer averaging period, larger scales of motion are included in the turbulence, and eddy coefficients are larger.

c. Hydrostatic Approximation

In a motionless fluid, the vertical equation of motion reduces to the hydrostatic equation expressing an exact balance between the vertical pressure gradient and the acceleration of gravity,

$$-\frac{1}{\rho_\infty}\frac{\partial \bar{p}}{\partial x_3} - g = 0$$

where ρ_∞ is a function of x_3 and is the density that would exist in the absence of motion. Whether or not this equation is valid as an approximation to a real flow system depends on the velocity, space, and time scales of concern in the vertical momentum equation. A simple scale analysis indicates that the hydrostatic assumption is generally a valid approximation except possibly in the treatment of high-velocity discharges, in which case the pressure field has a dynamic contribution that is not necessarily small compared with the hydrostatic component.

In general, then, the vertical pressure gradient in Eq. (B-13) may be written

$$\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x_3} = -\frac{\rho_\infty}{\rho_0}g + \frac{1}{\rho}\frac{\partial \bar{p}^*}{\partial x_3} \qquad \text{(B-18)}$$

where $\bar{p}^*$ is the difference between the actual pressure and the hydrostatic pressure, and ρ_∞ g is the gradient of hydrostatic pressure. Substitution of Eq. (B-18) into Eq. (B-13) results in a vertical equation of motion in which buoyancy is contained explicitly in the gravity term:

$$\frac{\partial \bar{u}_3}{\partial t} + \bar{u}_j\frac{\partial \bar{u}_3}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial \bar{p}^*}{\partial x_3} - \frac{\rho - \rho_\infty}{\rho_0}g + \nu\nabla^2\bar{u}_3 - \frac{\partial}{\partial x_j}(\overline{u_3' u_j'}) \qquad \text{(B-19)}$$

d. Planetary Rotation

A simple scale analysis can provide an estimate of the importance of rotational effects for a particular dispersion problem. Theoretically, a parcel of water with an initial velocity V_0 moves in an inertia circle in the absence of external forces and confining boundaries. The radius of the inertia circle at mean latitude ϕ_0 is $V_0/2\Omega_3 \sin\phi_0$. The time required for the water to move around the circle (i.e., period) is $2\pi/2\Omega_3 \sin\phi_0$. If, for a particular problem, the distance and time scales of interest are small compared with the circumference and period of the local inertia circle, rotational effects may be neglected.

In the near-field, neglect of the Coriolis force is a valid approximation. In the far-field, Coriolis effects might become noticeable as a *cum sol* deflection of the thermal plume. The practical importance of the latter depends upon the lateral dimensions of the receiving water and the time scale of temperature decay relative to the inertia period.

e. Heat Exchange Coefficient

Equation (B-14) must satisfy the boundary condition that the heat flux is continuous across the water surface. On the basis of Eq. (B-17), this boundary condition may be written

$$D_3\frac{\partial \bar{T}}{\partial x_3}\bigg|_{x_3=0} = \text{surface heat flux.}$$

As discussed in Appendix A, the surface heat flux is taken to be proportional to the product of a surface heat exchange coefficient K and the difference between the actual temperature of the water surface $\bar{T}_s$ and its equilibrium temperature Θ. The above boundary condition then becomes

$$D_3\frac{\partial \bar{T}}{\partial x_3}\bigg|_{x_3=0} = K(\bar{T}_s - \Theta) \qquad \text{(B-20)}$$

Equations (B-14), (B-17), and (B-20) describe the temperature field in the presence of a thermal discharge. In the absence of a discharge, the equations would be identical in form, with the ambient temperature $\bar{T}_a$ replacing the general temperature $\bar{T}$. Note that the difference $\bar{T} - \bar{T}_a = T_e$ is the excess temperature defined in Appendix A. If the assumptions are made that (1) the spatial gradients of ambient temperature are small compared with those existing in the presence of the discharge, and (2) the heat exchange coefficient and the equilibrium temperature are the same with or without the discharge, then the following equation may be written to describe the excess temperature:

$$\frac{\partial T_e}{\partial t} + \bar{u}_j\frac{\partial T_e}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_j\frac{\partial T_e}{\partial x_j}\right) \qquad \text{(B-21)}$$

with the surface boundary condition

$$D_3\frac{\partial T_e}{\partial x_3}\bigg|_{x_3=0} = K(\bar{T}_s - T_a) = KT_e \qquad \text{(B-22)}$$

The form of Eq. (B-22), with the ambient temperature replacing the equilibrium temperature, is identical to that discussed in Section 3c of Appendix A. This form is preferred by many workers because of the practical difficulty of obtaining the equilibrium temperature. It should be noted, however, that this simpler form depends upon the validity of the two assumptions given above.

5. Techniques for Solving the Hydrodynamic and Thermodynamic Equations

The mathematical form of the hydrodynamic and thermodynamic equations that govern thermal dispersion having been discussed, it is desirable to review the methods most frequently applied to solve those equations. For the most part, such methods tend to be consequences of the scale size and simplifying approximations used to construct the model. The following discussion deals with common solution techniques as they are applied to the spatial dispersion fields of thermal discharges.

a. Near-Field Modeling

The case of a buoyant axisymmetric jet discharging into a cross-flowing ambient stream, while representing only one of a variety of possible discharge configurations, contains the important dynamical processes and illustrates the basic assumptions and techniques generally used in near-field modeling. The transformation of Eqs. (B-9) through (B-14) into simpler forms suitable for the axisymmetric jet is straightforward but tedious and is not shown here. A complete derivation of the governing equations has been given by Hirst,[4] whose notation, with slight modification, is used below. The important point to note is that the near-field approximation is basically equivalent to the Prandtl boundary layer approximation applied to free turbulent shear flows, viz., streamwise gradients are much smaller than radial gradients, streamwise velocities are much larger than radial velocities, and the deviation from hydrostatic pressure is approximately constant throughout the jet.

Subject to these approximations, the conservation equations may be written in a natural cylindrical coordinate system (s,r) denoting the local axial and radial directions, respectively. The angular dependence does not appear since the jet is axisymmetric. The equations of conservation of mass, heat energy, and axial momentum become

$$\frac{\partial \bar{u}}{\partial s} + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}) = 0 \qquad \text{(B-23)}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial s} + \bar{v}\frac{\partial \bar{T}}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}(r\overline{v'T'}) \qquad \text{(B-24)}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial s} + \bar{v}\frac{\partial \bar{u}}{\partial r} = \left[\frac{\rho_\infty - \bar{\rho}}{\rho_o}\right]_s g - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'v'}) \qquad \text{(B-25)}$$

where $\bar{u}$ = time-averaged axial velocity component
$\bar{v}$ = time-averaged radial velocity component
$\left[\frac{\rho_\infty - \bar{\rho}}{\rho_o}\right]_s g$ = the axial component of buoyancy acceleration.

Only the axial momentum equation is shown here, since this is sufficient to illustrate the approach used in near-field analysis. This approach consists essentially of three steps designed to reduce the complete set of partial differential equations to a set of ordinary differential equations expressing the centerline velocity components, temperature, and Cartesian position as functions of s. These three steps consist of (1) removal of the r-dependence by integration of the conservation equations over the jet cross section, (2) specification of the radial profiles of temperature and velocity, and (3) introduction of an entrainment function.

The radial integration of the first step places the equations in so-called "integral" form. These are not integral equations in the strict mathematical sense, but rather in the sense that radial distributions of properties have been integrated away. The simpler set of ordinary equations expresses only bulk properties of the flow as functions of jet trajectory. Retrieval of the lost details requires the second step, in which radial distributions of velocity and temperature are specified. These are usually assumed to have a Gaussian form, as discussed in Section 3a of Appendix A. The third step is an attempt to express the radial flux of dilution water into the jet in terms of the axial flow.

In accordance with the first step above, integration of Eqs. (B-23) through (B-25) over the area gives

$$\frac{d}{ds}\left[2\pi\int_0^\infty \bar{u}\, r\, dr\right] = \frac{dQ}{ds} = -2\pi \lim_{r\to\infty}(r\bar{v}) = E \qquad \text{(B-26)}$$

$$\frac{d}{ds}\left[2\pi\int_0^\infty \bar{u}(\bar{T} - T_\infty)\, r\, dr\right] = \frac{dG}{ds}$$

$$= -Q\frac{dT_\infty}{ds} - 2\pi \lim_{r\to\infty}(r\overline{v'T'}) \qquad \text{(B-27)}$$

$$\frac{d}{ds}\left[2\pi\int_0^\infty \bar{u}^2\, r\, dr\right] = \frac{dM_s}{ds}$$

$$= \left[2\pi\int_0^\infty \alpha g(\bar{T} - T_\infty)\, r\, dr\right]_s$$

[4] E. A. Hirst, "Analysis of Round, Turbulent, Buoyant Jets Discharged to Flowing Ambients," ORNL-4685, Oak Ridge National Laboratory, 1971.

$$+ E\,[u_\infty]_s - 2\pi \lim_{r\to\infty} (\overline{r u' v'}) \qquad \text{(B-28)}$$

where the subscript s indicates a component in the s-direction, and the upper integration limit, ∞, for practical purposes, is the physical edge of the jet.

Equation (B-26) represents conservation of mass and states that the rate of change of volume flux Q along the jet trajectory s is equal to the rate of entrainment

$$-2\pi \lim_{r\to\infty} (r\overline{v}) = E$$

of fluid at the jet edge.

Equation (B-27) states that the rate of change of flux of excess temperature G along the jet trajectory is balanced by the rate of entrainment of fluid of ambient temperature T_∞ and the turbulent flux

$$-2\pi \lim_{r\to\infty} (\overline{r v' T'})$$

of heat energy at the jet edge.

Equation (B-28) represents the conservation of bulk momentum. It states that the rate of change of s-momentum flux M_s along the jet path is balanced by (1) the s-component of buoyancy (with temperature replacing density), (2) the entrainment $E[u_\infty]_s$ of ambient momentum by the mean radial flow at the jet edge, and (3) the turbulent flux

$$-2\pi \lim_{r\to\infty} (\overline{r u' v'})$$

of momentum at the jet edge. Note that in the buoyancy term, the equation of state has been used to express the density in terms of the temperature.

Closure of the set of conservation equations requires that the radial distributions of jet velocity and temperature (step 2) and the entrainment function (step 3) be specified.

Beyond the zone of flow establishment (the region in which the jet possesses a potential core), all radial profiles are assumed to have Gaussian forms, as discussed in Appendix A:

$$\overline{u} = (u_m - [u_\infty]_s)\, e^{-(r/b)^2} + [u_\infty]_s$$

$$\overline{T} - T_\infty = (T_m - T_\infty)\, e^{-(r/\lambda b)^2} \qquad \text{(B-29)}$$

where u_m and T_m represent centerline values, b is the jet half-width ($b = \sigma\sqrt{2}$), λ is the spreading ratio introduced in Appendix A, and $[u_\infty]_s$ is the component of ambient velocity in the s-direction. Substitution of Eq. (B-29) into Eqs. (B-26) through (B-28) results in a set of equations in which the unknowns are u_m, T_m, and b (not considered here are the three additional equations relating the jet position to the fixed Cartesian coordinates).

In the third step, the entrainment function must be expressed in terms of the centerline value of jet velocity, u_m. The simplest relationship that has been used is of the form,

$$E = a u_m \;, \qquad \text{(B-30)}$$

which states that the entrainment rate is proportional to the centerline jet velocity. The constant of proportionality a is known as the entrainment coefficient. Such a relationship, at best, satisfies the intuitive expectation that a fast jet will have a higher entrainment rate than a slow jet. However, the correct choice of a for any given problem is open to considerable question. It is also unlikely that a simple proportional relationship such as Eq. (B-30) applies in nature. Considerable theoretical and experimental work over the past decade has resulted in increasingly more complex (and hopefully realistic) forms for the entrainment function. Hirst[4] presents the most general expression to date:

$$E = \left(0.057 + \frac{0.97}{F_L} \sin\theta_2\right)$$

$$\left[b\,|\,u_m - [u_\infty]_s\,| + 9.0 b \sqrt{u_\infty^2 - [u_\infty]_s^2}\right] \qquad \text{(B-31)}$$

where θ_2 is the elevation angle of the plume above the horizontal and F_L is the local densimetric Froude number. This expression is similar to but considerably more sophisticated than that shown in Eq. (B-30). From the Froude number and angular dependence, it is seen that the "entrainment coefficient" is a dynamic quantity that changes with the evolution of the jet. Also, the second term in braces gives an enhanced entrainment rate for the case of the jet and ambient flow not being collinear.

In keeping with the usual definition of the near-field as being the region where dilution by jet entrainment dominates the dilution due to turbulent diffusion, terms describing the latter are usually dropped from Eqs. (B-27) and (B-28). Formally, this amounts to the approximations

$$2\pi \lim_{r\to\infty} (\overline{r v' T'}) \ll Q \frac{dT_\infty}{ds}$$

$$2\pi \lim_{r\to\infty} (\overline{r u' v'}) \ll E[u_\infty]_s$$

However, since the turbulent transport terms involve undetermined eddy coefficients and since the form of the entrainment function E is not well established, a proper scale analysis to substantiate the above inequalities is very difficult. On physical grounds, it does not appear to be reasonable that the entrainment is always independent of the ambient turbulence level.

b. Far-Field Modeling

The far-field is generally assumed to be that region of the thermal plume sufficiently removed from the discharge that (1) ambient turbulence dominates the mixing process and (2) the excess temperature is transported by the ambient flow as a passive contaminant.

The first of these assumptions results in the use of eddy coefficients to represent the mixing. This introduces an indeterminancy into the problem in that the values of these coefficients are not known beforehand.

The second assumption eliminates the effects of discharge momentum and removes the nonlinearity due to buoyancy coupling between the heat energy and momentum equations. This results in a considerable mathematical simplification in that the convective velocity field in Eq. (B-14) is known through either direct observation or prior solution of the appropriately simplified momentum equations.

For discussion purposes, mathematical models of the far-field region may be divided roughly into deterministic, stochastic, and phenomenological types.

(1) Deterministic Models

Since the actual motion and temperature always have stochastic components, deterministic methods must relate these components to time-averaged quantities in order that closure of the governing equations can be achieved. This having been done, the solutions to the basic hydrothermal equations may be determined either analytically or numerically.

Analytical solution refers to the closed form integration of the governing equations. As discussed earlier, this method is possible only for highly simplified cases. It is seldom possible to obtain analytical solutions for time-dependent flow fields or complex receiving water geometry. Consequently, the utility of any analytical solution should be very carefully assessed by the modeler to ascertain the conditions under which the model might be a valid predictive tool. From a practical point of view, the attractiveness and elegance of analytical solutions are often vitiated by the fact that the models from which they stem have been simplified to the point that they no longer adequately simulate the prototype. Hence, in predictive far-field modeling of complex systems, physically realistic solutions might require the rather early abandonment of analytical solutions in favor of numerical methods.

The most widely used numerical solution technique for far-field model equations is the method of finite differences. The fundamental principle of this method is the subdivision of the solution region into a number of discrete grid points at which the derivatives in the governing equations are approximated by finite differences. There are several approximation schemes in current use, the most popular of which are truncated Taylor's series and the treatment of individual grid meshes as discrete control volumes. Sophisticated forms of finite difference techniques permit variable time steps and grid mesh sizes. This refinement provides considerable computational efficiency and added flexibility in the solution of time-dependent problems with irregular shoreline geometry.

In recent years, a somewhat different technique known as the method of finite elements has emerged as a powerful tool for the numerical solution of hydrodynamic transport equations. In this method, the domain of interest is subdivided into a number of "finite elements" interconnected at a discrete number of nodal points. Within each element the dependent variables are approximated by known shape functions whose magnitudes are determined by assumed nodal values. In early applications of the finite element method, solutions were obtained from a discrete set of linear algebraic equations derived through minimization of a functional for the governing differential equation. The application of this method to hydrothermal problems was limited because it was not always possible to find the proper functional.

Recently, however, this restriction has been removed through use of Galerkin's method,[5] in which the set of algebraic equations is obtained directly from the governing differential equation. The approximate solution is obtained not by a variational principle, but rather by orthogonalization of the solution error with respect to the known shape functions.

The finite elements, usually triangular in shape, may be arbitrary in size and arrangement. This flexibility provides a twofold advantage in that computational resolution can be varied at will throughout the region of interest, and almost any boundary shape can be approximated by the proper choice of triangular elements.

(2) Stochastic Models

In stochastic dispersion models, the probabilistic behavior of the flow field is handled directly. The most promising stochastic solution technique is the Monte Carlo method, in which the probabilistic behavior of the

[5] L. A. Loziuk, J. C. Anderson, and T. Belytschko, "Hydrothermal Analysis by the Finite Element Method," ASCE Journal of Hydraulics Division, Vol. 98, No. HY11, November 1972.

flow field is modeled directly by the assignment of stochastic properties to dispersing "packets" of particles as they are tracked from the release point. The excess temperature within any given region is proportional to the local particle density.

The model associates a temperature contribution, a deterministic velocity, and a dispersion rate with each moving particle. As each particle is tracked over a given time interval, it is assigned a total displacement consisting of the sum of two vectors. The first vector displacement is caused by the known deterministic flow field and is given by $[\bar{u}_1\Delta t, \bar{u}_2\Delta t]$. The second displacement, which represents dispersion, is a random Gaussian function with zero mean and is derived from the relationship of the eddy coefficient D_i to the dispersion rate:

$$D_i = \frac{1}{2}\frac{d}{dt}(\sigma_i^2) \qquad \text{(B-32)}$$

where σ_i^2 is the particle variance in the i-direction.

From Eq. (B-32) the random displacement vector is given by $[(2D_1\Delta t)^{1/2}, (2D_2\Delta t)^{1/2}]$.

By use of a digital computer, many particles are released from the source, allowing the stochastic behavior of each particle to be simulated until it passes beyond the region of interest.

The Monte Carlo method has the advantage of direct dispersion simulation, and also the very appealing features of conceptual and programmatic simplicity. It should be noted, however, that with this technique it is still necessary to specify eddy diffusion coefficients.

(3) Phenomenological Models

In the sense used here, phenomenological far-field models differ from either deterministic or stochastic models in that they do not derive directly from solution of the basic transport equations, but rather from a combination of theory and correlations of observed plume behavior with known laboratory and field flow conditions. These models are relatively lacking in mathematical elegance and sophistication, and it is difficult to extract basic information from them concerning the relative importance of individual transport processes within the framework of the governing equations. However, an important advantage is that they are relatively easy to use and are based on a compilation of observed phenomena and a minimum number of simplifying assumptions.